Available online at http://scik.org

Adv. Fixed Point Theory, 2022, 12:5

https://doi.org/10.28919/afpt/7256

ISSN: 1927-6303

# FIXED POINT THEORY FOR SIMULATION FUNCTIONS IN G-METRIC SPACES: A NOVEL APPROACH

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**Abstract:** In this paper, with the aid of simulation mapping $\eta : [0,\infty)\times[0,\infty) \to \mathbb{R}$, we prove some Lemmas and fixed point result for generalized $\mathcal{Z}-$ contraction of the mapping $g : X \to X$ satisfying the following conditions:

$\eta\big(\mathcal{G}(gx,gy,gz),\mathcal{M}(x,y,z)\big)\geq 0,$

for all $x,y,z\in X$, where

$\mathcal{M}(x,y,z)=\max\{\mathcal{G}(x,gy,gy),\mathcal{G}(y,gx,gx),\mathcal{G}(y,gz,gz),\mathcal{G}(z,gy,gy),\mathcal{G}(z,gx,gx),\mathcal{G}(x,gz,gz)\}.$

and $(X,\mathcal{G})$ is a $\mathcal{G}-$ metric space. An example is also given to support our results.

**Keywords:** fixed point; generalized $\mathcal{Z}-$ contraction; simulation function; $\mathcal{G}-$ metric spaces.

**2010 AMS Subject Classification:** 47H10, 54H25.

## 1. INTRODUCTION

A metric space is a nonempty set $X$ with a two-variable map $d$ that allows us to calculate the distance between two points. We must find the distance not just between integers and vectors,

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Received February 10, 2022

but also between sequences and functions in higher mathematics. Numerous approaches exist in this sector in order to discover a suitable concept of a metric space. Many renowned mathematicians have considered various generalizations of a metric space. Mustafa and Sims [1] presented $G-$metric space in 2006 and provided an essential generalization of a metric space as follows:

**Definition 1.1. [1]** Let $X$ be a non empty set and $G: X^3 \to [0,\infty)$ be a map which satisfies the following properties:

1. $G(x,y,z) = 0$ if $x = y = z$,
2. $0 < G(x,x,y)$ whenever $x \neq y$,
3. $G(x,x,y) \leq G(x,y,z), y \neq z$,
4. $G(x,y,z) = G(x,z,y) = G(y,x,z) = G(z,x,y) = G(y,z,x) = G(z,y,x)$,
5. $G(x,y,z) \leq G(x,a,a) + G(a,y,z), \ \forall\, x,y,z,a \in X$.

Then, the function $G$ is said to be $G-$metric on $X$ and the pair $(X,G)$ is known as $G-$ metric space.

Banach [2] established the Banach contraction principle, a useful conclusion in fixed point theory involving a contraction mapping, in 1922.

**Definition 1.2. [2]** Let $(X,d)$ be a complete metric space and let $f: X \to X$ be a self-mapping. Let $d(fx,fy) < d(x,y)$ holds for all $x,y \in X$ with $x \neq y$.Then, $f$ is called a contraction known as Banach contraction.

Following this approach, a number of scholars expanded on it by offering various contractions on metric spaces [3, 4-9]. We introduce a mapping, namely the simulation function, and the concept of generalized $\mathcal{Z}-$ contraction in this paper. Khojasteh et al. [10] have proposed a new class of mappings known as simulation functions. Later, Argoubi et al. [11] made a minor change to the definition of simulation functions by removing a constraint.

**Definition 1.3. [11]** A simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

$(\zeta_1)$ $\zeta(t,s) < s - t$for all $t,s > 0$

$(\zeta_2)$ if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty}\{t_n\} = \lim_{n\to\infty}\{s_n\} = l \in (0,\infty),$$

then

$$\lim_{n\to\infty} sup\zeta(t_n, s_n) < 0.$$

## 2. Main Results

In this section, we prove certain Lemmas and some fixed point results for generalized $\mathcal{Z}-$ contraction in $\mathcal{G}-$ metric space.

**Definition 2.1.** Let $(X,\mathcal{G})$ be a $\mathcal{G}-$ metric space, $g : X \to X$ a mapping and $\eta \in \mathbb{Z}$. Then $g$ is called a generalized $\mathcal{Z}-$ contraction with respect to $\eta$ if the following condition is satisfied

$$\eta\big(\mathcal{G}(gx,gy,gz),\mathcal{M}(x,y,z)\big) \geq 0, \qquad (1)$$

for all $x,y,z \in X$, where $\mathcal{M}(x,y,z) =$
$\max\{\mathcal{G}(x,gy,gy),\mathcal{G}(y,gx,gx),\mathcal{G}(y,gz,gz),\mathcal{G}(z,gy,gy),\mathcal{G}(z,gx,gx), \mathcal{G}(x,gz,gz)\}$.

