Available online at http://scik.org
Adv. Fixed Point Theory, 2022, 12:8
https://doi.org/10.28919/afpt/7320
ISSN: 1927-6303

CONVERGENCE THEOREMS AND STABILITY RESULTS OF A TWO-STEP ITERATION SCHEME FOR POINTWISE ASYMPTOTICALLY NONEXPANSIVE SELF MAPPINGS AND POINTWISE ASYMPTOTICALLY NONEXPANSIVE NONSELF MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. We propose a two-step generalised lshikawa iteration scheme of hybrid mixed-type for two pointwise asymptotically nonexpansive self mappings and two pointwise asymptotically nonexpansive nonself mappings. Under the condition that pointwise asymptotically nonexpansive self mappings and pointwise asymptotically nonexpansive nonself mappings are compact, we proved demiclosedness principle for pointwise asymptotically nonexpansive nonself mappings; in addition, we established stability results and weak convergence theorems of the scheme to the common fixed point of the mappings in uniformly convex Banach spaces. Our results modify, improve and generalise numerous results currently existing in literature.

Keywords: pointwise asymtotically nonexpansive mappings; pointwise asymptotically nonexpansive nonself mapping; hybrid mixed type iteration scheme; common fixed point; uniformly convex banach space; weak convergence.

2010 AMS Subject Classification: 47H10, 37C25, 35M20.

1. Introduction

Let C be a nonempty subset of a real Banach space E. Let $T : C \longrightarrow C$ be a nonlinear mapping. We denote the set of all fixed points of T by $F(T)$. The set of common fixed point of six

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Received March 02, 2022

mappings S_1, S_2, T_1 *and* T_2 will be denoted by $\mathscr{F} = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Throughout this paper, the symbol $\mathbb{N}$ will denote the set of natural numbers.

Definition 1.1. *A mapping $T : C \longrightarrow C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that, for all $x, y \in C$,*

$$\|T^n(x) - T^n(y)\| \leq k_n \|x - y\|, \forall n \in \mathbb{N}. \tag{1.1}$$

Definition 1.2. *A mapping $T : C \longrightarrow C$ is said to be pointwise asymptotically nonexpansive* [24] *if*

$$\|T^n(x) - T^n(y)\| \leq k_n(x)\|x - y\|, \forall x, y \in C, \forall n \in \mathbb{N}, \tag{1.2}$$

where $\{k_n(x)\}$ is a sequence in $[0, \infty)$ and $k_n \to 1$ pointwise on C. If $k_n(x) \leq 1$ and $\lim_{n\to\infty} k_n(x) = k \in [0, 1)$ *in* (1.2), *then T is called pointwise asymptotically contraction.*

ln 2008, the class of pointwise asymptotically nonexpansive mapping was introduced by Kirk and Xu [25]. They proved that if C is a bounded uniformly convex Banach space and $T$is a pointwise asymptotically nonexpansive mapping of C, then T has a fixed point. Since then, many results on pointwise asymptotically nonexpansive mappings and stability of iterative schemes have been obtained in literature (See [22], [23], [24], [25], etc for details).

Remark 1.1. *lf $\{k_n\}$ in* (1.2) *is independent of x, then the mapping T is called asymptotically nonexpansive (see [22] for details). Thus, it is clear that the class of asymptotically nonexpansive mapping is a subclass of the class of pointwise asymptotically nonexpansive mapping.*

The example below shows a practical application of pointwise asymptotically nonexpansive mappings:

Example 1.1(See [22]). Let C be a nonempty closed subset of the complex set R. Let f be a continously differentiable self-mapping of C. Let $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ be two systems with initial point x_0 and y_0 respectively. lf the initial error $|x_0 - y_0|$ is significantly small, then f is pointwise asymptotically nonexpansive mapping. It was remarked in [22] that if the function f is not constant, then f is not asymptotically nonexpansive mapping (See [22] for details).

Definition 1.3. *A subset C of a Banach space E is said to be a retract of E if there exists a continous mapping $P : E \longrightarrow C$ (called retraction) such that $P(x) = x$ for all $x \in K$. If, in addition P is nonexpansive, then P is said to be nonexpansive retraction of E.*

If $P : E \longrightarrow C$ is a retraction, then $P^2 = P$. A retract of a Hausdorff space must be a closed subset. Every closed convex subset of a uniformly convex Banach space is a retract.

Definition 1.4. *Let C be a nonempty, closed and convex subset of a Banach space E. A nonself mapping $T : C \to E$ is said to be pointwise asymptotically nonexpansive mapping in the sense of Kirk and Xu [25] if*

$$\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \leq k_n(x)\|x-y\|, \forall x, y \in C, \forall n \in \mathbb{N}, \tag{1.3}$$

where $\{k_n(x)\}$ is a sequence in $[0,\infty)$ and $k_n \to 1$ pointwise on C.

From definitions (1.2) and (1.3), we see that if the retraction map $P : E \longrightarrow C$ is an identity, then (1.3) reduces to (1.2). Hence, the class of pointwise asymptotically nonexpansive nonself mappings includes the class of pointwise asymptotically nonexpansive mapping as a special case; that is, each pointwise asymptotically nonexpansive self mapping is pointwise asymptotically nonexpansive nonself mapping but the reverse is not true. Now, we consider the case of $m_n(x) = \max_{n\geq 1}\{k_n(x), 1\}$. Denote $\Gamma(C)$ as the class of pointwise asymptotically nonexpansive nonself mapping and pointwise asymptotically nonexpansive self mapping T, S respectively satisfying $\lim_{n\to\infty} m_n(x) = 1$. Define $Q_n(x) = m_n(x) - 1$. It is easy to see that $\lim_{n\to\infty} Q_n(x) = 0$.

Definition 1.5. *Let $\Gamma_r(C)$ be a class of all $T, S \in \Gamma(C)$ such that $\sum_{n=1}^{\infty} Q_n(x) < \infty, \forall x \in C$. Then,* $\lim_{n\to\infty} m_n(x) = 1$.

Chidume et al. [3] studied the following iterative scheme in 2004:

$$\begin{aligned} x_1 &= x \in C \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1}x_n + (1-\alpha_n)x_n), n \geq 1 \end{aligned} \tag{1.4}$$

where α_n is a sequence in $(0,1)$ and C is a nonempty closed convex subset of of a real uniformly convex Banach space E, P is a nonexpansive retraction of E onto C, and proved some strong and weak convergence theorems for asymptotically nonexpansive nonself mappings in the intermediate sense in the framework of uniformly convex Banach spaces.

ln 2017, Feng, Jiang and Su [22] introduced the following generalised lshikawa iteration process:

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n)T^n y_n), \\ y_n &= \beta_n x_n + (1-\beta_n)T^n x_n), n \geq 1 \end{aligned} \tag{1.5}$$

where $T : C \longrightarrow C$ is a pointwise asymptotically nonexpansive mapping and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0,1)$, and proved some weak convergence theorems and stability results for pointwise asymptotically nonexpansive mappings in the setting of a uniformly convex Banach space.

