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GLOBAL EXISTENCE FOR NEUTRAL PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY

SYLVAIN KOUMLA[1,*], NELIO N'DOGOTAR[2], BOUBACAR DIAO[3]

[1]Department of Mathematics, Adam Barka University, Abeche, BP. 1173, Chad

[2]Department of Mathematics, University of Sarh, Sarh, BP. 105, Chad

[3]Department of Mathematics, Ckeikh Anta Diop University, Dakar, BP. 5005, Senegal

Abstract. The aim of this work is to prove some results about the existence and regularity of solutions for some neutral partial functional integrodifferential equations with infinite delay in Banach spaces. The results are based on semigroup theory and on Banach's fixed point theorems. The method used treats the equations in its subspace $\mathscr{D}(A)$. One example is given to illustrate the theory.

Keywords: global solutions; neutral equations; partial integrodifferential equations; c_0-semigroups; fixed point theorem; infinitesimal generator; infinite delay; phase space.

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1. INTRODUCTION

In this paper, we are interested in the existence and regularity of solutions for the following neutral partial functional integrodifferential equation with infinite delay

$$(1)\qquad \begin{cases} \dfrac{d}{dt}\mathfrak{g}(t,u_t) &= A\mathfrak{g}(t,u_t)+\displaystyle\int_0^t \mathscr{K}(t-s,\mathfrak{g}(s,u_s))ds+F(t,u_t), \quad t\geq 0 \\ u_0 &= \varphi\in\mathscr{B}, \end{cases}$$

*Corresponding author

E-mail address: skoumla@gmail.com

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where A is the infinitesimal generator of a strongly continuous semigroup of $(T(t))_{t\geq 0}$ on a Banach space E with domain $\mathscr{D}(A)$, and $\mathscr{K}$ is, in general, a nonlinear operator from $\mathbb{R}^+ \times \mathscr{D}(A)$ to E. The phase space $\mathscr{B}$ is the space of continuous functions from $]-\infty,0]$ into $\mathscr{D}(A)$, where $\mathscr{D}(A)$ is endowed with the graph norm, namely for $x \in \mathscr{D}(A)$, $|x|_{\mathscr{D}(A)} = |x|_E + |Ax|_E$. His know that $\left(\mathscr{D}(A), |.|_{\mathscr{D}(A)}\right)$ is a Banach space. Also $\mathfrak{g}$ is a function defined from $\mathbb{R}^+ \times \mathscr{C}$ into $\mathscr{D}(A)$ by

$$\mathfrak{g}(t,\varphi) = \varphi(0) - G(t,\varphi), \tag{2}$$

and $G,\ F$ are two continuous functions from $\mathbb{R}^+ \times \mathscr{B}$ into E. For $u \in \mathscr{B}$ and for every $t \geq 0$, the history function $u_t \in \mathscr{B}$ is defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for} \quad \theta \in]-\infty,0].$$

In the case where $\mathscr{K} = 0$ and A is the infinitesimal generator of a strongly continuous semigroup, the mild solution of $Eq.(1)$ is given by the following variation of constants formula

$$\mathfrak{g}(t,u_t) = T(t)\mathfrak{g}(0,\varphi) + \int_0^t T(t-s)F(s,u_s)ds \quad for \quad t \geq 0.$$

In the literature devoted to partial functional integrodifferential equations with finite delay, the phase space is often the space of all continuous functions on $[-r,0],\ r > 0$, endowed with the uniform norm topology. When the delay is infinite, the notion of the phase space $\mathscr{B}$ plays an important role in the study of both qualitative and quantitative theory, so a usual choice is a normed space satisfying suitable axioms, which was introduced by Hale and Kato [10] (see also Kappel and Schappacher [13] and Chumacher [21]). For detailed discussion on this topic, we refer the reader to the book by Hino and *al.* [12].

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last few decades. In the following, we provide several examples studied in the literature resulting from various physical systems. The work for neutral functional differential equations with infinite delay was initiated by Hernàndez and Henriquez in [11]. $Eq.(1)$ have been investigated by many authors using the semigroups theory; many results of existence, regularity of solutions have been obtained. For more details, we refer the reader to $[5,6,16]$. Recently, Koumla and *al.*, in $[17]$, are investigated several results on the global existence, uniqueness of mild solutions of $Eq.(1)$ with values in the Banach space and in its subspace $\mathscr{D}(A)$. The results are based on Banach's and Schauder's fixed point theorems and on the technique of equivalent norms.

Our work has a double motivated. First, it is motivated by the first author's recent paper $[5]$, which mainly inspires the operator technique of proof, and secondly, by the paper of Webb $[22]$ for the class of integrodifferential equation in the following form

$$\begin{cases} u'(t) &= Au(t) + \int_0^t g(t-s, u(s))\,ds + f(t) \quad \text{for} \quad t \geq 0, \\ u_0 &= x \in \mathscr{X}. \end{cases} \tag{3}$$

The goal of this work is to extend this problem to neutral type equations and to discuss the existence and regularity results of solutions for $Eq.(1)$ by using the semigroup theory. The result obtained is a generalization and a continuation of Webb $[22]$.

The paper is organized as follows. In Section II, we recall some preliminary results of $Eq.(3)$. In Section III, we study the existence of mild solutions for the neutral system (1) using the theory of semigroup and Banach's fixed point theorem. Sufficient conditions for the existence of mild and strict solutions are also established. Finally, we present in Section IV an example which illustrates our results.

2. PRELIMINARIES

In this section, we recall some notions and results that we need in the following. Throughout the paper, E is a Banach space, $A : \mathscr{D}(A) \subset E \to E$ is closed linear operator which generates a c_0-semigroup $(T(t))_{t\geq 0}$ on E. For more details, we refer to [20].

Recall that for such a semigroup, there exists $M > 0$ and $\omega \in \mathbb{R}$ such that

$$|T(t)| \leq Me^{\omega t}, \quad t \geq 0,$$

where $|T(t)|$ is the norm of the bounded linear operator $T(t)$.