**Lemma 2.2.** Let $(X,\mathcal{G})$ denote a $\mathcal{G}-$ metric space and $g : X \to X$ denote a generalized $\mathcal{Z}-$ contraction with regard to $\mathbb{Z}$. Then, for all $x \in X, g$ is asymptotically regular.

**Proof:** Let $x \in X$ be arbitrary. If for some $k \in \mathbb{N}, g^k x = g^{k-1}x$, then $g^{k-1}x$ is a fixed point of $g$. Therefore, we have

$$\begin{aligned}\mathcal{G}(g^n x, g^{n+1}x, g^{n+1}x) &= \mathcal{G}(g^{n-k+1}g^{k-1}x, g^{n-k+2}g^{k-1}x, g^{n-k+2}g^{k-1}x)\\ &= \mathcal{G}(g^{n-k+1}y, g^{n-k+2}y, g^{n-k+2}y)\\ &= \mathcal{G}(y,y,y) = 0.\end{aligned}$$

Therefore, $\lim_{n\to\infty}\mathcal{G}(g^n x, g^{n+1}x, g^{n+1}x) = 0$.

So, let us suppose that $g^n x \neq g^{n-1}x$ for all $n \in \mathbb{N}$, then it follows from (1) that

$$\eta(\mathcal{G}(g^{n+1}x, g^n x, g^n x), \mathcal{M}(g^n x, g^{n-1}x, g^{n-1}x)) \geq 0,$$

since $g$ is a generalized contraction, where

$$\begin{aligned}\mathcal{M}(g^n x, g^{n-1}x, g^{n-1}x) = \max\{&\mathcal{G}(g^n x, g^n x, g^n x), \mathcal{G}(g^{n-1}x, g^{n+1}x, g^{n+1}x),\\ &\mathcal{G}(g^{n-1}x, g^{n+1}x, g^{n+1}x), \mathcal{G}(g^{n-1}x, g^n x, g^n x),\end{aligned}$$

$$\mathcal{G}(g^{n-1}x, g^nx, g^nx), \mathcal{G}(g^{n-1}x, g^{n+1}x, g^{n+1}x),$$
$$\mathcal{G}(g^nx, g^nx, g^nx)\}$$
$$= \max\{\mathcal{G}(g^nx, g^{n-1}x, g^{n-1}x), \mathcal{G}(g^{n+1}x, g^nx, g^nx)\}, \text{ since}$$

$\mathcal{G}(g^{n+1}x, g^{n-1}x, g^{n-1}x) \leq \mathcal{G}(g^{n+1}x, g^nx, g^nx) + \mathcal{G}(g^nx, g^{n-1}x, g^{n-1}x)$.

If $\max\{\mathcal{G}(g^nx, g^{n-1}x, g^{n-1}x), \mathcal{G}(g^{n+1}x, g^nx, g^nx)\} = \mathcal{G}((g^{n+1}x, g^nx, g^nx)$, then

$$\eta\big(\mathcal{G}(g^{n+1}x, g^nx, g^nx), M(g^nx, g^{n-1}x, g^{n-1}x)\big)$$
$$= \eta(\mathcal{G}(g^{n+1}x, g^nx, g^nx), \mathcal{G}(g^{n+1}x, g^nx, g^nx) \quad \geq 0,$$

which is a contradiction. So, $\mathcal{G}(g^{n+1}x, g^nx, g^nx) < \mathcal{G}(g^nx, g^{n-1}x, g^{n-1}x)$ holds. This shows that $\mathcal{G}(g^nx, g^{n-1}x, g^{n-1}x)$ is monotonically decreasing sequence of non-negative reals and so it must be convergent.

Let $\lim_{n\to\infty} \mathcal{G}(g^nx, g^{n-1}x, g^{n-1}x) = s$.

If $s > 0$, then by contraction condition

$$0 \leq \lim_{n\to\infty} Sup\, \eta\left(\mathcal{G}(g^{n+1}x, g^nx, g^nx), M(g^nx, g^{n-1}x, g^{n-1}x)\right)$$
$$= \lim_{n\to\infty} Sup\, \eta\left(\mathcal{G}(g^{n+1}x, g^nx, g^nx), M(g^nx, g^{n-1}x, g^{n-1}x)\right) < 0,$$

a contradiction and thus $s > 0$ and $g$ is asymptotically regular.