Hybrid Mixed-Type Iteration Scheme

Let E be a real uniformly convex Banach space, C a nonempty, closed and convex subset of E and $P : E \longrightarrow C$ a nonexpansive retraction of E onto C. Let $S_1, S_2 : C \longrightarrow C$ be two pointwise asymptotically nonexpansive self mappings and $T_1, T_2 : C \longrightarrow E$ be two pointwise asymptotically nonexpansive nonself mappings. then, the hybrid iteration scheme for the above mentioned mappings is as follows:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1} y_n), \\ y_n &= P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1} x_n), \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$.

The aim of this paper is to study this new hybrid mixed-type iteration scheme (1.6), prove stability results for the scheme and establish some convergence theorems for mixed-type mappings in the setting of uniformly convex Banach spaces.

2. PRELIMINARY

For the sake of convenience, we restate the following concepts and results:

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is a function $\delta_E(\varepsilon) : (0,2] \longrightarrow (0,2]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x-y\|\}.$$

A Banach space E is uniformly convex if and if $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0,2]$.

We recall the following:

Definition 2.1. *The space E has Opial condition* [10] *if for any sequence* $\{x_n\}$ *in E,* x_n *converges to x weakly,it follows that* $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ *for all* $y \in E$ *with* $x \neq y$.

Examples of Banach spaces satisfying Opial conditions are Hilbert spaces and all spaces $l^p(1 < p < \infty)$. On the other hand, $L^p[0,\pi]$ with $1 < p \neq 2$ fails to satify Opial condition.

Definition 2.2. *: A mapping* $T : K \longrightarrow K$ *is said to be demiclosed at* 0 *if for any sequence* $\{x_n\}$ *in K, the condition that* x_n *converges weakly to* $x \in K$ *and* Tx_n *converges strongly to 0 implies* $Tx = 0$.

Definition 2.3. *A sequence* $\{t_n\} \subset (0,1)$ *is called bounded away from* 0 *if there exists* $0 < a < 1$ *such that* $t_n > a$ *for every* $n \in N$. *Similarlu, a sequence* $\{t_n\} \subset (0,1)$ *is called bounded away from* 1 *if there exists* $0 < b < 1$ *such that* $t_n < b$ *for every* $n \in N$.

Next, we state the following useful lemmas which will help us to prove our main results.

Lemma 2.1. *(see*[16]*): Let* $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ *and* $\{\gamma_n\}_{n=1}^{\infty}$ *be sequences of nonnegative numbers satisfying the inequality:*

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + \gamma_n, \forall n \geq 1. \tag{2.1}$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$, *then*

(1) $\lim_{n\to\infty} \alpha_n$ *exists*

(2) ln particular, if $\{\alpha_n\}_{n=1}^{\infty}$ *has a subsequence which converges strongly to* 0, *then* $\lim_{n\to\infty} = 0$.

Lemma 2.2. $(see[14])$*: Let E be a uniformly convex Banach space and* $0 < p \leq t_n \leq q < 1$ *for each* $n \geq 1$*. Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in E such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \limsup_{n\to\infty} \|y_n\| \leq r \text{ and } \lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r, \tag{2.2}$$

hold for some $r \geq 0$*. Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$*.*

Lemma 2.3. *(See [22]) Assume X is a uniformly convex Banch space and C is a bounded closed convex subset of X. Then every pointwise asymptotically nonexpansive mapping* $T : C \longrightarrow C$ *has a fixed point. Moreover, the set of fixed point of T is closed and convex.*

Lemma 2.4. *(See [22]) Let C be a nonempty closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition and let* $T \subset T_r(C)$*. Then,* $I - T$ *is demiclosed at zero; that is, if* $\{x_n\}$*is a sequence in C such that* $x_n \to p$ *and* $\lim \|x_n - Tx_n\| = 0$*, then* $(I-T)p = 0$

Lemma 2.5. *(see [2]) Let E be a uniformly convex Banach space, K a nonempty bounded close convex subset of E. Then, there exists a strictly increasing continuous convex function* $\phi : [0,\infty) \longrightarrow [0,\infty)$ *with* $\phi(0) = 0$ *such that for any Lipschitizian mapping* $T : K \longrightarrow E$ *with Lipschitz constant* $L \geq 1$ *and elements* $\{x_n\}_{j=i}^{n}$ *in K and any nonnegative numbers* $\{t_j\}_{j=1}^{n}$ *with* $\sum_{j=1}^{n} t_j = 1$*, the following inequality holds:*

$$\|T(\sum_{j=1}^{n} t_j x_j) - \sum_{j=1}^{n} t_j T x_j\| \leq L\phi^{-1}\{max_{1\leq j,k\leq n}(\|x_j - x_k\| - L^{-1}\|Tx_j - Tx_k\|)\}$$

Lemma 2.6. *(see [21]) If the sequence* $\{x_n\}_{n=1}^{\infty}$ *converges weakly to x, then there exists a sequence of convex combination* $y_j = \sum_{k=1}^{n(j)} \lambda_k^{(j)} x_{k+j}$*,* $\lambda_k^{(j)} \geq 0$ *and* $\sum_{k=1}^{n(j)} \lambda^{(j)} = 1$*, such that* $\|y_j - x\| \to 0$*. as* $n \to \infty$*.*

Lemma 2.7. $(\, Demiclosedness\ Principle\ for\ Pointwise\ Asymptotically\ Nonexpansive$
$Nonself\ Mappings\,)$

Let K be a nonempty, closed, convex and bounded subset of a uniformly convex Banach space E and $T : K \longrightarrow E$ *be L-Lipschitz continuous and pointwise asymptotically nonexpansive mapping with nonnegative sequence* $k_n(x)\}$ *such that* $k_n(x) \to 0$ *as* $n \to \infty$*. Then,* $I - T$ *is demiclosed at 0.*

Proof. Let $\{x_n\}$ converge weakly to $\omega \in K$ and $\{x_n - Tx_n\}$ converge strongly to 0. We prove that $(I-T)\omega = 0$. Clearly, $\{x_n\}$ is bounded. So, there exists $\rho > 0$ such that $\{x_n\} \subset C = K \cap \overline{B_\rho}(0)$, where $\overline{B_\rho}(0)$ is a closed ball in E with centre 0 and radius ρ. Thus, C is nonempty, closed , bounded and convex subset in K.