We denote by Y the space $\mathscr{D}(A)$ equipped with the graph norm defined by

$$|y|_{\mathscr{D}(A)} = |y|_E + |Ay|_E .$$

It is well-known that $\mathscr{D}(A)$ equipped with norm $|\cdot|_{\mathscr{D}(A)}$ is a Banach space.

Definition 2.1. A continuous function $u : [0,+\infty[\to E$ is said to be a strict solution of $Eq.(3)$ if (i) $u \in \mathscr{C}^1([0,+\infty[;E)\cap\mathscr{C}([0,+\infty[;Y)$; (ii) u satisfies $Eq.(3)$ for all $t \geq 0$.

Remark 1. *From this definition, we deduce that $u(t) \in Y$ and the function $s \mapsto g(t-s,u(s))$ is integrable on $[0,t]$ for every $t \geq 0$.*

Proposition 2.2. [22] *If u is a strict solution of $Eq.(3)$, then u satisfies the integral equation*

$$u(t) = T(t)x + \int_0^t T(t-s)\int_0^s g(s-r,u(r))\,drds + \int_0^t T(t-s)f(s)ds. \tag{4}$$

Remark 2. *If u satisfies the equation $Eq.(4)$, it is not necessarily a strict solution. That is why we give the next definition.*

Definition 2.3. *A continuous function $u : [0,+\infty[\to E$ is called a mild solution of $Eq.(3)$ if it $Eq.(4)$.*

In this work, we assume that the phase space $(\mathscr{B},|.|_{\mathscr{B}})$ is a normed linear space of functions mapping $]-\infty,0]$ into $\mathscr{D}(A)$ and satisfying the following fundamental axioms (cf. Hale and Kato in [10]).

$(\mathbf{A_1})$ There exist positive constant H and functions $\mu(.),\eta(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with μ continuous and η locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $u :]-\infty,\sigma+a] \to \mathscr{D}(A)$, $u_\sigma \in \mathscr{B}$, and $u(.)$ is continuous on $[\sigma,\sigma+a]$, then for every $t \in [\sigma,\sigma+a]$ the following conditions holds

$(\mathbf{i})$ $u_t \in \mathscr{B}$,

$(\mathbf{ii})$ $|u(t)|_Y \leq \mu\,|u_t|_{\mathscr{B}}$, which is equivalent to $|\varphi(0)|_Y \leq \mu\,|\varphi|_{\mathscr{B}}$ for every $\varphi \in \mathscr{B}$,

$(\mathbf{iii})$ $|u_t|_{\mathscr{B}} \leq \mu(t-\sigma)\sup_{\sigma\leq s\leq t}|u(s)|_Y + \eta(t-\sigma)\,|u_t|_{\mathscr{B}}$,

$(\mathbf{A_2})$ For the function $u(.)$ in $(\mathbf{A_1})$, $t \mapsto u_t$ is a $\mathscr{B}-$valued continuous function for $t \in [\sigma,\sigma+a]$.

$(\mathbf{B})$ The space $\mathscr{B}$ is a Banach space.

$(\mathbf{H_0})$ A is the infinitesimal generator of a c_0-semigroup $(T(t))_{t\geq 0}$ on E.

3. MAIN RESULTS

In this section, we prove the global existence, uniqueness and regularity of the solution to $Eq.(1)$. Firstly, we show the existence and uniqueness of the mild solution. Secondly, we give sufficient conditions ensuring that the mild solution is a strict solution of the problem, in the sens of the following definition.

3.1 Global existence and regularity of solutions

3.1.1 Global existence of solutions

Definition 3.1. *We say that a continuous function* $u :]-\infty, +\infty[\to Y$ *is a strict solution of* $Eq.(1)$ *if the following conditions hold*

(i) $u \in \mathscr{C}^1([0,+\infty[, E) \cap \mathscr{C}([0,+\infty[, Y)$;

(ii) *u satisfies* $Eq.(1)$ *on* $[0,+\infty[$;

(iii) $u(\theta) = \varphi(\theta)$ *for* $-\infty \leq \theta \leq 0$.

Remark 3. *Form this definition, we deduce that* $u(t) \in Y$ *and the function* $s \mapsto \mathfrak{g}(t-s, u(s))$ *is integrable on* $[0,t]$ *for the every* $t \geq 0$.

From Proposition II.2 we have the following.

Proposition 3.2. *If u is a strict solution of* $Eq.(1)$, *then u satisfies the integral equation*

$$(5) \qquad \begin{aligned} u(t) &= T(t)\mathfrak{g}(0,\varphi) + G(t,u_t) + \int_0^t T(t-s)\int_0^s \mathscr{K}(s-r, \mathfrak{g}(r,u_r))drds \\ &+ \int_0^t T(t-s)F(s,u_s)ds. \end{aligned}$$

Proof. *It is just a consequence of Proposition. 2.3 and of* $Eq.(3)$ ◊

Remark 4. *The converse in not true. In fact if u satisfies* $Eq.(5)$, *u it is necessary to be a strict solution. That is why we give the following definition.*

Definition 3.3. *We say that a continuous function* $u :]-\infty, +\infty[\to Y$ *is a mild solution of* $Eq.(1)$ *if it satisfies the equation* $Eq.(5)$.

In addition, we introduce the following hypotheses:

$(\mathbf{H_1})$ $F, G : \mathbb{R}^+ \times \mathscr{B} \to Y$ are continuous and lipschitzian with respect to the second argument, namely, then exist constants $L_F > 0$ and $L_G > 0$ such that

$$|F(t,u) - F(t,v)|_Y \leq L_F \, |u - v|_{\mathscr{B}} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad u, v \in \mathscr{B},$$

$$|G(t,u) - G(t,v)|_Y \leq L_G \, |u - v|_{\mathscr{B}} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad u, v \in \mathscr{B}.$$

$(\mathbf{H_2})$ The derivative $\frac{\partial \mathscr{K}}{\partial t}(t,u)$ exists and is continuous from $\mathbb{R}^+ \times Y$ into E, moreover there exist continuous and increasing functions $b : \mathbb{R}^+ \to \mathbb{R}^+$ and $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

$$|\mathscr{K}(s,u) - \mathscr{K}(s,v)|_E \leq b(s) \, |u - v|_Y$$

and

$$\left| \frac{\partial \mathscr{K}}{\partial s}(s,u) - \frac{\partial \mathscr{K}}{\partial s}(s,v) \right|_E \leq c(s) \, |u - v|_Y$$

for all $s \in \mathbb{R}+$, and $u, v \in Y$.

