Available online at http://scik.org
Advances in Fixed Point Theory, 3 (2013), No. 1, 213-225
ISSN: 1927-6303

STRONG CONVERGENCE THEOREMS FOR TWO FINITE FAMILIES OF ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPINGS IN BANACH SPACES

G. S. SALUJA

Department of Mathematics, Govt. Nagarjuna P.G. College of Science, Raipur - 492010 (C.G.), India

Abstract. The purpose of this paper is to establish strong convergence theorems of finite step iteration process with errors for two finite families of non-Lipschitzian asymptotically quasi-nonexpansive type mappings to converge to common fixed point in the framework of Banach spaces. The results presented in this paper improve and extend some results in Chen and Guo (2011) [1], Sitthikul and Saejung (2009) [18] and many others.

Keywords: Asymptotically quasi-nonexpansive type mapping, finite-step iteration process with errors, common fixed point, strong convergence, Banach space.

2000 AMS Subject Classification: 47H09, 47H10, 47J25.

1. Introduction and Preliminaries

Let K be a nonempty subset of a real Banach space E. Let $T\colon K \to K$ be a mapping, then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$. A mapping $T\colon K \to K$ is said to be:

Received December 11, 2012

(i) nonexpansive if

$$\|Tx - Ty\| \;\leq\; \|x - y\| \tag{1}$$

for all $x, y \in K$;

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \;\leq\; \|x - p\| \tag{2}$$

for all $x \in K$, $p \in F(T)$;

(iii) asymptotically nonexpansive [5] if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \;\leq\; k_n \, \|x - y\| \tag{3}$$

for all $x, y \in K$ and $n \geq 1$;

(iv) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \;\leq\; k_n \, \|x - p\| \tag{4}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$;

(v) uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \;\leq\; L \, \|x - y\| \tag{5}$$

for all $x, y \in K$ and $n \geq 1$;

(vi) asymptotically nonexpansive type [7], if

$$\limsup_{n\to\infty} \left\{ \sup_{x,\, y \in K} \Big(\|T^n x - T^n y\| - \|x - y\| \Big) \right\} \;\leq\; 0; \tag{6}$$

(vii) asymptotically quasi-nonexpansive type [12], if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \left\{ \sup_{x \in K,\;\, p \in F(T)} \Big(\|T^n x - p\| - \|x - p\| \Big) \right\} \;\leq\; 0. \tag{7}$$

Remark 1.1. It is easy to see that if $F(T)$ is nonempty, then asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping and asymptotically nonexpansive type mapping are the special cases of asymptotically quasi-nonexpansive type mappings.

The class of asymptotically nonexpansive self-mappings was introduced by Goebel and Kirk [5] in 1972 as an important generalization of the class of nonexpansive self-mappings, who proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Since then, iteration processes for asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings in Banach spaces have studied extensively by many authors (see [2],[4],[6]-[17]). In 2002, Xu and Noor [20] introduced and studied a three-step iteration scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. Cho et al. [3] extended the work of Xu and Noor to a three-step iterative scheme with errors in Banach space and proved the weak and strong convergence theorems for asymptotically nonexpansive mappings. In 2003, Sahu and Jung [12] studied Ishikawa and Mann iteration process in Banach spaces and they proved some weak and strong convergence theorems for asymptotically quasi-nonexpansive type mapping. In 2006, Shahzad and Udomene [17] gave the necessary and sufficient condition for convergence of common fixed point of two-step modified Ishikawa iterative sequence for two asymptotically quasi-nonexpansive mappings in real Banach space. In 2009, Sitthikul and Saejung [18] introduced and studied a finite-step iteration scheme for a finite family of nonexpansive and asymptotically nonexpansive mappings and proved some weak and strong convergence theorems in the setting of Banach spaces. Recently, Chen and Guo [1] introduced and studied a new finite-step iteration scheme with errors for two finite families of asymptotically nonexpansive mappings as follows:

Let K be a nonempty convex subset of a Banach space E with $K + K \subset K$. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N \colon K \to K$ be $2N$ asymptotically nonexpansive mappings. Then the sequence $\{x_n\}$ defined by

$$
\begin{aligned}
x_1 &= x \in K, \\
x_n^{(0)} &= x_n, \\
x_n^{(1)} &= \alpha_n^{(1)} T_1^n x_n^{(0)} + (1 - \alpha_n^{(1)}) S_1^n x_n + u_n^{(1)}, \\
x_n^{(2)} &= \alpha_n^{(2)} T_2^n x_n^{(1)} + (1 - \alpha_n^{(2)}) S_2^n x_n + u_n^{(2)}, \\
&\vdots \\
x_n^{(N-1)} &= \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + (1 - \alpha_n^{(N-1)}) S_{N-1}^n x_n + u_n^{(N-1)}, \\
x_n^{(N)} &= \alpha_n^{(N)} T_N^n x_n^{(N-1)} + (1 - \alpha_n^{(N)}) S_N^n x_n + u_n^{(N)}, \\
x_{n+1} &= x_n^{(N)}, \ \forall\, n \geq 1, \qquad (8)
\end{aligned}
$$

where $\{\alpha_n^{(i)}\} \subset [0, 1]$ and $\{u_n^{(i)}\}$ are bounded sequences in K for all $i \in I = \{1, 2, \ldots, N\}$, and the weak and strong convergence theorems are proved, which improve and generalize some results in [18].

The aim of this paper is to establish some strong convergence of the iteration scheme (8) to converge to common fixed points for two finite families of non-Lipschitzian asymptotically quasi-nonexpansive type mappings in the framework of Banach spaces. The results presented in this paper improve and extend the corresponding results of Chen and Guo (2011) [1], Sitthikul and Saejung (2009) [18] and many others.

In order to prove the main results of this paper, we need the following concepts and lemma:

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \Big\|\frac{1}{2}(x+y)\Big\| : \|x\| = 1,\ \|y\| = 1,\ \varepsilon = \|x-y\|\Big\}.$$

A mapping $T\colon K \to K$ is said to be semi-compact [2] if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ strongly.

Lemma 1.1. (See [19]) Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \;\le\; a_n + b_n,\ n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

2. Main results

In this section, we first prove the following lemma in order to prove our main theorems.

Lemma 2.1. Let E be a real Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N\colon K \to K$ be $2N$ asymptotically quasi-nonexpansive type mappings with $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (8), where $\{\alpha_n^{(i)}\} \subset [0,1]$ for all $i \in I$ with $\sum_{n=1}^{\infty} \Big\|u_n^{(i)}\Big\| < \infty$ for all $i \in I$. Put

$$\begin{aligned} A_{in} \;=\; & \max\Big\{ \sup_{p\in F,\ n\ge 1} \Big(\|T_i^n x_n - p\| - \|x_n - p\| \Big) \vee \\ & \sup_{p\in F,\ n\ge 1} \Big(\|S_i^n x_n - p\| - \|x_n - p\| \Big) \vee 0 :\ 1 \le i \le N \Big\} \end{aligned} \tag{9}$$

such that $\sum_{n=1}^{\infty} A_{in} < \infty$ for all $1 \le i \le N$. Then the limit $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$.

Proof. Let $q \in F$. Then from (8) and (9), we have

$$\begin{aligned}
\left\|x_n^{(1)} - q\right\| &= \left\|\alpha_n^{(1)} T_1^n x_n + (1-\alpha_n^{(1)}) S_1^n x_n + u_n^{(1)} - q\right\| \\
&\le \alpha_n^{(1)} \left\|T_1^n x_n - q\right\| + (1-\alpha_n^{(1)}) \left\|S_1^n x_n - q\right\| + \left\|u_n^{(1)}\right\| \\
&\le \alpha_n^{(1)}\Big[\|x_n - q\| + A_{1n}\Big] + (1-\alpha_n^{(1)})\Big[\|x_n - q\| + A_{1n}\Big] + \left\|u_n^{(1)}\right\| \\
&\le \|x_n - q\| + A_{1n} + \left\|u_n^{(1)}\right\| \\
&= \|x_n - q\| + d_{1n} \qquad (10)
\end{aligned}$$