Claim: $T(PT)^{n-1}\omega \to \omega$ as $n \to \infty$. In fact, since $\{x_n\}$ converges weakly to ω, by Lemma 2.6(see [21]), we have for all $n > 1$, there exists a convex combination

$$y_n = \sum_{i=1}^{m(n)} t_i^{(n)} x_{i+n},\ t_i^{(n)} \geq 0 \text{ and } \sum_{i=1}^{m(n)} t_i^{(n)} = 1 \text{ such that } \|y_n - \omega\| \to 0 \text{ as } n \to \infty. \tag{2.3}$$

Also, since $\{x_n - Tx_n\}$ converges to 0, then for any $\varepsilon > 0$ and a positive integer $m \geq 1$, there exists $N_1 = N(\varepsilon) > 0$ such that

$$\|(I-T)x_n\| < \frac{\varepsilon}{1+m}, \forall n \geq N_{1.} \tag{2.4}$$

Hence, $\forall n \geq N_1$, using Definition 1.3 and the fact that P is nonexpansive , we have the following estimates:

For arbitrary but fixed $j \geq 1$, we have

$$\begin{aligned}
\|x_n - T(PT)^{(j-1)}x_n\| &\leq \|(I-T)x_n\| + \|(T-T(PT))x_n\| \\
&\quad + \|(T(PT) - T(PT)^2)x_n\| \\
&\quad + \|(T(PT)^2 - T(PT)^3)x_n\| \\
&\quad + \cdots + \|(T(PT)^{j-2} - T(PT)^{j-1}))x_n\| \\
&\leq \|(I-T)x_n\| + k_n^{(1)}(x)\|(I-T)x_n\| + k_n^{(2)}(x)\|(I-T)x_n\| \\
&\quad + k_n^{(3)}(x)\|(I-T)x_n\| \\
&\quad + \cdots + k_n^{(j)}(x)\|(I-T)x_n\| \\
&= \|(I-T)x_n\| + \sum_{j=1}^{m-1} k_n^{(j)}\|(I-T)x_n\| \\
&\leq (1 + \sum_{n=1}^{m-1} k_n(x))\|(I-T)x_n\|, \qquad (2.5)
\end{aligned}$$

where $k_n(x) = \max_{1\leq j\leq m-1}\{k_n^{(j)}(x)\}$.
From (2.4), (2.5) and the fact that $\sum_{n=1}^{\infty} k_n(x)) < \infty$, we get

$$\|x_n - T(PT)^{j-1}x_n\| < \varepsilon. \tag{2.6}$$

Now, since $T : C \longrightarrow E$ is L-Lipschitizian and pointwise asymptotically nonexpansive , so is $T : K \longrightarrow E$. Therefore, $\forall j \geq 1, T(PT)^{j-1} : C \longrightarrow E$ is Lipschitizian mapping with the Lipschiz constant $\mu_j \geq 1$.
In addition,

$$\begin{aligned}
\|T(PT)^{j-1}y_n - y_n\| &= \|T(PT)^{j-1}y_n - \sum_{i=1}^{m(n)} t_i^{(n)} T(PT)^{j-1}x_{i+n} + \sum_{i=1}^{m(n)} t_i^{(n)} T(PT)^{j-1}x_{i+n} \\
&\quad - \sum_{i=1}^{m(n)} t_i^{(n)} x_{i+n}\| \\
&\leq \|T(PT)^{j-1}y_n - \sum_{i=1}^{m(n)} t_i^{(n)} T(PT)^{j-1}x_{i+n}\| \\
&\quad + \sum_{i=1}^{m(n)} t_i^{(n)} \|T(PT)^{j-1}x_{i+n} - x_{i+n}\|. \qquad (2.7)
\end{aligned}$$

Using (2.6), we get

$$\sum_{i=1}^{m(n)} t_i^{(n)} \|T(PT)^{j-1}x_{i+n} - x_{i+n}\| < \varepsilon, \forall n \geq N. \tag{2.8}$$

Furthermore, by Lemma 2.7, there exists a strictly increasing continuous function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ such that for all $n \geq N$, we have

$$\begin{aligned}
\|T(PT)^{j-1}y_n - \sum_{i=1}^{m(n)} t_i^{(n)} T(PT)^{j-1}x_{i+n}\| &= \|T(PT)^{j-1}(\sum_{i=1}^{m(n)} t_i^{(n)} x_{i+n}) - \sum_{i=1}^{m(n)} t_i^{(n)} T(PT)^{j-1}x_{i+n}\| \\
&\leq \mu_j \phi^{-1}\{max_{1\leq j,k\leq n}(\|x_{i+n} - x_{i+k}\| \\
&\quad -\mu_j^{-1}\|T(PT)^{J-1}x_{i+n} - T(PT)^{J-1}x_{k+n}\|)\} \\
&= \mu_j \phi^{-1}\{max_{1\leq j,k\leq n}(\|x_{i+n} - T(PT)^{J-1}x_{i+n} \\
&\quad +T(PT)^{J-1}x_{i+n} - T(PT)^{J-1}x_{k+n} \\
&\quad +T(PT)^{J-1}x_{k+n} - x_{i+k}\| \\
&\quad -\mu_j^{-1}\|T(PT)^{J-1}x_{i+n} - T(PT)^{J-1}x_{k+n}\|)\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \mu_j\phi^{-1}\{max_{1\leq j,k\leq n}(\|x_{i+n}-T(PT)^{J-1}x_{i+n}\| \\
&\quad +\|T(PT)^{J-1}x_{i+n}-T(PT)^{J-1}x_{k+n}\| \\
&\quad +\|T(PT)^{J-1}x_{k+n}-x_{i+k}\| \\
&\quad -\mu_j^{-1}\|T(PT)^{J-1}x_{i+n}-T(PT)^{J-1}x_{k+n}\|)\} \\
&\leq \mu_j\phi^{-1}\{max_{1\leq j,k\leq n}(\varepsilon+\varepsilon+(1-\mu_j^{-1}) \\
&\quad \times\|T(PT)^{J-1}x_{i+n}-T(PT)^{J-1}x_{k+n}\|)\} \\
&\leq \mu_j\phi^{-1}\{max_{1\leq j,k\leq n}(\varepsilon+\varepsilon+(1-\mu_j^{-1})\mu_j \\
&\quad \times\|x_{i+n}-x_{k+n}\|\} \\
&\leq \mu_j\phi^{-1}\{max_{1\leq j,k\leq n}(\varepsilon+\varepsilon+(1-\mu_j^{-1})\mu_j \\
&\quad \times(\|x_{i+n}\|+\|x_{k+n}\|\}.
\end{aligned}
$$

Thus,

$$
\|T(PT)^{j-1}y_n-\sum_{i=1}^{m(n)}t_i^{(n)}T(PT)^{j-1}x_{i+n}\|\leq(\varepsilon+\varepsilon+2r(1-\mu_j^{-1})\mu_j), \tag{2.9}
$$

since x_{i+n} and x_{k+n} are both in C.

