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SOME NEW GEOMETRIC CONSTANTS IN BANACH SPACES

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Abstract. In this article, inspired by the geometric constant $T(X)$, we introduce two geometric constants $T_G(X)$ and $T_\perp(X)$. Firstly, we give some basic properties of these two geometric constants. Then the relationship between these constants and other geometric constants can also be obtained. Moreover, we discuss the relation of constant $T_G(X)$ to the geometric properties of the Banach space.

Keywords: Banach space; geometric constant; Birkhoff orthogonality; uniformly nonsquareness.

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1. Introduction

The study of geometric constants on Banach spaces is very interesting. This is because it can help us to quantify some geometric properties of the Banach space. The best known of these are the von Neumann-Jordan constant $C_{NJ}(X)$ and the James constant $J(X)$. The study of these two geometric constants has attracted the attention of many scholars. One of the main concerns is how to calculate the values of the two geometric constants in Banach space. In [1], a new geometric constant $T(X)$ was introduced by Alonso et al., which studied the geometric mean of the variable lengths of the sides of a triangle. And this constant can also be used to estimate the specific values of the von Neumann-Jordan and James constant.

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Motivated by the characterizations of geometric properties in terms of $T(X)$, we will introduce two brand-new geometric constants $T_G(X)$ and $T_\perp(X)$ based on two unit vectors. The aim of this paper is to present the estimation of upper and lower bounds of these new constants, and then conduct an incisive investigation of the properties such as uniformly non-square property, value of inner product space, and uniformly normal structure, etc. In addition, we will clarify several relationships among these new constants and other geometric constants through inequalities.

2. PRELIMINARIES

Throughout the paper, we assume that all the discussed spaces X are real and have dimension at least two. Let $B_X = \{x \in X : \|x\| \leq 1\}$ be the unit ball and $S_X = \{x \in X : \|x\| = 1\}$ be the unit sphere.

A Banach space X is called uniformly non-square [9] if there exists a $\delta \in (0,1)$ such that either $\|x+y\| \leq 2(1-\delta)$ or $\|x-y\| \leq 2(1-\delta)$ for any $x,y \in S_X$. Conversely, if there exists $\|x\| = \|y\| = 1$ such that $\|x+y\| = 2$ and $\|x-y\| = 2$, then we say that X is not uniformly non-square.

Let X be a real normed space, the von Neumann-Jordan constant $C_{NJ}(X)$ is defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x,y \in X, (x,y) \neq (0,0)\right\}.$$

The von Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space was introduced by Clarkson [6], and an equivalent definition of it is found in [11] based on the characterization of the connection with James constant $J(X)$ as well as normal structure coefficient of X in the context of the fixed point property. More concisely, we present the trivial form $C'_{NJ}(X)$ of the von Neumann-Jordan constant.

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{4} \ : \ x,y \in S_X\right\}.$$

Later many scholars have studied the relationship between these two constants, such as $1 \leq C'_{NJ}(X) \leq C_{NJ}(X) \leq 2$, see [2] for more detail.

To measture the uniformly nonsquareness of the unit ball, Gao and Lau [8] introduced the James constant of a Banach space X, which is defined as

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

Another equivalent definition of James constant can be expressed as

$$J(X) = \sup\{\|x+y\| : x, y \in S_X, x \perp_I y\}.$$

There is a close relationship between the James constant and von Neumann-Jordan constant, more information about the James constant can see [11].

The constant $A_2(X)$ is given by

$$A_2(X) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} \ : \ x, y \in S_X\right\},$$

was first stuied in Baronti et al. [3]. They give the value of this constant on some specific spaces.

In [1], the authors introduced the following geometric constant:

$$T(X) = \sup\{\sqrt{\|x+y\|\|x-y\|} : x, y \in S_X\},$$

which can help us estimate the exact values of the James and von Neumann–Jordan constants on some Banach spaces.