Theorem 3.4. *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. If* $\varphi \in \mathscr{B}$, *then there exist a unique continuous function* $u :]-\infty, +\infty[\to Y$ *which is a mild solution of* $Eq.(1)$

Proof. We define the set $[M_{t_1}(\varphi) := \{\mathrm{u} \in \mathscr{C}([0,t_1]; \mathrm{Y}) : u(0) = \varphi(0)\}]$. $M_{t_1}(\varphi)$ is a closed subset of $\mathscr{C}([0,t_1]; \mathrm{Y})$, where $\mathscr{C}([0,t_1]; \mathrm{Y})$ is the space of continuous functions from $[0,t_1]$ to Y equipped with the uniform norm topology. Next, for each $u \in M_{t_1}(\varphi)$ we define

$$\tilde{u}(t) = \begin{cases} \varphi(t) & \text{for} \quad t \in (-\infty, 0], \\ u(t) & \text{for} \quad t \in [0, t_1]. \end{cases}$$

Consider the operator $\mathrm{P} : M_{t_1}(\varphi) \to \mathscr{C}((-\infty, 0], \mathrm{E})$ defined by

$$\begin{aligned} (Pu)(t) \quad &= G(t, \tilde{u}_t) + T(t)\mathfrak{g}(0, \varphi) + \int_0^t T(t-s) \int_0^s \mathscr{K}(s - r, \mathfrak{g}(r, \tilde{u}_r)) \, dr \\ &+ \int_0^t T(t-s) F(s, \tilde{u}_s) ds. \end{aligned}$$

The first steep is to show that $P(M_{t_1}(\varphi)) \subset M_{t_1}(\varphi)$. On the other hand we have

$$
\begin{aligned}
(APu)(t) &= AG(t,\tilde{u}_t) + AT(t)\mathfrak{g}(0,\varphi) + A\int_0^t T(t-s)\int_0^s \mathscr{K}(s-r,\mathfrak{g}(r,\tilde{u}_r))drds \\
&+ A\int_0^t T(t-s)F(s,\tilde{u}_s)ds \quad 0 \le t \le t_1.
\end{aligned}
$$

Since A is closed, then

$$
\begin{aligned}
(APu)(t) &= AG(t,\tilde{u}_t) + AT(t)\mathfrak{g}(0,\varphi) + A\int_0^t T(t-s)\int_0^s \mathscr{K}(s-r,\mathfrak{g}(r,\tilde{u}_r))drds \\
&+ \int_0^t T(t-s)AF(s,\tilde{u}_s)ds \quad 0 \le t \le t_1.
\end{aligned}
$$

For the next, we need the following Lemmas.

Lemma 3.5. [5] Let $u : [0,t_1] \to E$ be continuously differentiable. Assume that $(\mathbf{H_2})$ hold. Then,

$k(t) = \int_0^t \mathscr{K}(t-s,u(s))ds$ is continuously differentiable from $[0,t_1]$ to E.

Lemma 3.6. $[14, p.488]$ *Let $k : [0,t_1] \to E$ be continuously differentiable, and q be defined by*

$$
q(t) = \int_0^t T(t-s)k(s)ds, \quad \text{for} \quad t \in [0,t_1].
$$

Then $q(t) \in Y$, for every $[0,t_1]$, q is continuously differentialble, and

$$
Aq(t) = q'(t) - k(t) = \int_0^t T(t-s)k'(s)ds + T(t)k(0) - k(t).
$$

By virtue of the hypothesis we have on B, by Lemma III.5 and III.6, for $u \in Y$,

we have

$$
\begin{aligned}
(APu)(t) &= AG(t,\tilde{u}_t) + AT(t)\mathfrak{g}(0,\varphi) + \int_0^t T(t-s)\mathscr{K}(0,\mathfrak{g}(s,\tilde{u}_s))\,ds \\
&+ \int_0^t T(t-s)\int_0^s \frac{\partial \mathscr{K}}{\partial s}(s-r,\mathfrak{g}(r,\tilde{u}_r))\,drds \\
&- \int_0^t \mathscr{K}(t-s,\mathfrak{g}(s,\tilde{u}_s))\,ds + \int_0^t T(t-s)AF(s,\tilde{u}_s)ds.
\end{aligned}
$$

From the axioms $(\mathbf{A_1} - \mathbf{i}), \mathbf{A_2}$ and assumption $(\mathbf{H_1})$, it follows that the maps $t \mapsto f(t,\tilde{u}_t)$ is continuous. Moreover, from $(\mathbf{H_2})$ and $(\mathbf{A_1})$ we infer that for every $u \in M_{t_1}(\varphi)$ the function $s \mapsto \mathscr{K}(s,\tilde{u})$ is continuous on $[0,t_1]$ and so by assumption $(\mathbf{H_0})$ that $t \mapsto \int_0^t T(s)f(s,\tilde{u}_s)ds$ is continuous on $[0,t_1]$. Thus, for $u \in M_{t_1}(\varphi)$, Pu and APu are both continuous from $[0,t_1]$ to E, P maps $M_{t_1}(\varphi)$ into $M_{t_1}(\varphi)$. Then $Pu \in \mathscr{C}([0,t_1];Y)$ and consequently $P(M_{t_1}(\varphi)) \subset M_{t_1}(\varphi)$.