Also, (2.7), (2.8) and (2.9) imply that

$$
\|T(PT)^{j-1}y_n-y_n\|\leq\mu_j\phi^{-1}(\varepsilon+\varepsilon+2r(1-\mu_j^{-1})\mu_j). \tag{2.10}
$$

Taking $\limsup_{n\to\infty}$ on both sides of (2.10), and noting that $\varepsilon>0$ is arbitrary, we have

$$
\limsup_{n\to\infty}\|T(PT)^{j-1}y_n-y_n\|\leq\mu_j\phi^{-1}(2r(1-\mu_j^{-1})\mu_j). \tag{2.11}
$$

On the other hand, for any $j\geq 1$, it follows from (2.3) that

$$
\begin{aligned}
\|T(PT)^{j-1}\omega-\omega\| &\leq \|T(PT)^{j-1}\omega-T(PT)^{j-1}y_n\|+\|T(PT)^{j-1}y_n-y_n\|+\|y_n-\omega\| \\
&\leq \mu_j\|y_n-\omega\|+\|T(PT)^{j-1}y_n-y_n\|+\|y_n-\omega\|. \qquad (2.12)
\end{aligned}
$$

Taking $\limsup_{n\to\infty}$ in the above inequality and using (2.11), we have

$$
\|T(PT)^{j-1}\omega-\omega\|\leq\mu_j\phi^{-1}(2r(1-\mu_j^{-1})\mu_j).
$$

Again, taking $\limsup_{j\to\infty}$ in the above inequality, we have

$$\limsup_{j\to\infty}\|T(PT)^{j-1}\omega-\omega\|\leq\phi^{-1}(0)=0,$$

which implies that $\|T(PT)^{j-1}\omega-\omega\|\to 0$ as $j\to\infty$, and hence proving our claim. By continuity of TP, we have that

$$\lim_{j\to\infty}TP(T(PT)^{j-1}\omega)=TP\omega=T\omega=\omega.$$

This completes the proof. □

3. Main Results

Lemma 3.1. *Let E be a uniformly convex Banach space and C a nonempty closed convex subset of E. Let $S_1,S_2\in\Gamma_r(C)$ and $S_1,S_2:C\longrightarrow C$ be two pointwise asymptotically nonexpansive self mappings with sequences $\{k_n^{(1)}(x)\},\{k_n^{(3)}(x)\}\in[1,\infty):\sum_{n\to\infty}(k_n^{(1)}(x)-1)<\infty,\sum_{n\to\infty}(k_n^{(3)}(x)-1)<\infty$. Let $T_1,T_2\in\Gamma_r(C)$ and $T_1,T_2:C\longrightarrow E$ be two pointwise asymptotically nonexpansive nonself mappings with sequences $\{k_n^{(2)}(x)\},\{k_n^{(4)}(x)\}\in[1,\infty):\sum_{n\to\infty}(k_n^{(2)}(x)-1)<\infty,\sum_{n\to\infty}(k_n^{(4)}(x)-1)<\infty$. Let $\{x_n\}$ be a sequence defined by*

$$(3.1)\qquad \left.\begin{array}{l} x_1\in C \\ \qquad x_{n+1}=P((1-\gamma_n)S_1^nx_n+\gamma_nT_1(PT_1)^{n-1}y_n \\ \qquad y_n=P((1-\alpha_n)S_2^nx_n+\alpha_nT_2(PT_2)^{n-1}x_n \end{array}\right\}$$

, where $\{\gamma_n\}$ and $\{\alpha_n\}$ are real sequences $\in[0,1)$. Suppose $\mathscr{F}=F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)\neq\emptyset$. Then, $\lim_{n\infty}\|x_n-q\|$ and $\lim_{n\infty}d(x_n-\mathscr{F})$ both exist for all $q\in\mathscr{F}$.

Proof. For any $q\in\mathscr{F}$, it follows from (3.1) that

$$\begin{aligned}\|y_n-q\| &= |P((1-\alpha_n)S_2^nx_n+\alpha_nT_2(PT_2)^{n-1}x_n)-P(q)\| \\ &\leq \|(1-\alpha_n)S_2^nx_n+\alpha_nT_2(PT_2)^{n-1}x_n-q\| \\ &\leq (1-\alpha_n)\|S_2^nx_n-q\|+\alpha_n\|T_2(PT_2)^{n-1}x_n-q\| \\ &\leq (1-\alpha_n)k_n^{(3)}(x)\|x_n-q\|+\alpha_nk_n^{(4)}(x)\|x_n-q\|\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)m_n(x)\|x_n-q\|+\alpha_n m_n(x)\|x_n-q\| \\
&= (1+Q_n(x))\|x_n-q\| \qquad (3.2)
\end{aligned}
$$

Again, using (3.1), we have

$$
\begin{aligned}
\|x_{n+1}-q\| &= |P((1-\gamma_n)S_1^n x_n+\gamma_n T_1(PT_1)^{n-1}y_n)-P(q)\| \\
&\leq \|(1-\gamma_n)S_1^n x_n+\gamma_n T_1(PT_1)^{n-1}y_n-q\| \\
&\leq (1-\gamma_n)\|S_1^n x_n-q\|+\gamma_n\|T_1(PT_1)^{n-1}y_n-q\| \\
&\leq (1-\gamma_n)k_n^{(1)}(x)\|x_n-q\|)+\gamma_n k_n^{(2)}(x)\|y_n-q\| \\
&\leq (1-\gamma_n)m_n(x)\|x_n-q\|)+\gamma_n m_n(x)\|y_n-q\| \qquad (3.3)
\end{aligned}
$$

Putting (3.2) into (3.3), we obtain

$$
\begin{aligned}
\|x_{n+1}-q\| &\leq (1-\gamma_n)m_n(x)\|x_n-q\|+\gamma_n(1+Q_n(x))\|x_n-q\| \\
&\leq (1-\gamma_n)(1+Q_n(x))\|x_n-q\|+\gamma_n(1+Q_n(x))\|x_n-q\| \\
&= (1+\delta_n(x))\|x_n-q\|, \qquad (3.4)
\end{aligned}
$$

where $\delta_n(x)=Q_n(x)$. Since $\sum_{n=1}^{\infty}\delta_n(x)<\infty$, it follows from lemma 2.1 that $\lim_{n\to\infty}\|x_n-q\|$ exists.