To describe the difference between Birkhoff orthogonality and isosceles orthogonality, Ji et al. [10] introduced a geometric constant $D(X)$, which is defined as follows:

$$D(X) = \inf\left\{\inf_{\lambda \in \mathbb{R}} \|x+\lambda y\| \ : \ x, y \in S_X, x \perp_I y\right\}.$$

Definition 2.1. [5] *We define $diamA = \sup\{\|x-y\| : x, y \in A\}$ to represent diameter of A and $r(A) = \inf\{\sup\{\|x-y\|\} : y \in A\}$ is called Chebyshev radius of A. A Banach space X has normal structure provided*

$$r(A) < diamA$$

for every bounded closed convex subset A of X with $diamA > 0$. A Banach space X is said to have uniform normal structure if

$$\inf\left\{\frac{diamA}{r(A)}\right\} > 1,$$

with diamA > 0.

3. Properties of $T_G(X)$

Motivated by the geometric constant $T(X)$ introduced by Alonso, et al. [1], we define the constant $T_G(X)$ of a Banach space X as follows

Definition 3.1.

$$T_G(X) = \sup\{\sqrt{\|x+y\| \cdot \|2x-y\|} : x, y \in S_X, \|x-y\| = 1\}.$$

First, we give the bounds of the geometric constant $T_G(X)$ on Banach space in the following theorem.

Theorem 3.2. *For any Banach space X, we have* $1 \leq T_G(X) \leq 2$.

Proof. Since $2 = \|2x\| \leq \|x+y\| + \|x-y\|$, we have $\|x+y\| \geq 1$. By $\|2x-y\| \geq \|2x\| - \|y\| = 1$, we obatin $T_G(X) \geq 1$.

It is easy to see $\|2x-y\| \leq \|x-y\| + \|x\| = 2$. □

The following example shows the case where the upper bound of the geometric constant $T_G(X)$ is reachable.

Example 3.3. *Let X be* $\mathbb{R}^2$ *endowed with the* ℓ_1 *norm* $\|x\| = \max\{|x_1|, |x_2|\}$. *Then* $T_G(X) = 2$.

Let $x = (1,1)$, $y = (1,0)$, then $\|x\| = \|y\| = \|x-y\| = 1$. It is easy to obtain

$$\|2x-y\| = \|2(1,1) - (1,0)\| = \|(1,2)\| = 2$$

and

$$\|x+y\| = \|(1,1) + (1,0)\| = 2,$$

hence

$$T_G(X) \geq \sqrt{\|x+y\| \cdot \|2x-y\|} = 2,$$

since $T_G(X) \leq 2$, then we get

$$T_G(X) = 2.$$

In the following example, we give an estimate of the constant $T_G(X)$ on l_p spaces.

Example 3.4. *For any $p \geq 2$, then $T_G(l_p) \leq (2^p - 1)^{\frac{1}{p}}$. For any $1 \leq p < 2$, then $T_G(l_p) \leq 3^{\frac{1}{p}}$.*

In fact, by Clarkson inequality, for $p \geq 2, x, y \in l_p$, we have

$$\begin{aligned} 2(\|x\|^p + \|y\|^p) &\leq \|x+y\|^p + \|x-y\|^p \\ &\leq (\|x\| + \|y\|)^p + |\ \|x\| - \|y\|\ |^p. \end{aligned}$$

Let $1 \leq p < 2$, for any $x, y \in l_p$, we have

$$\begin{aligned} &(\|x\| + \|y\|)^p + |\ \|x - \|y\|\ |^p \\ \leq &\|x+y\|^p + \|x-y\|^p \\ \leq &2(\|x\|^p + \|y\|^p). \end{aligned}$$

It follows that

$$\|x+y\|^p \leq 2^p - 1,\ \|2x-y\|^p \leq 2^p - 1, (\forall p \geq 2)$$

and

$$\|x+y\|^p \leq 3,\ \|2x-y\|^p \leq 3, (\forall 1 \leq p < 2).$$

Hence, for any $p \geq 2, T_G(l_p) \leq (2^p - 1)^{\frac{1}{p}}$. For any $1 \leq p < 2, T_G(l_p) \leq 3^{\frac{1}{p}}$.