where $d_{1n} = A_{1n} + \left\|u_n^{(1)}\right\|$. Since by assumption of the theorem $\sum_{n=1}^{\infty} \left\|u_n^{(1)}\right\| < \infty$ and $\sum_{n=1}^{\infty} A_{1n} < \infty$, it follows that $\sum_{n=1}^{\infty} d_{1n} < \infty$. Again using (8), (9) and (10), we obtain

$$\begin{aligned}
\left\|x_n^{(2)} - q\right\| &= \left\|\alpha_n^{(2)} T_2^n x_n^{(1)} + (1-\alpha_n^{(2)}) S_2^n x_n + u_n^{(2)} - q\right\| \\
&\le \alpha_n^{(2)} \left\|T_2^n x_n^{(1)} - q\right\| + (1-\alpha_n^{(2)}) \left\|S_2^n x_n - q\right\| + \left\|u_n^{(2)}\right\| \\
&\le \alpha_n^{(2)}\Big[\left\|x_n^{(1)} - q\right\| + A_{2n}\Big] + (1-\alpha_n^{(2)})\Big[\|x_n - q\| \\
&\quad + A_{2n}\Big] + \left\|u_n^{(2)}\right\| \\
&\le \alpha_n^{(2)}\Big[\|x_n - q\| + d_{1n}\Big] + (1-\alpha_n^{(2)}) \|x_n - q\| \\
&\quad + A_{2n} + \left\|u_n^{(2)}\right\| \\
&\le \|x_n - q\| + \alpha_n^{(2)} d_{1n} + A_{2n} + \left\|u_n^{(2)}\right\| \\
&\le \|x_n - q\| + d_{1n} + A_{2n} + \left\|u_n^{(2)}\right\| \\
&= \|x_n - q\| + d_{2n} \qquad (11)
\end{aligned}$$

where $d_{2n} = d_{1n} + A_{2n} + \left\|u_n^{(2)}\right\|$. Since by assumption of the theorem $\sum_{n=1}^{\infty} \left\|u_n^{(2)}\right\| < \infty$, $\sum_{n=1}^{\infty} A_{2n} < \infty$ and $\sum_{n=1}^{\infty} d_{1n} < \infty$, it follows that $\sum_{n=1}^{\infty} d_{2n} < \infty$. Continuing the above process, we get that

$$\left\|x_n^{(i)} - q\right\| \le \|x_n - q\| + d_{in} \qquad (12)$$

with $\sum_{n=1}^{\infty} d_{in} < \infty$ for all $n \ge 1$ and $1 \le i \le N$. In particular,

$$\|x_{n+1} - q\| = \left\|x_n^{(N)} - q\right\| \le \|x_n - q\| + d_{Nn}. \qquad (13)$$

Since $\sum_{n=1}^{\infty} d_{in} < \infty$ for all $n \geq 1$ and $1 \leq i \leq N$, it follows by Lemma 1.1, we have that $\lim_{n\to\infty} \|x_n - q\|$ exists. This completes the proof.

Theorem 2.1. Let E be a real Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N \colon K \to K$ be $2N$ non-Lipschitzian asymptotically quasi-nonexpansive type mappings with $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (8), where $\{\alpha_n^{(i)}\} \subset [a, 1-a]$ for some $a \in (0,1)$ and all $i \in I$ with $\sum_{n=1}^{\infty} \left\| u_n^{(i)} \right\| < \infty$ for all $i \in I$. Put

$$\begin{aligned} A_{in} &= \max\Big\{ \sup_{p\in F,\ n\geq 1} \Big(\|T_i^n x_n - p\| - \|x_n - p\| \Big) \vee \\ &\qquad \sup_{p\in F,\ n\geq 1} \Big(\|S_i^n x_n - p\| - \|x_n - p\| \Big) \vee 0 : \ 1 \leq i \leq N \Big\} \end{aligned}$$

such that $\sum_{n=1}^{\infty} A_{in} < \infty$ for all $1 \leq i \leq N$. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$ in K if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{\|x - y\| : y \in F\}$.