Now, taking the infimum over all $q\in F$ in (3.4), we get

$$d(x_{n+1},F)\leq(1+\delta_{n)}d(x_n,F),\forall n\in N.$$

Again, since $\sum_{n=1}^{\infty}\delta_n x<\infty$, it follows from Lemma 2.1 that $\lim_{n\to\infty}d(x_n,F)$ exists. This completes the proof. □

Lemma 3.2. *Let E be a uniformly convex Banach space and C a nonempty closed convex subset of E. Let $S_1,S_2\in\Gamma_r(C)$ and $S_1,S_2:C\longrightarrow C$ be two pointwise asymptotically nonexpansive self mapping with the sequences $\{k_n^{(1)}(x)\},\{k_n^{(3)}(x)\}\in[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(1)}-1)<\infty,\sum_{n=1}^{\infty}(k_n^{(3)}-1)<\infty$. Let $T_1,T_2\in\Gamma_r(C)$ and $T_1,T_2:C\longrightarrow E$ be two pointwise asymptotically nonexpansive nonself mappings with sequences $\{k_n^{(2)}(x)\},\{k_n^{(4)}(x)\}\in[1,\infty)$ such that*

$(k_n^{(2)}-1)<\infty, \sum_{n=1}^{\infty}(k_n^{(4)}-1)<\infty$. *Let* $\{x_n\}$ *be a sequence defined by*

$$\left.\begin{aligned} & x_1 \in C \\ & \qquad x_{n+1} = P((1-\gamma_n)S_1^n x_n + \gamma_n T_1(PT_1)^{n-1}y_n \\ & \qquad y_n = P((1-\alpha_n)S_2^n x_n + \alpha_n T_2(PT_2)^{n-1}x_n \end{aligned}\right\} \tag{3.5}$$

, where $\{\gamma_n\}$ *and* $\{\alpha_n\}$ *are real sequences* $\in [0,1)$. *Suppose* $\mathscr{F} = F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2) \neq 0$. *If the following conditions hold:*

i. $\sum_{n=1}^{\infty} Q_n(x) < \infty$;
ii. $\|x - T_1(PT_1)^{n-1}y\| \leq \|S_1^n x - T_1(PT_1)^{n-1}y\|, \|x - T_2(PT_2)^{n-1}y\| \leq \|S_2^n x - T_2(PT_2)^{n-1}y\|$;

Then, $\lim_{n\infty}\|x_n - S_i x_n\| = 0$ *and* $\lim_{n\infty}\|x_n - T_i x_n\| = 0$*, for* $i = 1,2$.

Proof. For any given $q \in \mathscr{F}$, $\lim_{n\infty}\|x_n - q\|$ exists by Lemma 3.1. Now, assume that $\lim_{n\infty}\|x_n - q\| = c$. Then, it follows from (3.3) and the fact that $\sum_{n=1}^{\infty} Q_n(x) < \infty$ that

$$\lim\|(1-\gamma_n)(S_1^n x_n - q) - \gamma_n T_1(PT_1)^{n-1}y_n - q)\| = c. \tag{3.6}$$

ln addition, we have

$$\begin{aligned} \|S_1^n x_n - q\| &\leq k_n^{(1)}(x)\|x_n - q\| \\ &\leq m_n(x)\|x_n - q\| \\ &= (1+Q_n(x))\|x_n - q\|. \end{aligned}$$

$$\Rightarrow \limsup\|S_1^n x_n - q\| \leq \limsup(1+Q_n(x))\|x_n - q\| = c. \tag{3.7}$$

Furthermore,

$$\begin{aligned} \|T_1(PT_1)y_n - q\| &\leq k_n^{(2)}(x)\|y_n - q\| \\ &\leq m_n(x)\|y_n - q\| \\ &\leq (1+Q_n(x))\|y_n - q\|. \end{aligned}$$

(3.8)

Using (3.2) and the fact that

$$\limsup_{n\to\infty} \|y_n - q\| \leq c,$$

we obtain from (3.8) that

$$\limsup \|T_1(PT_1)y_n - q\| \leq \limsup[(1+Q_n(x)K)\|y_n - q\| + \theta_n(x)] \leq c \tag{3.9}$$

Thus, from (3.6), (3.7), (3.9) and Lemma 2.2, we get

$$\lim_{n\to\infty} \|S_1^n x_n - T_1(PT_1)^{n-1} y_n\| = 0. \tag{3.10}$$

Condition (ii) and (3.10) imply that

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1} y_n\| = 0. \tag{3.11}$$

Again, we have

$$\begin{aligned} \|S_2^n x_n - q\| &\leq k_n^{(2)}(x)\|x_n - q\| \\ &\leq m_n(x)\|x_n - q\| \\ &\leq (1+Q_n(x))\|x_n - q\|. \\ \Rightarrow \limsup_{n\to\infty} \|S_2^n x_n - q\| &\leq \limsup(1+Q_n(x))\|x_n - q\| = c. \end{aligned} \tag{3.12}$$

Furthermore,

$$\begin{aligned} \|T_2(PT_2)x_n - q\| &\leq k_n^{(3)}(x)\|x_n - q\| \\ &\leq m_n(x)\|x_n - q\| \\ &\leq (1+Q_n(x)K)\|x_n - q\|. \end{aligned}$$

$$\Rightarrow \limsup_{n\to\infty} \|T_2(PT_2)x_n - q\| \leq \limsup(1+Q_n(x)K)\|x_n - q\| \leq c. \tag{3.13}$$

From (3.12), (3.13) and the fact that

$$\lim \|(1-\gamma_n)(S_2^n x_n - q) - \gamma_n T_2(PT_2)^{n-1} x_n - q)\| = c,$$

we get, using lemma 2.2, that

$$\lim_{n\to\infty}\|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.14}$$

By Condition (ii), it follows that $\|x_n - T_2(PT_2)^{n-1}x_n\| \leq \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\|$, and so from (3.14), we have

$$\lim_{n\to\infty}\|x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.15}$$

Now, we prove that

$$\lim_{n\to\infty}\|x_n - T_1(PT_1)^{n-1}x_n\| = \lim_{n\to\infty}\|x_n - T_2(PT_2)^{n-1}x_n\| = 0.$$