We claim that P is a strict contraction in $M_{t_1}(\varphi)$. In fact, let $u,v \in M_{t_1}(\varphi)$. In fact,

$$\begin{aligned}
|(Pu)(t)-(Pv)(t)|_E &\le |G(t,\tilde{u}_t)-G(t,\tilde{v}_t)|_E \\
&+\left|\int_0^t T(t-s)\int_0^s \left(\mathscr{K}(s-r,\mathfrak{g}(r,\tilde{u}_r))-\mathscr{K}(s-r,\mathfrak{g}(r,\tilde{v}_r)\right)drds\right|_E \\
&+\left|\int_0^t T(t-s)\left(F(s,\tilde{u}_s)-F(s,\tilde{v}_s)\right)ds\right|_E \\
&\le |G(t,\tilde{u}_t)-G(t,\tilde{v}_t)|_E \\
&+\int_0^t e^{w(t-s)}\int_0^s |\mathscr{K}(s-r,\mathfrak{g}(r,\tilde{u}_r))-\mathscr{K}(s-r,\mathfrak{g}(r,\tilde{v}_r))|_E\,drds \\
&+\int_0^t e^{w(t-s)}|F(s,\tilde{u}_s)-F(s,\tilde{v}_s)|_E\,ds \\
&\le |G(t,\tilde{u}_t)-G(t,\tilde{v}_t)|_Y \\
&+\int_0^t e^{w(t-s)}\int_0^s |\mathscr{K}(s-r,\mathfrak{g}(r,\tilde{u}_r))-\mathscr{K}(s-r,\mathfrak{g}(r,\tilde{v}_r))|_Y\,drds \\
&+\int_0^t e^{w(t-s)}|F(s,\tilde{u}_s)-F(s,\tilde{v}_s)|_Y\,ds
\end{aligned}$$

Without loss of generality, we assume that $w>0$. By $(\mathbf{H_1})$ and $(\mathbf{H_2})$, we obtain that

$$\begin{aligned}
|(Pu)(t)-(Pv)(t)|_E &\le L_G\,|\tilde{u}_t-\tilde{v}_t|_{\mathscr{C}((-\infty,0],Y)} \\
&+e^{wt_1}\int_0^t\int_0^s b(s-r)\,|\tilde{u}_r-\tilde{v}_r|_{\mathscr{C}((-\infty,0],Y)}\,drds \\
&+L_F e^{wt_1}\int_0^t |\tilde{u}_s-\tilde{v}_s|_{\mathscr{C}((-\infty,0],Y)}\,ds,
\end{aligned} \tag{6}$$

$$\begin{aligned}
|(APu)(t)-(APv)(t)|_E &\le |AG(t,\tilde{u}_t)-AG(t,\tilde{v}_t)| \\
&+\int_0^t e^{w(t-s)}|\mathscr{K}(0,\mathfrak{g}(s,\tilde{u}_s))-\mathscr{K}(0,\mathfrak{g}\tilde{v}_s))|\,ds \\
&+\int_0^t e^{w(t-s)}\int_0^s \left|\frac{\partial\mathscr{K}}{\partial s}(s-r,\mathfrak{g}(r,\tilde{u}_r))-\frac{\partial\mathscr{K}}{\partial s}(s-r,\mathfrak{g}(r,\tilde{v}_r))\right|drds \\
&+\int_0^t |\mathscr{K}(t-s,\mathfrak{g}(s,\tilde{u}_s))-\mathscr{K}(t-s,\mathfrak{g}(s,\tilde{v}_s))|\,ds \\
&+\int_0^t e^{w(t-s)}|AF(s,\tilde{u}_s)-AF(s,\tilde{v}_s)|\,ds
\end{aligned}$$

$$\begin{aligned}
|(APu)(t)-(APv)(t)|_E &\le L_G\,|\tilde u_t-\tilde v_t|_{\mathscr{C}((-\infty,0],Y)}\\
&+b(0)e^{wt_1}\int_0^t |\tilde u_s-\tilde v_s|_{\mathscr{C}((-\infty,0],Y)}\,ds\\
&+e^{wt_1}\int_0^t\int_0^s c(s-r)\,|\tilde u_r-\tilde v_r|_{\mathscr{C}((-\infty,0],Y)}\,drds\\
&+\int_0^t b(t-s)\,|\tilde u_s-\tilde v_s|_{\mathscr{C}((-\infty,0],Y)}\,ds\\
&+L_F e^{wt_1}\int_0^t |\tilde u_s-\tilde v_s|_{\mathscr{C}((-\infty,0],Y)}\,ds.
\end{aligned} \tag{7}$$

From (6) and (7), we have

$$\begin{aligned}
|(Pu)(t)-(Pv)(t)|_Y &\le L_G\,|\tilde u_t-\tilde v_t|_{\mathscr{C}((-\infty,0],Y)}\\
&+b(0)e^{wt_1}\int_0^t |\tilde u_s-\tilde v_s|_{\mathscr{C}((-\infty,0],Y)}\,ds\\
&+e^{wt_1}\int_0^t\int_0^s [b(s-r)+c(s-r)]\,|\tilde u_r-\tilde v_r|_{\mathscr{C}((-\infty,0],Y)}\,drds\\
&+\int_0^t b(t-s)\,|\tilde u_s-\tilde v_s|_{\mathscr{C}((-\infty,0],Y)}\\
&+L_F e^{wt_1}\int_0^t |\tilde u_s-\tilde v_s|_{\mathscr{C}((-\infty,0],Y)}\,ds.
\end{aligned}$$

Define $\alpha(t)=\displaystyle\int_0^t e^{-ws}(b(s)+c(s))ds$ and $\beta(t)=\displaystyle\max_{0\le s\le t} e^{-ws}b(s)$ for $t\ge 0$.