Proof. The necessity of Theorem 2.1 is obvious. So, we will prove the sufficiency. Assume that $\liminf_{n\to\infty} d(x_n, F) = 0$. Taking the infimum over all $q \in F$ in (13), we have

$$d(x_{n+1}, F) \leq d(x_n, F) + d_{Nn}.$$

By assumption of the theorem and Lemma 1.1, we know that $\lim_{n\to\infty} d(x_n, F)$ exists and so $\lim_{n\to\infty} d(x_n, F) = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in K. In fact, For any positive integers m, n, $m > n$, and (13), we have

$$\begin{aligned} \|x_m - q\| &\leq \|x_{m-1} - q\| + d_{N(m-1)} \\ &\leq \cdots \\ &\leq \|x_n - q\| + \sum_{k=n}^{m-1} d_{Nk}. \end{aligned}$$

Thus for any $q \in F$, we have

$$\begin{aligned} \|x_m - x_n\| &\leq \|x_m - q\| + \|x_n - q\| \\ &\leq 2\|x_n - q\| + \sum_{k=n}^{m-1} d_{Nk}. \end{aligned}$$

Taking the infimum over all $q \in F$, we obtain that

$$\|x_m - x_n\| \leq 2d(x_n, F) + \sum_{k=n}^{m-1} d_{Nk}.$$

It follows from $\sum_{k=n}^{\infty} d_{Nk} < \infty$ and $\lim_{n\to\infty} d(x_n, F) = 0$ that $\{x_n\}$ is a Cauchy sequence, K is a closed subset of E and so $\{x_n\}$ converges strongly to $q_0 \in K$. Further, $F(T_i)$ and $F(S_i)$ $(i = 1, 2, \dots, N)$ are closed sets, and so F is a closed subset of K. Therefore, $q_0 \in F$, that is, $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \dots, T_N, S_1, S_2, \dots, S_N\}$ in K. This completes the proof.

Theorem 2.2. Let E be a real Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N \colon K \to K$ be $2N$ uniformly L-Lipschitzian and non-Lipschitzian asymptotically quasi-nonexpansive type mappings with $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (8), where $\{\alpha_n^{(i)}\} \subset [a, 1-a]$ for some $a \in (0,1)$ and all $i \in I$ with $\sum_{n=1}^{\infty} \left\| u_n^{(i)} \right\| < \infty$ for all $i \in I$. Put

$$\begin{aligned} A_{in} &= \max\Big\{ \sup_{p\in F,\ n\geq 1} \Big(\|T_i^n x_n - p\| - \|x_n - p\| \Big) \vee \\ &\quad \sup_{p\in F,\ n\geq 1} \Big(\|S_i^n x_n - p\| - \|x_n - p\| \Big) \vee 0 :\ 1 \leq i \leq N \Big\} \end{aligned}$$

such that $\sum_{n=1}^{\infty} A_{in} < \infty$ for all $1 \leq i \leq N$. Suppose $\lim_{n\to\infty} \|x_n - S_i x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in I$. If there exists a T_i or S_i, $i \in I$, which is semi-compact. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \dots, T_N, S_1, S_2, \dots, S_N\}$ in K.

Proof. Without loss of generality, we can assume that T_1 is semi-compact. From Lemma 2.1, we know that the sequence $\{x_n\}$ is bounded and by assumption of the theorem, we know that $\lim_{n\to\infty} \|x_n - S_i x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in I$. Since T_1

is semi-compact and $\lim_{n\to\infty}\|x_n - T_1x_n\| = 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in K$ as $i \to \infty$. Thus

$$\|x^* - T_ix^*\| = \lim_{i\to\infty}\|x_{n_i} - T_ix_{n_i}\| = 0$$

and

$$\|x^* - S_ix^*\| = \lim_{i\to\infty}\|x_{n_i} - S_ix_{n_i}\| = 0$$

for all $i \in I$. Which implies that $x^* \in F = \bigcap_{i=1}^N F(S_i) \cap F(T_i)$ and so $\liminf_{n\to\infty} d(x_n, F) \leq \liminf_{i\to\infty} d(x_{n_i}, F) \leq \lim_{i\to\infty}\|x_{n_i} - x^*\| = 0$. It follows from Theorem 2.1 that $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$ in K. This completes the proof.