**Lemma 2.3.** Every Picard sequence converges to its unique fixed point, which is found in every generalized $\mathcal{Z}-$ contraction mapping on a complete $\mathcal{G}-$ metric space where $x_n = gx_{n-1}$ for all $n \in \mathbb{N}$.

**Proof:** Let $(X, \mathcal{G})$ denote a $\mathcal{G}-$ metric space and $g : X \to X$ a mapping and $\zeta \in \mathbb{Z}$.

Let us first demonstrate that if $g$ has a fixed point, it is unique.

If the mapping $g$ has two fixed points $p, r \in X$, then $d(p, r) > 0$.

By (1), we get

$$\eta(\mathcal{G}(gp, gr, gr), \mathcal{M}(p, r, r)) > 0,$$

where

$\mathcal{M}(p, r, r)$

$= \max\{\mathcal{G}(p, gr, gr), \mathcal{G}(r, gp, gp), \mathcal{G}(r, gr, gr), \mathcal{G}(r, gr, gr), \mathcal{G}(p, gr, gr), \mathcal{G}(p, gr, gr)\}.$

$= \mathcal{G}(p, r, r)$, which contradicts$(\zeta_2)$.

As a result, there is just one fixed point.

Now, we shall show that if $\{x_n\}$ is a Picard sequence created by $g$ then $\lim_{n\to\infty} x_n = z$ is only fixed point.

Let $x_0 \in X$ be any number, and $\{x_n\}$ be the Picard sequence, with $x_n = g$ for all $n \in \mathbb{N}$. Assume, on the other hand, that $\{x_n\}$ is not bounded. We can assume that $x_{n+k} \neq x_n$ for any $n, k \in \mathbb{N}$ without losing generality. Because $\{x_n\}$ is unbounded, there is a subsequence $\{x_{n_k}\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}, n_{k+1}$ is the smallest integer.

$\mathcal{G}(x_{n_{k+1}}, x_{n_k}, x_{n_k}) > 1$and

$\mathcal{G}\left(x_m, x_{n_k}, x_{n_k}\right) \leq 1$ for $n_k \leq m \leq n_{k+1} - 1$.

Therefore, by triangle inequality, we have

$$
\begin{aligned}
1 &< \mathcal{G}\left(x_{n_{k+1}}, x_{n_k}, x_{n_k}\right) \\
&\leq \mathcal{G}\left(x_{n_{k+1}}, x_{n_{k+1}} - 1, x_{n_{k+1}} - 1\right) + \mathcal{G}\left(x_{n_{k+1}} - 1, x_{n_k}, x_{n_k}\right) \\
&\leq \mathcal{G}\left(x_{n_{k+1}}, x_{n_{k+1}} - 1, x_{n_{k+1}} - 1\right) + 1.
\end{aligned}
$$

Letting $k \to \infty$ and using Lemma 2.2, we get

$$\lim_{k\to\infty} \mathcal{G}\left(x_{n_{k+1}}, x_{n_k}, x_{n_k}\right) = 1,$$

$$
\begin{aligned}
&\mathcal{M}\left(x_{n_{k+1}} - 1, x_{n_{k-1}} - 1, x_{n_{k-1}} - 1\right) \\
&\quad = \max \{\mathcal{G}\left(x_{n_{k+1}} - 1, gx_{n_{k-1}} - 1, gx_{n_{k+1}} - 1\right), \mathcal{G}\left(x_{n_{k-1}} - 1, x_{n_{k+1}} - 1, x_{n_{k+1}} - 1\right), \\
&\qquad \mathcal{G}\left(x_{n_{k-1}} - 1, x_{n_{k-1}} - 1, x_{n_{k-1}} - 1\right), \mathcal{G}\left(x_{n_{k-1}} - 1, x_{n_{k-1}} - 1, x_{n_{k-1}} - 1\right), \\
&\qquad \mathcal{G}\left(x_{n_{k-1}} - 1, gx_{n_{k+1}} - 1, gx_{n_{k+1}} - 1\right), \mathcal{G}\left(x_{n_{k+1}} - 1, gx_{n_{k-1}} - 1, gx_{n_{k-1}} - 1\right).
\end{aligned}
$$