Theorem 3.5. *If X is an inner product space, then $T_G(X) = \sqrt{3}$.*

Proof. For any $x, y \in S_X$ satisfying $\|x-y\| = 1$, by utilizing parallelogram law, we can obtain

$$\|x+y\| = \sqrt{2\|x\|^2 + 2\|y\|^2 - \|x-y\|^2} = \sqrt{3}.$$

Since

$$\begin{aligned} \|2x-y\|^2 + \|y\|^2 &= \|(x-y)+x\|^2 + \|(x-y)-x\|^2 \\ &= 2\|x-y\|^2 + 2\|x\|^2 = 4. \end{aligned}$$

Therefore, $\|2x-y\| = \sqrt{3}$.

From this, we get

$$T_G(X) = \sup\{\sqrt{\|x+y\| \cdot \|2x-y\|} : x, y \in S_X, \|x-y\| = 1\} = \sqrt{3}.$$

□

4. Some Inequalities Related to New Constant $T_G(X)$

In this section, we established some inequalities to get the connection between $T_G(X)$ and other geometric constants.

Proposition 4.1. *Let X be a Banach space. Then* $\sqrt{2(J(X)-1)} \leq T_G(X) \leq \sqrt{2}T(X)$.

Proof. For any $x, y \in S_X, \|x-y\| = 1$, we get

$$\begin{aligned}
2\min\{\|x+y\|, \|x-y\|\} &= \min\{\|2x+2y\|, \|2x-2y\|\} \\
&\leq \min\{\|x+y\| + \|x\| + \|y\|, \|2x-y\| + \|-y\|\} \\
&\leq \min\{\|x+y\| + 2, \|2x-y\| + 1\} \\
&\leq 2 + \min\{\|x+y\|, \|2x-y\|\}.
\end{aligned}$$

By

$$\|2x-y\| \geq \|2x\| - \|y\| = 1$$

and

$$2 = \|2x\| \leq \|x+y\| + \|x-y\| = \|x+y\| + 1,$$

we get

$$\begin{aligned}
2 + \min\{\|x+y\|, \|2x-y\|\} &\leq 2 + \min\{\|x+y\| \cdot \|2x-y\|, \|2x-y\| \cdot \|x+y\|\} \\
&= 2 + \|x+y\| \cdot \|2x-y\|,
\end{aligned}$$

which implies that

$$2J(X) \leq 2 + T_G(X)^2,$$

namely,

$$T_G(X) \geq \sqrt{2(J(X)-1)}.$$

On the other hand, for any $x, y \in S_X$, since $\|x-y\| = 1$, we have

$$\|x+y\| \cdot \|2x-y\| \leq \|x+y\|(\|x\| + \|x-y\|) \leq 2\|x+y\| \cdot \|x-y\|.$$

Therefore,

$$\begin{aligned} T_G(X) &= \sup\{\sqrt{2\|x+y\|\cdot\|x-y\|} : \|x\|=\|y\|=\|x-y\|=1\} \\ &\leq \sup\{\sqrt{2\|x+y\|\cdot\|x-y\|} : \|x\|=\|y\|=1\} = \sqrt{2}T(X). \end{aligned}$$

□

Proposition 4.2. *Let X be a Banach space. Then*

$$T_G(X)^2 \leq 4C'_{NJ}(X).$$

Proof. For any $x, y \in S_X$, $\|x-y\|=1$, we have

$$\begin{aligned} \|x+y\|\cdot\|2x-y\| &\leq \|x+y\|(\|x-y\|+\|x\|) \\ &= \|x+y\|\cdot\|x-y\|+\|x+y\| \\ &= 2\|x+y\|\cdot\|x-y\| \\ &\leq 4\frac{\|x+y\|^2+\|x-y\|^2}{4} \\ &\leq 4C'_{NJ}(X). \end{aligned}$$