Then

$$\begin{aligned}
|(Pu)(t)-(Pv)(t)|_Y &\le L_G\,|\tilde u_t-\tilde v_t|_{\mathscr{C}((-\infty,0],Y)}\\
&+b(0)e^{wt_1}\int_0^{t_1} |\tilde u_s-\tilde v_s|_{\mathscr{C}((-\infty,0],Y)}\,ds\\
&+e^{wt_1}\alpha(t)\int_0^{t_1} |\tilde u_s-\tilde v_s|_{\mathscr{C}((-\infty,0],Y)}\,ds\\
&+e^{wt_1}\beta(t)\int_0^{t_1} |\tilde u_s-\tilde v_s|_{\mathscr{C}((-\infty,0],Y)}\,ds\\
&+L_F e^{wt_1}\int_0^{t_1} |\tilde u_s-\tilde v_s|_{\mathscr{C}((-\infty,0],Y)}\,ds,
\end{aligned}$$

and finally

$$|(Pu)(t)-(Pv)(t)|_Y \le \left[L_G+t_1e^{wt_1}(b(0)+\alpha(t)+\beta(t)+L_F)\right]|\tilde u-\tilde v|_{\mathscr{C}((-\infty,0],Y)}.$$

If we choose t_1 small enough and $L_G<1$ such that

$$\left[L_G+t_1e^{wt_1}(b(0)+\alpha(t)+\beta(t)+L_F)\right]<1,$$

then P is a strict contraction in $M_{t_1}(\varphi)$, and by applying Banach's fixed point theorem, we deduce that there exists a unique fixed point $u = u(.,\varphi)$ for P in $M_{t_1}(\varphi)$, which implies that $Eq.(1)$ has a unique mild solution on $(-\infty, t_1]$. A similar argument can be used for $[t_1, 2t_1]$,...,$[nt_1, (n+1)t_1]$, for all $n \geq 0$, which implies that the mild solution exists uniquely in $(-\infty, +\infty)$. This completes the proof. $\Diamond$

3.1.2 Existence of strict solutions

In this section we recall some fundamental results needed to establish our results. We consider the inhomogeneous initial value problem

$$\left\{ \begin{array}{l} u'(t) = Au(t) + h(t) \quad \text{for} \quad t \geq 0, \\ \\ u(0) = x \in \mathscr{X} \end{array} \right. \tag{8}$$

where $h : [0,a] \longrightarrow \mathscr{X}$, is continuous.

Definition 3.7. *A continuous function $u : [0,+\infty[\rightarrow \mathscr{X}$ is said to be strict solution of $Eq.(8)$ if*

(i) $u \in \mathscr{C}^1([0,+\infty[;\mathscr{X}) \cap \mathscr{C}([0,+\infty[;\mathscr{D}(A))$

(ii) u satisfies $Eq.(8)$ for all $t \geq 0$.

Proposition 3.8. [20]. *If u is a strict solution of $Eq.(8)$, then u is given by*

$$u(t) = T(t)x + \int_0^a T(t-s)h(s)ds \quad \text{for} \quad t \in [0,a]. \tag{9}$$

The next Theorem provides sufficients conditions for the regularity of solution to $Eq.(8)$.

Theorem 3.9. [20]. *Let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Let $h \in L^1(0,a;\mathscr{X})$ be continuous on $[0,a]$ and let*

$$v(t) = \int_0^t T(t-s)h(s)ds \quad t \in [0,a].$$

The $Eq.(8)$ has a strict solution u on $[0,a]$ for every $x \in \mathscr{D}(A)$ if one of the following conditions is satisfied;

(1) $v(t)$ is continuously differentiable on $[0,a]$.

(2) $v(t) \in \mathscr{D}(A)$ for $0 < t < a$ and $Av(t)$ is continuous on $[0,a]$.

• *If Eq.*(8) *has a strict solution u on* $[0,a]$ *for some* $x \in \mathscr{D}(A)$ *then v satisfies both* (1) *and* (2).

From Theorem III.9 we dram the following useful Lemma.

Lemma 3.10. $[20]$. *Let A be the infinitesimal generator of a* C_0*-semigroup* $(T(t))_{t\geq 0}$. *Let* $h \in L^1([0,a];\mathscr{D}(A))$ *be continuous on* $[0,a]$. *If* $h(s) \in \mathscr{D}(A)$ *for* $0 < s < a$ *and* $Ah \in L^1([0,a];\mathscr{D}(A))$ *then for every* $x \in \mathscr{D}(A)$ *the Eq.*(8) *has a strict solution u on* $[0,a]$.

Now we give some sufficient conditions for the existence of a strict solution. To do this, we suppose the following condition on G.

$(\mathbf{H_3})$ $G : \mathbb{R}^+ \times \mathscr{C}((-\infty,0],Y) \to Y$ is continuously differentiable with respect to the first variable on $\mathbb{R}^+$.

we posit $h(t) = \int_0^t \mathscr{K}(t-s,\mathfrak{g}(s,u_s))ds + F(t,u_t) \quad \text{for} \quad t \geq 0.$

Theorem 3.11. *Let* $u \in \mathscr{C}([0,t_1],Y)$ *the mild solution of Eq.*(1). *If* $\mathfrak{g}(.,\varphi) \in Y$ *and* $h \in L^1([0,t_1];\mathscr{D}(A))$ *is continuous from* $[0,t_1]$ *to* $\mathscr{D}(A)$, *then* u *is a strict solution of Eq.*(1).

Proof. It is just a consequence of Theorem III.9.

Here

$$h(t) = \int_0^t \mathscr{K}(t-s,\mathfrak{g}(s,u_s))ds + F(t,u_t)ds \quad \text{for} \quad t \geq 0$$

and

$$v(t) = \int_0^t T(t-s)\int_0^s \mathscr{K}(s-r,\mathfrak{g}(s,u_r))drds + \int_0^t T(t-s)F(s,u_s)ds \quad \text{for} \quad t \geq 0.$$

We show that v satisfies the following two conditions

(i) $v(t)$ is continuously differentiable on $[0,t_1]$;

(ii) $v(t) \in Y$ on $[0,t_1]$ and $Av(t) \in L^1([0,t_1],E)$.