Since, $P(S^n x_n) = S^n x_n$ and $P : E \longrightarrow K$ is a nonexpansive retraction of E onto C, we get

$$\begin{aligned}
\|y_n - S_2^n x_n\| &= \|P((1-\alpha_n)S_2^n x_n + \alpha_n T_2(PT_2)^{n-1}x_n) - S_2^n x_n\| \\
&\leq \|(1-\alpha_n)S_2^n x_n + \alpha_n T_2(PT_2)^{n-1}x_n - S_2^n x_n\| \\
&= \alpha_n\|(S_2^n x_n - \alpha_n T_2(PT_2)^{n-1}x_n)\|
\end{aligned} \tag{3.16}$$

(3.15) and (3.16) imply that

$$\lim_{n\to\infty}\|y_n - S_2^n x_n\| = 0. \tag{3.17}$$

Furthermore, we have

$$\begin{aligned}
\|y_n - x_n\| &= \|y_n - S_2^n x_n + S_2^n x_n - T_2(PT_2)^{n-1}x_n + T_2(PT_2)^{n-1}x_n - x_n\| \\
&\leq \|y_n - S_2^n x_n\| + \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| + \|T_2(PT_2)^{n-1}x_n - x_n\|.
\end{aligned}$$

Thus, it follows from (3.14), (3.15) and (3.17) that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.18}$$

Observe that

$$
\begin{aligned}
& \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| \\
\leq\ & \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\| \\
\leq\ & \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + (\|y_n - x_n\| + \nu_n^{(1)}(x)\phi(\|y_n - x_n\|) + \omega_n^{(1)}(x) \\
\leq\ & \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + \|y_n - x_n\| + KQ_n(x)\|y_n - x_n\| + \theta_n(x) \\
=\ & \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + (1 + KQ_n(x))\|y_n - x_n\| + \theta_n(x). \qquad (3.19)
\end{aligned}
$$

lt follows from (3.10),(3.18), 3.19 and the fact that $\sum_{n=1}^{\infty}\Theta_n(x) < \infty$ that

$$
\lim_{n\to\infty}\|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| = 0. \qquad (3.20)
$$

Again, since $\|x_n - T_1(PT_1)^{n-1}x_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\|$, (by condition (ii), we obtain, using (3.20), that

$$
\lim_{n\to\infty}\|x_n - T_1(PT_1)^{n-1}x_n\| = 0. \qquad (3.21)
$$

Now, it follows from

$$
\begin{aligned}
\|x_{n+1} - S_1^n x_n\| &= \|P[(1-\gamma_n)S_1^n x_n + \gamma_n T_1(PT_1)^{n-1}y_n] - S_1^n x_n\| \\
&\leq \|(1-\gamma_n)S_1^n x_n + \gamma_n T_1(PT_1)^{n-1}y_n - S_1^n x_n\| \\
&= \gamma_n\|S_1^n x_n - T_1(PT_1)^{n-1}y_n]\|,
\end{aligned}
$$

and (3.10) that

$$
\lim_{n\to\infty}\|x_{n+1} - S_1^n y_n\| = 0. \qquad (3.22)
$$

Also, from (3.10), (3.22) and the inequality

$$
\|x_{n+1} - T_1(PT_1)^{n-1}y_n\| \leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\|,
$$

we get

$$
\lim_{n\to\infty}\|x_{n+1} - T_1(PT_1)^{n-1}y_n\| = 0. \qquad (3.23)
$$

Futhermore, (3.10), (3.11) and the inequality:

$$
\|S_1^n x_n - x_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - x_n\|,
$$

imply that

$$\lim_{n\to\infty} \|S_1^n x_n - x_n\| = 0. \tag{3.24}$$

In addition, (3.14), (3.15) and the inequality:

$$\|S_2^n x_n - x_n\| \leq \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| + \|T_2(PT_2)^{n-1}x_n - x_n\|,$$

imply that

$$\lim_{n\to\infty} \|S_2^n x_n - x_n\| = 0. \tag{3.25}$$

Moreover, from (3.15), (3.24) and the inequality:

$$\|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| \leq \|S_1^n x_n - x_n\| + \|x_n - T_2(PT_2)^{n-1}x_n\|,$$

we obtain

$$\lim_{n\to\infty} \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.26}$$

Since

$$\begin{aligned} \|x_{n+1} - T_2(PT_2)^{n-1}y_n\| &\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| \\ &\quad + \|T_2(PT_2)^{n-1}x_n - T_2(PT_2)^{n-1}y_n\| \\ &\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| + k_n(x)\|x_n - y_n\| \\ &\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| + m_n(x)\|x_n - y_n\| \\ &= \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| + (1 + Q_n(x))\|x_n - y_n\|), \end{aligned}$$

it follows from (3.18), (3.22), (3.26) and the fact that $\sum_{n=1}^{\infty} \theta_n(x) < \infty$ that

$$\lim_{n\to\infty} \|x_{n+1} - T_2(PT_2)^{n-1}y_n\| = 0. \tag{3.27}$$

Now, since $(PTi)(PTI)^{n-2}y_{n-1}, x_n \in C$ for $i = 1,2$, and T_1, T_2, are two total pointwise asymptotically nonexpansive nonself mappings, we have

$$\begin{aligned} \|T_i(PT_i)^{n-1}y_{n-1} - T_ix_n\| &= \|T_i(PT_i)(PT_i)^{n-2}y_{n-1} - T_i(Px_n)\| \\ &\leq k_n^{(i)}(x)\|(PT_i)(PT_i)^{n-2}y_{n-1} - P(x_n)\| \\ &\leq m_n(x)\|(PT_i)(PT_i)^{n-2}y_{n-1} - P(x_n)\| \\ &\leq (1+Q_n(x))\|T_i(PT_i)^{n-2}y_{n-1} - x_n\|. \end{aligned} \tag{3.28}$$

For $i = 1.2$, it follows from (3.23), (3.27) and the fact that $\sum_{n=1}^{\infty} Q_n(x) < \infty$ that

$$\lim_{n\to\infty} \|Ti(PTi)^{n-1}y_{n-1} - Tix_n\| = 0. \tag{3.29}$$

Moreover, observe that

$$\|x_{n+1} - y_n\| \leq \|x_{n+1} - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - x_n\| + \|x_n - y_n\|,$$

which by (3.11), (3.18) and (3.23) gives

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.30}$$

Next, observe, for $i = 1,2$, that

$$\begin{aligned} \|x_n - T_ix_n\| &\leq \|x_n - T_i(PT_i)^{n-1}x_n\| + \|T_i(PT_i)^{n-1}x_n - T_i(PT_i)^{n-1}y_{n-1}\| \\ &\quad + \|T_i(PT_i)^{n-1}y_{n-1} - T_ix_n\| \\ &\leq \|x_n - T_i(PT_i)^{n-1}x_n\| + k_n^{(i)}(x)\|x_n - y_{n-1}\| + \|T_i(PT_i)^{n-1}y_{n-1} - T_ix_n\| \\ &\leq \|x_n - T_i(PT_i)^{n-1}x_n\| + m_n(x)\|x_n - y_{n-1}\| + \|T_i(PT_i)^{n-1}y_{n-1} - T_ix_n\| \\ &= \|x_n - T_i(PT_i)^{n-1}x_n\| + (1+Q_n(x))\|x_n - y_{n-1}\| + \|T_i(PT_i)^{n-1}y_{n-1} - T_ix_n\| \end{aligned}$$

Thus, it follows from (3.15), (3.21), (3.29),(3.30) and the fact that $\sum_{n=1}^{\infty} Q_n(x) < \infty$ that $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0$, for $i = 1,,2$.