Now, since $g$ is a generalized $\mathcal{Z}-$ contraction, so that

$$
\begin{aligned}
0 &\leq \lim_{k\to\infty} \sup \eta\, (\mathcal{G}\left(gx_{n_{k+1}} - 1, gx_{n_{k-1}} - 1, gx_{n_{k+1}} - 1\right) \\
&= \lim_{k\to\infty} \sup \eta\, (\mathcal{G}(x_{n_{k+1}}, x_{n_k}, x_{n_k}), \mathcal{G}(x_{n_{k+1}} - 1, x_{n_{k-1}}, x_{n_{k-1}})) < 0,
\end{aligned}
$$

a contradiction. This contradiction proves the result.

**Theorem 2.4.** Let $(X,\mathcal{G})$ be a complete $\mathcal{G}-$ metric space and $g: X \to X$ a mapping and $\eta \in \mathbb{Z}$ and this is ageneralized$\mathcal{Z}-$ contraction.Then, $g$ has a unique fixed point $u$ in $X$ and for every $x_0 \in X$ the Picard sequence $\{x_n\}$, where $x_n = gx_{n-1}$ for all $\in \mathbb{N}$ converges to the fixed point of $g$.

**Proof:** Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, i.e. $x_n = gx_{n-1} \forall\, n \in \mathbb{N}$. We shall show that this sequence is a Cauchy sequence.

For this, let

$$C_n = sup\{x_p, x_r, x_r : p, r \geq n\}.$$

Note that the sequence $\{x_n\}$ is monotonically decreasing sequence of the reals and by Lemma 2.3, the sequence $\{x_n\}$ is bounded, therefore $C_n < \infty$ for all $n \in \mathbb{N}$. Thus, $\{C_n\}$ is monotonic bounded sequence, therefore converges, that is $\exists\, C \geq 0$ such that $C_n = C$. We shall show that $C = 0$. If $C > 0$, then by the definition of $C_n$, for every $k \in \mathbb{N}$, $\exists\, m_k > n_k \geq k$ and $C_k - \frac{1}{k} < \mathcal{G}(x_{m_k}, x_{n_k}, x_{n_k}) \leq C_k$.

Hence,

$$\lim_{k\to\infty} \mathcal{G}(x_{m_k}, x_{n_k}, x_{n_k}) \leq C_k \tag{2}$$

Using (1) and the triangular inequality, we obtain

$$\mathcal{G}(x_{m_k}, x_{n_k}, x_{n_k}) \leq \mathcal{G}(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})$$

$$\leq \mathcal{G}(x_{m_{k-1}}, x_{m_k}, x_{m_k}) + \mathcal{G}(x_{m_k}, x_{n_k}, x_{n_k}) + \mathcal{G}(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}).$$

$\mathcal{G}(x_{m_{k-1}}, x_{m_k}, x_{m_k}) \to 0$, $\mathcal{G}(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}) \to 0$ as $k \to \infty$.

Then, by Squeeze Theorem, we have

$\lim_{k\to\infty} \mathcal{G}(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}) = C$ as well

$$\mathcal{M}(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})$$

$$= \max\{\mathcal{G}(x_{m_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}}), \mathcal{G}(x_{n_{k-1}}, gx_{m_{k-1}}, gx_{m_{k-1}}), \mathcal{G}(x_{n_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}}),$$
$$\mathcal{G}(x_{n_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}}), \mathcal{G}(x_{n_{k-1}}, gx_{m_{k-1}}, gx_{m_{k-1}}), \mathcal{G}(x_{m_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}})\}.$$

We know that $\mathcal{G}\left(x_{m_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}}\right) = C$ as well

$\mathcal{M}(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}) \to 0, \mathcal{G}(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}) \to 0$ as $k \to \infty$.

$0 \leq \lim_{k\to\infty} \sup \eta\left(\mathcal{G}\left(x_{m_k}, x_{n_k}, x_{n_k}\right), \mathcal{M}\left(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}\right) < 0.$

This contradiction proves that $C = 0$ and so $\{x_n\}$ is a Cauchy sequence. Since $X$ is a complete $\mathcal{G}-$ metric space, $\exists u \in X$ such that $\lim_{n\to\infty} x_n = u$. We shall show that the point u is a fixed point of $g$. Suppose $gu \neq u$, then $\mathcal{G}(u, gu, gu) > 0$.