Based on the formula (5) we have: $v(t) = u(t) - T(t)\mathfrak{g}(0,\varphi) - G(t,u_t)$ is differentiable for $t > 0$ as sum of three differentiable functions and $\frac{d}{dt}v(t) = \frac{d}{dt}u(t) - T(t)A\mathfrak{g}(0,\varphi) - \frac{d}{dt}G(t,u_t)$ is obviously continuous on $]0,t_1[$. Therefore (i) is satisfied. Also if $\mathfrak{g}(0,\varphi) \in Y$ then $T(t)\mathfrak{g}(0,\varphi) \in Y$ for $t \geq 0$ and therefore $v(t) = u(t) - T(t)\mathfrak{g}(0,\varphi) - G(t,u_t) \in Y$ for $t > 0$ and $Av(t) =$

$Au(t)-AT(t)\mathfrak{g}(0,\varphi)-AG(t,u_t)=\frac{d}{dt}u(t)-\int_0^t\mathscr{K}(t-s,\mathfrak{g}(s,u_s))ds-F(t,u_t)-T(t)A\mathfrak{g}(0,\varphi)-AG(t,u_t)$ is continuous on $]0,t_1[$. Thus also (ii) is satisfaied.

On the other hand, it is easy to verify for $h>0$ the identify

$$\begin{aligned}\left(\frac{T(h)-I}{h}\right)v(t) &= \frac{v(t+h)-v(t)}{h} \\ &\quad -\frac{1}{h}\int_t^{t+h}T(t+h-s)\left[k(s)+F(s,u_s\right]ds\end{aligned} \tag{10}$$

From the continuity of k and F it is clear that the second therm on the right-hand side of (10) has the limit $k(t)+F(t,u_t)$ as $h\to 0$. If $v(t)$ is continuously differentiable on $]0,t_1[$ then it follows from (10) that $v(t)\in Y$ for $0<t<t_1$ and $Av(t)=\dfrac{d}{dt}v(t)-[k(t)+F(t,u_t)]$. Since $v(0)=0$ it follows that $u(t)=T(t)\mathfrak{g}(0,\varphi)+v(t)+G(t,u_t)$ is the solution of $Eq.(1)$ for $\mathfrak{g}(0,\varphi)\in Y$. If $v(t)\in Y$ it follows from (10) that $v(t)$ is differentiable from the right at t and the right derivative $D^+v(t)$ of v satisfies $D^+v(t)=Av(t)+k(t)+F(t,u_t)$. Since $D^+v(t)$ is continuous, $v(t)$ is continuously differentiable and $\frac{d}{dt}v(t)=Av(t)+k(t)+F(t,u_t)$. Since $v(0)=0$, $u(t)=T(t)\mathfrak{g}(0,\varphi)+G(t,u_t)+v(t)$ is the solution of $Eq.(1)$ for $\mathfrak{g}(0,\varphi)\in Y$ and the proof is complete. $\Diamond$

3.2 Special case: The non-atomic case

In this section, we examine the existence and regularity of solutions to $Eq.(1)$ where $\mathfrak{g}$ is non-atomic. Precisely, we considere the following equation

$$\begin{cases}\dfrac{d}{dt}\tilde{\mathfrak{g}}(u_t) &= A\tilde{\mathfrak{g}}(u_t)+\displaystyle\int_0^t\mathscr{K}(t-s,\tilde{\mathfrak{g}}(u_s))ds+F(t,u_t) \quad \text{for} \quad t\geq 0 \\ \\ u_0 &= \varphi\in\mathscr{C}=\mathscr{C}\left((-\infty,0];\mathscr{D}(A)\right),\end{cases} \tag{11}$$

where $\tilde{\mathfrak{g}}:\mathscr{C}\to\mathscr{D}(A)$, is a bounded linear operator defined by $\tilde{\mathfrak{g}}(\varphi)=\varphi(0)-D_0(\varphi)$ for $\varphi\in\mathscr{C}$, which D_0 a bounded linear operator from $\mathscr{C}$ to E which has the following from

$$\mathscr{D}_0(\varphi)=\int_{-\infty}^0 d\eta(\theta)\varphi(\theta) \quad \text{for} \quad \varphi\in\mathscr{C},$$

for a mapping $\eta:(-\infty,0]\to\mathscr{B}(E)$, of bounded variation and nonatomic at zero, which means that

$$var_{[-\varepsilon,0]}\eta:=\int_{-\varepsilon}^0|d\eta(\theta)|\to 0 \quad \text{as} \quad \varepsilon\to 0.$$

Theorem 3.12. *Assume that* $(\mathbf{H_1})-(\mathbf{H_2})$ *hold. If* $\varphi \in \mathscr{C}((-\infty,0];\mathscr{D}(A))$, *then there exists a unique continuous function* $u:(-\infty,+\infty[\to Y$ *which is a mild solution of* $Eq.(11)$.

Proof. Using the same notations as in the proof of Theorem III.4, we can define the mapping $P: M_{t_1}(\varphi)\to M_{t_1}(\varphi)$ by

$$\begin{aligned}(Pu)(t) \quad &= D_0(\tilde{u}_t)+T(t)\tilde{\mathfrak{g}}(\varphi)+\int_0^t T(t-s)\int_0^s \mathscr{K}(s-r,\tilde{\mathfrak{g}}(\tilde{u}_r))drds \\ &+\int_0^t T(t-s)F(s,\tilde{u}_s)ds \quad \text{for} \quad t\in[0,t_1].\end{aligned}$$

The operator P is well defined. It is enough to prove that P has a unique fixed point on $M_{t_1}(\varphi)$. The proof is similar to that given for Theorem III.4. The only difference is to replace respectively G by D_0 and the assumption $0<L_G=var_{[-t_1,0]}\eta<1$ by nonatomicity of D_0. The rest of the proof is the same. ♢

For the strict solution we pose $h(t)=\displaystyle\int_0^t \mathscr{K}(t-s,\tilde{\mathfrak{g}}(u_s))ds+F(t,u_t) \quad \text{for} \quad t\geq 0.$

Theorem 3.13. *Let* u *be the mild solution of* $Eq.(11)$. *If* $\tilde{\mathfrak{g}}(0,\varphi)=\varphi(0)-D_0\in\mathscr{D}(A)$ *and* $h\in L^1([0,t_1];\mathscr{D}(A))$ *is continuous from* $[0,t_1]$ *to* $\mathscr{D}(A)$, *then* u *is a strict solution of* $Eq.(11)$.