Finally, we prove that $\lim_{n\to\infty} \|x_n - S_i^n x_n\| = 0$, for $i = 1,,2$.

Observe that

$$
\begin{aligned}
\|x_n - S_i x_n\| &\le \|x_n - S_i^n x_n\| + \|S_i(S_i^{n-1}x_n) - S_i^n\| \\
&\le \|x_n - S_i^n x_n\| + k_n^i(x)\|S_i^{n-1}x_n - x_n\| \\
&\le \|x_n - S_i^n x_n\| + m_n(x)\|S_i^{n-1}x_n - x_n\| \\
&\le \|x_n - S_i^n x_n\| + (1+Q_n(x))\|x_n - x_{n-1}\| + \|x_{n-1} - S_i^{n-1}x_{n-1}\| \\
&\quad + \|S_i^{n-1}x_{n-1} - S_i^{n-1}x_n\| \\
&\le \|x_n - S_i^n x_n\| + (1+Q_n(x))[\|x_n - x_{n-1}\| + \|x_{n-1} - S_i^{n-1}x_{n-1}\|] \\
&\quad + (1+Q_n(x))k_n^{(i)}(x)\|x_{n-1} - x_n\| \\
&\le \|x_n - S_i^n x_n\| + (1+Q_n(x))[\|x_n - x_{n-1}\| + \|x_{n-1} - S_i^{n-1}x_{n-1}\|] \\
&\quad + (1+Q_n(x))m_n(x)\|x_{n-1} - x_n\| \\
&\le \|x_n - S_i^n x_n\| + (1+Q_n(x))[\|x_n - x_{n-1}\| + \|x_{n-1} - S_i^{n-1}x_{n-1}\|] \\
&\quad + (1+Q_n(x))(1+Q_n(x))\|x_{n-1} - x_n\| \\
&= \|x_n - S_i^n x_n\| + (1+Q_n(x))[\|x_n - x_{n-1}\| + \|x_{n-1} - S_i^{n-1}x_{n-1}\|] \\
&\quad + (1+Q_n(x))^2\|x_{n-1} - x_n\| \qquad (3.31)
\end{aligned}
$$

Since,

$$\|x_n - x_{n-1}\| \le \|x_n - S_1^n x_n\| + \gamma_n\|S_1^n x_n - T_1(PT_1)^{n-1}y_n\|,$$

it follows from (3.10), (3.24), (3.25),(3.31) and the fact that $\sum_{n=1}^{\infty}\theta_n(x) < \infty$ that

$$\lim_{n\to\infty}\|x_n - S_i x_n\| = 0, i = 1,2. \qquad (3.32)$$

This completes the proof. □

Theorem 3.3. *Under the assumption of Lemma* 3.2*, if E satisfies Opial's condition and the mappings $I - S_i$ and $I - T_i$ for $i = 1,2$, where I denotes the identity mapping, are demiclosed at zero, then the sequence $\{x_n\}$ defined by* (3.1) *converges weakly to a common fixed point in $\mathscr{F} = F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)$.*

Proof. Let $q^\star \in \mathscr{F}$. From Lemma 3.1, the squence $\{\|x_n - p^\star\|\}$ is bounded. Since, E is uniformly convex , every bounded subset of E is weakly compact. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q^\star \in C$. By Lemma 3.2, we have $\lim_{n\to\infty} \|x_{n_k} - S_i x_{n_k}\| = 0$ and $\lim_{n\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$ for $i = 1,2$. Since by hypothesis, the mappings $I - S_i$ and $I - T_i$ for $i = 1,2$, where I denotes the identity mapping, are demiclosed at zero, $S_i q^\star = q^\star$ and $T_i q^\star = q^\star$ for $i = 1,2.$, which means that $q^\star \in \mathscr{F} = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Finally, we show that $\{x_n\}$ converges weakly to $q^\star$. Suppose on the contrary that $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ which converges weakly to $p^\star \in C$ and $q^\star \neq p^\star$ By Lemma 3.1, $\lim_{n\to\infty} \|x_n - q^\star\|$ and $\lim_{n\to\infty} \|x_n - p^\star\|$ exist. By virtue of Opial's condition on E, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - q^\star\| &= \lim_{n\to\infty} \|x_{n_k} - q^\star\| \\
&< \lim_{n\to\infty} \|x_{n_k} - p^\star\| \\
&= \lim_{n\to\infty} \|x_n - p^\star\| \\
&= \lim_{n\to\infty} \|x_{n_j} - p^\star\| \\
&< \lim_{n\to\infty} \|x_{n_j} - q^\star\| \\
&= \lim_{n\to\infty} \|x_n - q^\star\|,
\end{aligned}$$

which is a contradiction, so $q^\star = p^\star$ Therefore, the sequence $\{x_n\}$ defined by (3.1) converges weakly to $q^\star \in \mathscr{F}$. This completes the proof. □

Theorem 3.4. *Let E be a uniformly convex Banach space which satisfies Opial's conditon and C be a nonempty closed convex subset of E. Let $T,S \in T_r(C)$ with $T : C \longrightarrow E, S : C \longrightarrow C$ and $S_i, T_i, i = 1,2,$ be compact mappings. Let $gl[(T,S),z_n]$ be the generalised lshikawa-type iterative scheme of (3.1) and $\{\gamma_n\}, \{\alpha_n\}$ be sequence bounded away from 0 and 1. Then, the iterative scheme $gl[(T,S),z_n]$ is stable.*

Proof. Let $\{z_n\}$ be an arbitrary sequence such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. For each $p \in F$, it follows that

$$\begin{aligned}
\|z_{n+1} - p\| &= \|z_{n+1} - gl[(T,S),z_n] + gl[(T,S),z_n] - p\| \\
&\leq \|z_{n+1} - gl[(T,S),z_n]\| + \|gl[(T,S),z_n] - p\| \\
&\leq \varepsilon_n + \|gl[(T,S),z_n] - p\| \qquad (3.33)
\end{aligned}$$