Theorem 2.3. Let E be a real Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $\{S_i\}_{i=1}^N, \{T_i\}_{i=1}^N \colon K \to K$ be $2N$ uniformly L-Lipschitzian and non-Lipschitzian asymptotically quasi-nonexpansive type mappings with $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (8), where $\{\alpha_n^{(i)}\} \subset [a, 1-a]$ for some $a \in (0,1)$ and all $i \in I$ with $\sum_{n=1}^\infty \left\|u_n^{(i)}\right\| < \infty$ for all $i \in I$. Put

$$\begin{aligned} A_{in} &= \max\Big\{ \sup_{p\in F,\ n\geq 1}\Big(\|T_i^n x_n - p\| - \|x_n - p\|\Big) \vee \\ &\qquad \sup_{p\in F,\ n\geq 1}\Big(\|S_i^n x_n - p\| - \|x_n - p\|\Big) \vee 0 :\ 1 \leq i \leq N\Big\} \end{aligned}$$

such that $\sum_{n=1}^\infty A_{in} < \infty$ for all $1 \leq i \leq N$. Suppose that the mappings S_i and T_i for all $i \in I$ satisfy the following conditions:

(d_1) $\lim_{n\to\infty}\|x_n - S_ix_n\| = 0$ and $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0$, for all $i \in I$;

(d_2) there exists a constant $A > 0$ such that

$$\Big\{\|x_n - S_ix_n\| + \|x_n - T_ix_n\|\Big\} \geq A\, d(x_n, F)$$

for all $n \geq 1$ and $i \in I$.

Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \dots, T_N, S_1, S_2, \dots, S_N\}$ in K.

Proof. From conditions (d_1) and (d_2), we have $\lim_{n\to\infty} d(x_n, F) = 0$, it follows as in the proof of Theorem 2.1, that $\{x_n\}$ must converges strongly to a common fixed point of the mappings $\{T_1, T_2, \dots, T_N, S_1, S_2, \dots, S_N\}$ in K. This completes the proof.

Remark 2.1. (i) Since asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mappings are asymptotically quasi-nonexpasive type mappings. Hence our results improve and generalize the corresponding results of [1, 18] and many others.

(ii) Our results also extend the corresponding results of Sahu et al. [12] to the case of multi-step iteration process with errors considered in this paper.

Example 2.1. Let $E = [-\pi, \pi]$ and let T be defined by

$$Tx = x\, cos x$$

for each $x \in E$. Clearly $F(T) = \{0\}$. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$\|Tx - z\| = \|Tx - 0\| = |x||cos\, x| \leq |x| = \|x - z\|\,,$$

and T is asymptotically quasi-nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$. Hence by remark 1.1, T is asymptotically quasi-nonexpansive type mapping. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{\pi}{2}$ and $y = \pi$, then

$$\|Tx - Ty\| = \left\|\frac{\pi}{2}\, cos\, \frac{\pi}{2} - \pi\, cos\, \pi\right\| = \pi,$$

whereas

$$\|x - y\| = \left\|\frac{\pi}{2} - \pi\right\| = \frac{\pi}{2}.$$

Example 2.2. Let $E = \mathbb{R}$ and let T be defined by

$$T(x) = \begin{cases} \frac{x}{2} \, cos \, \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \neq 0$ and $Tx = x$, then $x = \frac{x}{2} \, cos \, \frac{1}{x}$. Thus $2 = cos \, \frac{1}{x}$. This is impossible. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$\|Tx - z\| = \|Tx - 0\| = |\frac{x}{2}||cos \, \frac{1}{x}| \leq \frac{|x|}{2} < |x| = \|x - z\| \, ,$$

and T is asymptotically quasi-nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$. Hence by remark 1.1, T is asymptotically quasi-nonexpansive type mapping. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{2}{3\pi}$ and $y = \frac{1}{\pi}$, then

$$\|Tx - Ty\| = \left\| \frac{1}{3\pi} \, cos \, \frac{3\pi}{2} - \frac{1}{2\pi} \, cos \, \pi \right\| = \frac{1}{2\pi},$$

whereas

$$\|x - y\| = \left\| \frac{2}{3\pi} - \frac{1}{\pi} \right\| = \frac{1}{3\pi}.$$

3. Conclusion

By Remark 1.1 it is clear that if $F(T)$ is nonempty, then asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mappings are asymptotically quasi-nonexpansive type mappings, thus our results are good improvement and generalization of corresponding results of [1, 18] and many others from the current literature.