The proof is similar to that given for Theorem III.12. The only difference is to replace respectively G by D_0 and the assumption $0<L_G<1$ by nonatomicity of D_0. The rest of the proof is the same. ♢

4. Application

For illustration, we propose to study the existence of solutions for the following model

$$(12)\qquad \begin{cases} \dfrac{\partial}{\partial t}\left[w(t,\xi)-\displaystyle\int_{-\infty}^0 h(t,w(t+\theta,\xi))d\theta\right]=\dfrac{\partial^2}{\partial\xi^2}\left[w(t,\xi)-\displaystyle\int_{-\infty}^0 h(t,w(t+\theta,\xi))d\theta\right]ds \\ +\displaystyle\int_0^t \delta\left(t-s,\frac{\partial^2}{\partial\xi^2}\left(w(t,\xi)-\int_{-\infty}^0 h(t,w(t+\theta,\xi))d\theta\right)\right)ds \\ \displaystyle\int_{-\infty}^0 f(t,w(t+\theta,\xi))d\theta \quad \text{for} \quad t\geq 0 \quad \text{and} \quad \xi\in[0,\pi] \\ w(t,0)-h(t,w(t+\theta,0))d\theta=0 \quad \text{for} \quad t\geq 0 \\ w(t,\pi)-h(t,w(t+\theta,\pi))d\theta=0 \quad \text{for} \quad t\geq 0 \\ w(\theta,\xi)=w_0(\theta,\xi) \quad \text{for} \quad \theta\in]-\infty,0] \quad \text{and} \quad \xi\in[0,\pi], \end{cases}$$

where $f,h:\mathbb{R}^-\times\mathbb{R}\to\mathbb{R}$ is continuous and Lipschitzian with respect to the second argument, the initial data function $\varphi_0:]-\infty,0]\times[0,1]\to\mathbb{R}$ is a given function, $\delta:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ is bounded uniformly continuous, continuously differentiable in its first place and the derivative $\frac{\partial\delta}{\partial t}$ exists and is lipschitzian continuous.

To rewrite $Eq.(12)$ in the abstract from, we introduce the space $E=L^2((0,1);\mathbb{R})$.

Let $A:\mathscr{D}(A)\to E$ be defined by

$$\begin{cases}\mathscr{D}(A)=H^2(0,1)\cap H_0^1(0,1),\\ \\ Az=z''.\end{cases}$$

It is well known that A is the generator of c_0-semigroup, which implies that $(\mathbf{H_0})$ is satisfied.

Let $\mathscr{K}:\mathbb{R}^+\times\mathscr{D}(A)\to E$ by $\mathscr{K}(t,z)=\delta(t,Az)$ for $t\geq 0$.

The phase space $\mathscr{B}=BUC(\mathbb{R}^-;\mathscr{D}(A))$ is the space of bounded uniformly continuous functions from $\mathbb{R}^-$ into $\mathscr{D}(A)$ provided with the following norm

$$|\varphi|_{\mathscr{B}}=\sup_{\theta\leq 0}|\varphi(\theta)|_{D(A)}=\sup_{\theta\leq 0}|\varphi(\theta)|_{L^2(0,1)}+\sup_{\theta\leq 0}\left|\frac{\partial^2}{\partial\xi^2}\varphi(\theta)\right|_{L^2(0,1)}.$$

Then $\mathscr{B}=BUC(\mathbb{R}^-;\mathscr{D}(A))$ satisfies axioms $(\mathbf{A_1})$, $(\mathbf{A_2})$.

Let $F,G:\mathbb{R}^+\times\mathscr{B}\to E$ be defined by

$$F(t,\varphi)(\xi)=\int_{-\infty}^0 f(\theta,\varphi(\theta)(\xi))d\theta \quad \text{for} \quad 0\leq\xi\leq 1 \quad \text{and} \quad t\geq 0,$$

$$G(t,\varphi)(\xi)=\int_{-\infty}^0 h(\theta,\varphi(\theta)(\xi))d\theta \quad \text{for} \quad 0\leq\xi\leq 1 \quad \text{and} \quad t\geq 0.$$

Let us suppose $v(t)=w(t,.)$ and the initial data φ be defined by

$$\varphi(\theta)(\xi)=\varphi_0(\theta,\xi), \quad \text{for} \quad \theta\leq 0 \quad \text{and} \quad \xi\in[0,1].$$

Then $Eq.(12)$ takes the following abstract from

$$(13)\qquad \begin{cases}\dfrac{d}{dt}[v(t)-G(t,v_t)]= & A[v(t)-G(t,v_t)]\\ & +\displaystyle\int_0^t\delta(t-s,(v(t)-G(t,v_s)))ds+F(t,v_t)\\ v_0=\varphi. & \end{cases}$$

($\mathbf{H_3}$) We suppose that there exists the functions $l_f, l_h(.) \in L^1(\mathbb{R}^-, \mathbb{R}^+)$ such that

$$|f(\theta,\xi_1) - f(\theta,\xi_2)| \leq l_f(\theta)\,|\xi_1 - \xi_2| \quad \text{for} \quad \theta \leq 0 \quad \text{and} \quad \xi_1, \xi_2 \in \mathbb{R},$$

$$|h(\theta,\xi_1) - h(\theta,\xi_2)| \leq l_h(\theta)\,|\xi_1 - \xi_2| \quad \text{for} \quad \theta \leq 0 \quad \text{and} \quad \xi_1, \xi_2 \in \mathbb{R}.$$

($\mathbf{H_4}$) $f(\theta,0) = h(\theta,0) = 0, \quad \text{for} \quad \theta \leq 0.$