Again, using (1), we have

$0 \leq \lim_{k\to\infty} \sup \eta\left(\mathcal{G}\left(x_{m_k}, x_{n_k}, x_{n_k}\right), \mathcal{M}\left(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}\right)\right) < 0.$

$\lim_{n\to\infty} \mathcal{G}(gz, gx_n, gx_n) = \lim_{n\to\infty} \mathcal{G}(gz, gx_{n+1}, gx_{n+1}) = \mathcal{G}(gz, z, z) > 0,$

and

$$\mathcal{M}(z, x_n, x_n) = \max\{\mathcal{G}(gz, gx_n, gx_n), \mathcal{G}(x_n, gz, gz), \mathcal{G}(x_n, gx_n, gx_n), \mathcal{G}(x_n, gx_n, gx_n), \mathcal{G}(x_n, gz, gz), \mathcal{G}(z, gx_n, gx_n)\}.$$

Therefore, $\mathcal{M}(z, x_n, x_n) \to \mathcal{G}(gz, z, z)$ as $n \to \infty$.

By contractive condition,

$0 \leq \eta(\mathcal{G}(gz, gx_n, gx_n), \mathcal{M}(z, x_n, x_n)) \to \eta\left(\mathcal{G}(gz, z, z), \mathcal{M}(gz, z, z)\right)$ as $n \to \infty$.

By $(\zeta_2)$,we have $\eta(\mathcal{G}(gz, z, z), \mathcal{M}(gz, z, z)) < 0$ which contradicts the contraction condition. That means $gz = z$ and $z$ is the unique fixed point of $g$.

**Example 2.5.** Let $X = [0, 1]$ and $g : X \to X \to \mathbb{R}$ be defined by

$$\mathcal{G}(x, y, z) = max\{|x - y|, |y - z|, |z - x|\}.$$

Then $(X, \mathcal{G})$ is a complete $\mathcal{G}$-Metric space.

Define a mapping $g : X \to X$ as $gx = \frac{x}{x+1}$ for all $x \in X$. $g$ is a continuous function but it is not a Banach contraction. But it is a generalized $\mathcal{Z}-$ contraction with respect to $\eta \in \mathcal{Z}$, where

$$\eta(t, s) = \frac{s}{s+1} - t \text{for all } t, s \in [0, \infty).$$

Indeed, $if\ x, y \in X$, then by a simple calculation it can be shown that $\eta\big(\mathcal{G}(gx, gy, gz), \mathcal{M}(x, y, z)\big) \geq 0$ for all $x, y \in X$,

where

$$\mathcal{M}(x, y, z) = \max\{\mathcal{G}(x, gy, gy), \mathcal{G}(y, gx, gx), \mathcal{G}(y, gz, gz), \mathcal{G}(z, gy, gy), \mathcal{G}(z, gx, gx), \mathcal{G}(x, gz, gz)\}.$$

Clearly, $0$ is the fixed point of $g$.

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

## REFERENCES

[1] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7(2006), 289-297.

[2] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrals, Fund. Math. 3(1922), 133-181.

[3] D.W. Boyd and J.S.W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20(1969), 458-464.

[4] F.E. Browder and W.V. Petrysyn, The solution by iteration of nonlinear functional equation in Banach spaces, Bull. Amer. Math. Soc. 72(1966), 571-576.

[5] N. Hussain, Z. Kadelburg, S. Radenović, F. Al-Solamy, Comparison Functions and Fixed Point Results in Partial Metric Spaces, Abstr. Appl. Anal. 2012 (2012), 605781.

[6] N. Hussain, M.A. Kutbi, S. Khaleghizadeh and P. Salim, Discussions on recent results for $\alpha - \psi -$contractive mappings, Abstr. Appl. Anal. 2014 (2014), 456482.

[7] F. Khojasteh, V. Rakocevic, Some new common fixed point results for generalizwd contractive multi - valued non - self - mappings, Appl. Math. Lett. 25 (2012), 287-293.

[8] N. Mizoguchi and W. Takahash, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl. 141 (1989), 177-188.

[9] B.E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 224 (1977), 257-290.

[10] B.E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. (TMA), 47 (2001), 2683-2693.

[11] F. Khojasteh, S. Shukla, S. Radenovic, A new approach to the study of fixed point theory for simulation functions, Filomat, 29 (2015), 1189-1194.

[12] H. Argoubi, B. Samet, C. Vetro, Nonlinear contractions involving simulation functions in a metric space with a partial order, J. Nonlinear Sci. Appl. 8 (2015), 1082-1094.