The proof is complete. □

Proposition 4.3. *Let X be a Banach space. Then $T_G(X) \leq \frac{3}{2}A_2(X)$.*

Proof. For any $x, y \in S_X$ and $\|x-y\|=1$, we can deduce

$$\begin{aligned} &\sqrt{\|x+y\|\|2x-y\|} \leq \frac{\|x+y\|+\|2x-y\|}{2} \\ &= \frac{\|x+y\|+\left\|\frac{3}{2}(x-y)+\frac{1}{2}(x+y)\right\|}{2} \\ &\leq \frac{3}{2}\left(\frac{\|x+y\|+\|x-y\|}{2}\right) \\ &\leq \frac{3}{2}A_2(X), \end{aligned}$$

which implies that $T_G(X) \leq \frac{3}{2}A_2(X)$. □

Recall that Banach space X is called uniformly convex, if, for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any $x, y \in S_X$ with $\|x-y\| > \varepsilon$, then $\|\frac{x+y}{2}\| < 1-\delta$. For $\varepsilon \in [0,2]$, the Clarkson modulus of convexity of X is defined in the following way:

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X, \|x-y\| = \varepsilon\right\},$$

and Banach space X is said to be uniformly convex if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. The following proposition provides the relation between $T_G(X)$ and modulus of convexity of X.

Proposition 4.4. *Let X be a Banach space. Then*

$$T_G(X)^2 \leq 3(1-\delta_X(1)) + \frac{1}{2}(1-\delta_X(1))^2.$$

Proof. For any $x, y \in S_X, \|x-y\| = 1$, we can deduce

$$\begin{aligned}\|x+y\| \cdot \|2x-y\| &= \|x+y\| \cdot \left\|\frac{3}{2}(x-y) + \frac{1}{2}(x+y)\right\| \\ &= \frac{3}{2}\|x+y\| \cdot \|x-y\| + \frac{1}{4}\|x+y\|^2 \\ &\leq 3(1-\delta_X(1)) + \frac{1}{2}(1-\delta_X(1))^2,\end{aligned}$$

which shows that $T_G(X)^2 \leq 3(1-\delta_X(1)) + \frac{1}{2}(1-\delta_X(1))^2$. □

We will bring out the connection with geometric constant $D(X)$, for the sake of reaching the lower bound of $T_G(X)$ in the following theorem.

Theorem 4.5. *For any Banach space, we have*

$$T_G(X) \geq \sqrt{2}D(X).$$

Proof. For any $x, y \in S_X, \|x-y\| = 1$, since

$$\left\|x - \frac{1}{2}y\right\| \geq \inf\{\|x+\lambda y\| : \lambda \in \mathbb{R}\} \geq D(X),$$

hence

$$\|2x-y\| \geq 2D(X).$$

It is easy to see

$$\|x+y\| \geq \inf\{\|x+\lambda y\| : \lambda \in \mathbb{R}\} \geq D(X),$$

therefore

$$
\begin{aligned}
T_G(X)^2 &= \sup\{\|x+y\|\cdot\|2x-y\| : \|x\|=\|y\|=1, \|x-y\|=1\} \\
&\geq 2D(X)D(X) \\
&= 2D(X)^2.
\end{aligned}
$$

□

Next, we give the connection between geometric constant $T_G(X)$ and geometric properties of the Banach space.