References

[1] W. Chen and W. Guo, Convergence theorems for two finite families of asymptotically nonexpansive mappings, Math. Comput. Modelling 54 (2011), 1311-1319. (Zbl 1228.47057)

[2] C.E. Chidume and B. Ali, Weak and strong convergence theorems for finite families of asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 330 (2007), 377-387. (Zbl 1141.47039)

[3] Y.J. Cho, H. Zhou and G. Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, Comput. Math. Appl. 47 (2004), 707-717. (Zbl 1081.47063)

[4] H. Fukhar-ud-din and S.H. Khan, Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications, J. Math. Anal. Appl. 328 (2007), 821-829. (Zbl 1113.47055)

[5] K. Goebel and W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972), no.1, 171-174. (Zbl 0256.47045)

[6] S.H. Khan and W. Takahashi, Approximating common fixed points of two asymptotically nonexpansive mappings, Sci. Math. Jpn. 53 (2001), no.1, 143-148. (Zbl 0985.47042)

[7] W.A. Kirk, Fixed point theorems for non-lipschitzian mappings of asymptotically nonexpansive type, Israel J. Math. 17 (1974), 339-346. (Zbl 0286.47034)

[8] Q.H. Liu, Iterative sequences for asymptotically quasi-nonexpansive mappings, J. Math. Anal. Appl. 259 (2001), 1-7. (Zbl 1033.47047)

[9] Q.H. Liu, Iterative sequences for asymptotically quasi-nonexpansive mappings with error member, J. Math. Anal. Appl. 259 (2001), 18-24. (Zbl 1001.47034)

[10] M.O. Osilike and S.C. Aniagbosor, Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings, Math. and Computer Modelling 32 (2000), 1181-1191. (Zbl 0971.47038)

[11] B.E. Rhoades, Fixed point iteration for certain nonlinear mappings, J. Math. Anal. Appl. 183 (1994), 118-120. (Zbl 0807.47045)

[12] D.R. Sahu, J.S. Jung, Fixed point iteration processes for non-Lipschitzian mappings of asymptotically quasi-nonexpansive type, Int. J. Math. Math. Sci. 33 (2003), 2075-2081.

[13] G.S. Saluja, Strong convergence theorem for two asymptotically quasi-nonexpansive mappings with errors in Banach spaces, Tamkang J. Math. 38(1) (2007), 85-92. (MR 2008b:47097, Zbl 1143.47306)

[14] G.S. Saluja, Approximating fixed points of Noor iteration with errors for asymptotically quasi-nonexpansive mappings, Funct. Anal. Appro. Comput. 1(1) (2009), 31-48. (MR 2011d:47159)

[15] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43(1) (1991), 153-159. (Zbl 0709.47051)

[16] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 158 (1991), 407-413. (Zbl 0734.47036)

[17] N. Shahzad and A. Udomene, Approximating common fixed points of two asymptotically quasi nonexpansive mappings in Banach spaces, Fixed point Theory and Applications 2006(2006), Article ID 18909, Pages 1-10.

[18] K. Sitthikul and S. Saejung, Convergence theorems for a finite family of nonexpansive and asymptotically nonexpansive mappings, Acta Univ. Palack. Olomuc. Math. 48 (2009), 139-152. (Zbl 1192.47063)

[19] K.K. Tan and H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993), 301-308. (MR1238879 (94g:47076), Zbl 0895.47048)

[20] B.L. Xu and M.A. Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 267(2) (2002), 444-453. (Zbl 1011.47039)