Assumptions ($\mathbf{H_3}$) and ($\mathbf{H_4}$) imply that $f(t,\varphi),\ h(t,\varphi) \in \mathscr{D}(A)$ for $\varphi \in \mathscr{B}$. In fact, let $\varphi \in \mathscr{B}$. Then

$$F(t,\varphi)(\xi) = \int_{-\infty}^{0} f(\theta,\varphi(\theta)(\xi))d\theta \quad \text{for} \quad \xi \in [0,1],$$

$$G(t,\varphi)(\xi) = \int_{-\infty}^{0} h(\theta,\varphi(\theta)(\xi))d\theta \quad \text{for} \quad \xi \in [0,1],$$

and

$$|F(t,\varphi)(\xi)| \leq \int_{-\infty}^{0} l_f(\theta)\,|\varphi(\theta)(\xi)|\,d\theta \leq \sup_{\theta\leq 0}|\varphi(\theta)(\xi)| \int_{-\infty}^{0} l_f(\theta)d\theta,$$

$$|F(t,\varphi)(\xi)|_{L^2(0,1)} \leq \sup_{\theta\leq 0}|\varphi(\theta)|_{L^2(0,1)}\,|l_f|_{L^1(-\infty,0)}.$$

$$|G(t,\varphi)(\xi)| \leq \int_{-\infty}^{0} l_h(\theta)\,|\varphi(\theta)(\xi)|\,d\theta \leq \sup_{\theta\leq 0}|\varphi(\theta)(\xi)| \int_{-\infty}^{0} l_h(\theta)d\theta,$$

$$|G(t,\varphi)(\xi)|_{L^2(0,1)} \leq \sup_{\theta\leq 0}|\varphi(\theta)|_{L^2(0,1)}\,|l_h|_{L^1(-\infty,0)}.$$

On the other hand, we have

$$\begin{aligned} |Af(t,\varphi)(x)| \quad &\leq \int_{-\infty}^{0} b_1(\theta)\,|A\varphi(\theta)(x)|\,d\theta \\ &\leq \sup_{\theta\leq 0}|A\varphi(\theta)(x)| \int_{-\infty}^{0} b_1(\theta)d\theta \\ |Af(t,\varphi)(x)|_{L^2(0,1)} \quad &\leq \sup_{\theta\leq 0}|A\varphi(\theta)|_{L^2(0,1)}\,|b_1|_{L^1(-\infty,0)}. \end{aligned}$$

Consequently

$$|f(t,\varphi)(x)|_{\mathscr{D}(A)} \leq \mathsf{C}\,|\varphi|_{\mathscr{B}}.$$

Moreover assumption ($\mathbf{H_4}$) implies that $f(t,\varphi)(0) = f(t,\varphi)(1) = 0.$

Using the dominated convergence theorem, one can show that $f(t,\varphi)$ is a continuous function on $[0,1]$. Moreover, for every $\varphi_1,\varphi_2\in\mathscr{B}$, we have

$$\begin{aligned}
|(f(t,\varphi_1)-f(t,\varphi_2))(x)| &\leq \int_{-\infty}^{0}|h(\theta,\varphi_1(\theta)(x))-h(\theta,\varphi_2(\theta)(x))|\,d\theta\\
&\leq \int_{-\infty}^{0} b_1(\theta)\,|\varphi_1(\theta))(x)-\varphi_2(\theta)(x)|\,d\theta\\
|(f(t,\varphi_1)-f(t,\varphi_2))(x)|_{L^2(0,1)} &\leq \sup_{-\infty\leq\theta\leq 0}|\varphi_1(\theta)(x)-\varphi_2(\theta)(x)|\,|b_1|_{L^1(\mathbb{R}^-)}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|A(f(t,\varphi_1)-f(t,\varphi_2))(x)| &\leq \int_{-\infty}^{0} b_1(\theta)\,|A(\varphi_1(\theta))(x)-\varphi_2(\theta))(x)|\,d\theta\\
|A(f(t,\varphi_1)-f(t,\varphi_2))(x)|_{L^2(0,1)} &\leq \sup_{-\infty\leq\theta\leq 0}|A(\varphi_1(\theta)(x)-\varphi_2(\theta))(x)|\,|b_1|_{L^1(\mathbb{R}^-)}
\end{aligned}$$

Consequently

$$|f(t,\varphi_1)-f(t,\varphi_2)|_{\mathscr{D}(A)}\leq \mathsf{C}\,|\varphi_1-\varphi_2|_{\mathscr{B}}.$$

We conclude that f is Lipschitz continuous.

In addition, we suppose that

(i) δ is bounded uniformly continuous, continuously differentiable in its first place and the derivative $\frac{\partial\delta}{\partial t}$ exists and is lipschitzian continuous.

(ii) The initial data $\varphi\in\mathscr{B}=BUC\left(]-\infty,0]\times[0,1];\mathscr{D}(A)\right)$,

$\varphi_0(0,0)=\varphi_0(0,1)=0$ is continuous from $]-\infty,0]\times[0,1]$ to $\mathscr{D}(A)$.

From the assumption (i), $\mathscr{K}$ satisfies the hypothesis $(\mathbf{H_2})$. Finally, from assumption (ii) and Theorem III.4, we deduce that $\varphi\in\mathscr{B}, Eq.(13)$ has a unique mild solution which is defined for all $t\geq 0$.

To prove that the mild solution of $Eq.(13)$ is a strict one, we need the following assumption.

(iii) $h\in L^1(\mathbb{R}^-\times\mathbb{R};\mathbb{R})$ be continuous on $\mathbb{R}^-\times\mathbb{R}$.

(iv) $\varphi_0 \in \mathscr{B}$ such that $\varphi_0(0,.) \in \mathscr{D}(A)$. Consequently, by Theorem III.11, we obtain the following existence result.

Proposition 4.1. Under the above assumptions, $Eq.(13)$ has a unique strict solution v and the solution u defined by $u(t,x) = v(t)(x)$ for $t \geq 0$ and $x \in [0,1]$ is a solution $Eq.(13)$.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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