Theorem 4.6. *Let Banach space X be finite-dimensional, if X is rotund, then $T_G(X) < 2$.*

Proof. Suppose that $T_G(X) = 2$, Since the closed unit ball of finite-dimensional Banach space is compact, so there exist $\|x_0\| = \|y_0\| = \|x_0 - y_0\| = 1$, such that $\sqrt{\|x_0+y_0\|\cdot\|2x_0-y_0\|} = 2$. Thus $\|x_0+y_0\| = 2$ and $x_0 \neq y_0$. It contradicts to X is rotund. □

Lemma 4.7. *Let X be a Banach space with $J(X) < \frac{1+\sqrt{5}}{2}$. Then X has uniformly normal structure.*

Theorem 4.8. *Let X be a Banach space, if $T_G(X) < \frac{3}{2}$. Then X has uniformly normal structure.*

Proof. Since $\sqrt{2(J(X)-1)} \leq T_G(X) < \frac{3}{2} < 5^{\frac{1}{4}}$, we get

$$J(X) \leq \frac{1}{2}(T_G(X)^2+1).$$

We thus have $J(X) < \frac{1+\sqrt{5}}{2}$, which implies that X has uniformly structure. □

The proof technique of the following proposition is from the Gao's paper [7]. For completeness, we give detailed proof in here.

Proposition 4.9. *Let X be a Banach space with $T_G(X) < 2$. Then X is uniformly nonsquare.*

Proof. Assume that X is not uniformly nonsquare, then for any $0 < \varepsilon < \frac{1}{2}$, there exists $\|x\| = \|y\| = 1$, such that

$$\|(x+y)/2\| > 1-\varepsilon \text{ and } \|(x-y)/2\| > 1-\varepsilon.$$

Let $\|z_1\| = \|z_2\| = 1$ such that $y = z_2 - z_1$. Then $\|z_2 - y\| = \|z_1\| = 1$ and $\|y-(-z_1)\| = \|z_2\| = 1$.

Since $\|y-x\|>2-\varepsilon>1$, there exists $0<c<1$, such that $z_2=\alpha[cx+(1-c)y]$, where $\alpha=\frac{1}{\|cx+(1-c)y\|}<1+2\varepsilon$.

Similarly we can show that there exists $d, 0<d<1$, such that

$$z_1=\beta(d(-y)+(1-d)x),$$

where $\beta=\frac{1}{\|d(-y)+(1-d)x\|}<1+2\varepsilon$.

Thus

$$\begin{aligned}\|y+z_2\| &\geq \|y+(cx+(1-c)y)\|-\|(cx+(1-c)y)-z_2\| \\ &=2\left\|\frac{c}{2}x+\frac{2-c}{2}y\right\|-(\alpha-1) \\ &\geq 2-4\varepsilon-2\varepsilon=2-6\varepsilon,\end{aligned}$$

and

$$\begin{aligned}\|2y-z_2\| &= \|y-z_1\| \\ &\geq \|y-(d(-y)+(1-d)x)\|-\|(d(-y)+(1-d)x)-z_1\| \\ &\geq 2\left\|\frac{1-d}{2}x+\frac{1+d}{2}(-y)\right\|-(\beta-1) \\ &\geq 2-4\varepsilon-2\varepsilon=2-6\varepsilon.\end{aligned}$$

Hence

$$T_G(X)=\sup\{\sqrt{x+y\|\cdot\|2x-y\|}\ :\ \|x\|=\|y\|=\|x-y\|=1\}\geq 2-6\varepsilon.$$

Since ε can be arbitrarily small, we know $T_G(X)=2$. $\square$

5. THE $T(X)$ TYPES CONSTANT RELATE TO BIRKHOFF ORTHOGONALITY

In a normed linear space X, a vector x is said to be Birkhoff orthogonal to a vector y [4] ($x\perp_B y$) if the inequality $\|x+\alpha y\|\geq\|x\|$ holds for any real number α.

We may guess $sup\{\sqrt{\|x+y\|\cdot\|2x-y\|}: x,y\in S_X, \|x-y\|\geq 1\}$ is very close to $T_G(X)$. Inspired by $\|x\|=\|y\|=1$ and $x\perp_B y$ implies $\|x-y\|\geq 1$, we consider the following constant:

$$T_\perp(X)=\sup\{\sqrt{\|x+y\|\cdot\|2x-y\|}: x,y\in S_X, x\perp_B y\}.$$

In this section, we show some properties of the geometric constant $T_{\perp}(X)$.