By the definition of $gl[(T,S),z_n]$ and pointwise asymptotically nonexpansiveness of S and T, we obtain

$$\begin{aligned}
\|y_n - p\| &= \|P((1-\alpha_n)S_2^n z_n + \alpha_n T_2(PT_2)^{n-1}z_n) - p\| \\
&\leq \|(1-\alpha_n)S_2^n z_n + \alpha_n T_2(PT_2)^{n-1}z_n - p\| \\
&\leq (1-\alpha_n)\|S_2^n z_n - p\| + \alpha_n\|T_2(PT_2)^{n-1}z_n - p\| \\
&\leq (1-\alpha_n)k_n^{(3)}(x)\|z_n - p\|) + \alpha_n k_n^{(4)}(x)\|z_n - p\| \\
&\leq (1-\alpha_n)m_n(x)\|z_n - p\|) + \alpha_n m_n(x)\|z_n - p\| \\
&= (1+Q_n(x))\|z_n - p\| \qquad (3.34)
\end{aligned}$$

and

$$\begin{aligned}
\|gl[(T,S),z_n] - p\| &= \|P((1-\gamma_n)S_1^n x_n + \gamma_n T_1(PT_1)^{n-1}y_n) - p\| \\
&\leq \|(1-\gamma_n)S_1^n x_n + \gamma_n T_1(PT_1)^{n-1}y_n - p\| \\
&\leq (1-\gamma_n)\|S_1^n x_n - p\| + \gamma_n\|T_1(PT_1)^{n-1}y_n - p\| \\
&\leq (1-\gamma_n)k_n^{(1)}(x)\|z_n - p\| + \gamma_n k_n^{(2)}(x)\|y_n - p\| \\
&\leq (1-\gamma_n)m_n(x)\|z_n - p\| + \gamma_n m_n(x)\|y_n - p\| \\
&\leq (1-\gamma_n)(1+Q_n(x))\|z_n - p\| + \gamma_n(1+Q_n(x))\|y_n - p\|)
\end{aligned}$$

(3.35)

(3.34) and (3.35) imply that

$$\begin{aligned}
\|gl[(T,S),p] - p\| &\leq (1-\gamma_n)(1+Q_n(x))\|z_n - p\| + (1+Q_n(x))(1+Q_n(x)) \\
&\quad \times\|z_n - p\| \\
&= [1+(1+(1+Q_n(x)))Q_n(x))]\|z_n - p\| \qquad (3.36)
\end{aligned}$$

Putting (3.36) into (3.33), we get

$$\|z_{n+1} - p\| \leq \varepsilon_n + [1+(1+(1+Q_n(x)))Q_n(x))]\|z_n - p\|$$

Since $\sum_{n=1}^{\infty} Q_n(x) < \infty$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, it follows from Lemma 2.1 that $\{z_n\}$ is bounded and that $\lim_{n\to\infty} \|z_n - p\|$ exists. Let $\lim_{n\to\infty} \|z_n - p\| = c$, where c is any real number. From (3.34) and the fact that $\sum_{n=1}^{\infty} Q_n(x) < \infty$, we have

$$\lim_{n\to\infty} \|y_n - p\| \leq c.$$

Actually, $\lim_{n\to\infty} \|y_n - p\| = c$. For if $\lim_{n\to\infty} \|y_n - p\| < c$, then from (3.35), we get

$$\lim_{n\to\infty} \|gl[(T,S),z_n] - p\| < c \tag{3.37}$$

(3.37) together with (3.33) imply that

$$\begin{aligned} c &= \lim_{n\to\infty} |z_{n+1} - p\| \\ &\leq \lim_{n\to\infty} \varepsilon_n + \lim_{n\to\infty} \|gl[(T,S),z_n] - p\| \\ &< c, \end{aligned}$$

which is a contradiction, so

$$\lim_{n\to\infty} \|y_n - p\| = c \tag{3.38}$$

Following the same argument as in Lemma 3.2 above, we obtain

$$\lim_{n\to\infty} \|z_n - Siz_n\| = 0 \tag{3.39}$$

and

$$\lim_{n\to\infty} \|z_n - Tiz_n\| = 0, \tag{3.40}$$

for i=1,2. Again, since T_i and S_i are compact, there exists a subsequence $\{z_{n_n}\}$ and $p \in E$ such that

$$\lim_{n\to\infty} \|T_i z_n - p\| = 0$$

and

$$\lim_{n\to\infty} \|S_i z_n - p\| = 0.$$

Furthermore, from (3.39) and (3.40),

$$\begin{aligned} \lim_{k\to\infty} \|z_{n_k} - p\| &\leq \lim_{k\to\infty} \|z_{n_k} - T_i z_{n_k}\| + \lim_{k\to\infty} \|T_i z_{n_k} - p\| \\ &= 0 \end{aligned} \tag{3.41}$$

By Lemma 2.4, we obtain that $p \in F$. Since, $\lim_{n\to\infty} \|y_n - p\| = 0$, it follows that$\lim_{n\to\infty} y_n = p$. □

Corollary 3.5. *Let E be a uniformly convex Banach space and C a nonempty closed convex subset of E. Let $T_1, T_2 \in \Gamma_r(C)$ and $T_1, T_2 : C \longrightarrow E$ be two pointwise asymptotically nonexpansive nonself mappings with sequences $\{k_n\}^{(1)}(x), \{k_n\}^{(2)}(x) \in [1,\infty) : \sum_{n\to\infty}(k_n^{(1)}(x) - 1) < \infty, \sum_{n\to\infty}(k_n^{(2)} - 1)(x) < \infty$. Let $\{x_n\}$ be a sequence defined by*

$$
\left.\begin{array}{l} x_1 \in C \\ x_{n+1} = P((1-\gamma_n)x_n + \gamma_n T_1 (PT_1)^{n-1} y_n \\ y_n = P((1-\alpha_n)x_n + \alpha_n T_2 (PT_2)^{n-1} x_n \end{array}\right\}, \tag{3.42}
$$

where $\{\gamma_n\}$ and $\{\alpha_n\}$ are real sequences $\in [0,1)$. Suppose $\mathscr{F} = F(T_1) \cap F(T_2) \neq \emptyset$. Then, $\lim_{n\infty} \|x_n - q\|$ *and* $\lim_{n\infty} d(x_n - \mathscr{F})$ *both exist for all $q \in F$.*

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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