Theorem 5.1. *Let X be Banach space. Then $\sqrt{2} \leq T_{\perp}(X) \leq \sqrt{6}$.*

Proof. Since $x \perp y$, then $\|x+y\| \geq \|x\| = 1$ and $\|2x-y\| = 2\|x - \frac{1}{2}y\| \geq 2\|x\| = 2$. We thus get

$$T_{\perp}(X) \geq \sqrt{\|x+y\| \cdot \|2x-y\|} \geq \sqrt{2}.$$

Since

$$\|x+y\| \cdot \|2x-y\| \leq (\|x\|+\|y\|)(\|2x\|+\|y\|) = 6,$$

then we have $T_{\perp}(X) \leq \sqrt{6}$. □

Example 5.2. *Let X be $\mathbb{R}^2$ endowed with the ℓ_1 norm $\|x\| = \max\{|x_1|, |x_2|\}$. Then $T_{\perp}(X) = \sqrt{6}$.*

Let $x = (1,1)$, $y = (1,-1)$, then $\|x\| = \|y\| = 1, x \perp_B y$. It is easy to see

$$\|2x-y\| = \|2(1,1)-(1,-1)\| = \|(1,3)\| = 3,$$

$$\|x+y\| = \|(1,1)+(1,-1)\| = 2,$$

hence

$$T_{\perp}(X) \geq \sqrt{\|x+y\| \cdot \|2x-y\|} = \sqrt{6},$$

since $T_{\perp}(X) \leq \sqrt{6}$, therefore

$$T_{\perp}(X) = \sqrt{6}.$$

Theorem 5.3. *If X is an inner product space, then $T_{\perp}(X) = 10^{\frac{1}{4}} \approx 1.778$.*

Proof. For any $x, y \in S_X$ satisfying $x \perp_B y$, we can obtain

$$\|x+y\| = \sqrt{2}.$$

Since

$$\|2x-y\|^2 = 2^2 + 1.$$

Therefore, $\|2x-y\| = \sqrt{5}$.

Hence,

$$T_{\perp}(X) = 10^{\frac{1}{4}}.$$

Proposition 5.4. *Let X be a non-trivial Banach space, then $T_{\perp}(X) \geq 2\sqrt{J(X)-1}$.*

Proof. For any $x, y \in S_X, x \perp_B y$, we get

$$\begin{aligned}
2\min\{\|x+y\|, \|x-y\|\} &= \min\{\|2x+2y\|, \|2x-2y\|\} \\
&\leq \min\{\|x+y\| + \|x\| + \|y\|, \|2x-y\| + \|-y\|\} \\
&\leq \min\{\|x+y\| + 2, \|2x-y\| + 1\} \\
&\leq 2 + \min\{\|x+y\|, \|2x-y\|\}.
\end{aligned}$$

By $x \perp_B y$, we get $\|2x-y\| = 2\|x - \frac{1}{2}y\| \geq 2\|x\| = 2 \geq \|x+y\|$, we get

$$\begin{aligned}
2 + \min\{\|x+y\|, \|2x-y\|\} &= 2 + \|x+y\| \\
&\leq 2 + \frac{1}{2}\|x+y\| \cdot \|2x-y\|,
\end{aligned}$$

which implies that

$$2J(X) \leq 2 + \frac{1}{2}T_G(X)^2,$$

namely,

$$T_{\perp}(X) \geq 2\sqrt{J(X)-1}.$$

6. Conclusion

In this paper, based on the definition of the geometric constant $T(X)$, we introduce two geometric constants, $T_G(X)$ and $T_{\perp}(X)$. These two geometric constants are analogous and study respectively the unit vectors on which different conditions are satisfied. First, we characterise the upper and lower bounds of them. Secondly, these constants are closely related to other geometric constants, such as the James constant. Furthermore, we give a connection between the constant $T_G(X)$ and the geometric structure of space. However, we did not obtain values for thees geometric constants on some specific spaces, and this is something we need to go into in future.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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