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ρ-ORTHOGONALITY PROPERTIES IN 2-NORMED SPACE

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Abstract. In this paper, we define two 2-norm derivatives in 2-normed space and give some results. We use 2-norm derivatives to study the ρ-orthogonality in 2-normed space. We define ρ-orthogonality, ρ_+-orthogonality, ρ_--orthogonality in 2-normed space and give some properties of it.

Keywords: orthogonality; 2-norm derivative; 2-normed space.

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1. Introduction

The study of orthogonal properties in normed space is an important research direction. There have been many studies in orthogonal properties on normed space. Many scholars [1, 2, 3, 4, 8, 9] have put forward a variety of orthogonal relations. For the study of ρ-orthogonality, the author [10] have defined it by norm derivative. The norm derivatives are also called superior and inferior semi-inner products. They also proved that when the derivatives of two norms are equal, it is equivalent to that the space is smooth. Other properties of ρ-orthogonality are also proved in [11, 12, 13, 19, 20].

In 1965, G*ä*hler [5] introduced the concept of 2-normed space, which is a generalization of normed space. However, there are few studies on the orthogonal properties of 2-normed space.

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In 1975, White and Diminnie [14] gave a characterization of 2-inner product space by using the partial derivatives of bifunctionals. In 2006, the concepts of P-orthogonality, I-orthogonality, BJ-orthogonality in 2-normed space have been given in [17]. In 2007, Mazaheri and Nezhad [15] gave the definition of b-orthogonality in 2-normed space and gave some results in this field.

In this paper, we mainly study ρ-orthogonality in 2-normed space. We define two 2-norm derivatives in 2-normed space. We prove some properties of 2-norm derivatives. We find the relationship between 2-norm derivative and 2-inner product and the relationship between 2-norm derivative and 2-semi-inner product. We also define ρ-orthogonality, ρ_+-orthogonality, ρ_--orthogonality in 2-normed space and give some properties of it.

2. Main Results

We first introduce the concepts of 2-normed space and 2-inner product space. The concept of 2-normed space was introduced by G*ä*hler, which was widely generalized by other scholars [5, 6, 7].

Let X be a real linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real valued function on $X \times X$ satisfying the following conditions:

(a) $\|x,y\| = 0$ if and only if x,y are linearly dependent;

(b) $\|x,y\| = \|y,x\|$;

(c) $\|\alpha x,y\| = |\alpha|\|x,y\|$ for $\alpha \in \mathbb{R}$;

(d) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$ for every $x,y,z \in X$.

$\|\cdot,\cdot\|$ is called a 2-norm and $(X,\|\cdot,\cdot\|)$ is called a 2-normed space. Some basic properties of 2-normed space, which are nonnegative and $\|x,y+\alpha x\| = \|x,y\|$ for all $x,y \in X$ and for each $\alpha \in \mathbb{R}$.

Ehret [6] gave the concept of 2-inner product space. Let $(\cdot,\cdot|\cdot)$ be a real valued function on $X \times X \times X$ which satisfies the following conditions:

(a) $(x,x|z) \geq 0$, $(x,x|z) = 0$, if and only if x and z are linearly dependent;

(b) $(x,x|z) = (z,z|x)$;

(c) $(x,y|z) = (y,x|z)$;

(d) $(\alpha x,y|z) = \alpha(x,y|z)$ for $\alpha \in \mathbb{R}$;

(e) $(x+x^{'},y|z) = (x,y|z) + (x^{'},y|z)$ for every $x,x^{'},y,z \in X$.

$(\cdot,\cdot|\cdot)$ is called a 2-inner product and $(X,(\cdot,\cdot|\cdot))$ is called a 2-inner product space.

Ehret [16] proved that if $(X,(\cdot,\cdot|\cdot))$ is a 2-inner product space, then $\|x,y\| = (x,x|y)^{\frac{1}{2}}$ defines a 2-norm.

Definition 2.1. *Let $(X,\|\cdot,\cdot\|)$ be a real 2-normed space. We define two mappings $\rho'_+(x,y;z)$, $\rho'_-(x,y;z) : X\times X\times X \to \mathbb{R}$*

$$\rho'_+(x,y;z) := \lim_{t\to 0^+} \frac{\|x+ty,z\|^2 - \|x,z\|^2}{2t}.$$

$$\rho'_-(x,y;z) := \lim_{t\to 0^-} \frac{\|x+ty,z\|^2 - \|x,z\|^2}{2t}.$$

The mappings $\rho'_+(x,y;z)$, $\rho'_-(x,y;z)$ are called 2-norm derivatives.

Remark 2.2. (a) *According to the above definition, we can verify that*

$$\begin{aligned}\rho'_\pm(x,y;z) &= \lim_{t\to 0^\pm} \frac{\|x+ty,z\|^2 - \|x,z\|^2}{2t}\\ &= \lim_{t\to 0^\pm} \frac{\|x+ty,z\| + \|x,z\|}{2}\,\frac{\|x+ty,z\| - \|x,z\|}{t}\\ &= \|x,z\| \lim_{t\to 0^\pm} \frac{\|x+ty,z\| - \|x,z\|}{t}.\end{aligned}$$

In particular, if $z=\alpha x$ or $z=\beta y$ for $\alpha,\beta\in\mathbb{R}$, then $\rho'_\pm(x,y;z)=0$.

Proof. (i) If $z=\alpha x$, then

$$\rho'_\pm(x,y;\alpha x) = \|x,\alpha x\| \lim_{t\to 0^\pm} \frac{\|x+ty,\alpha x\| - \|x,\alpha x\|}{t} = 0.$$

(ii) If $z=\beta y$, then

$$\rho'_\pm(x,y;\beta y) = \|x,\beta y\| \lim_{t\to 0^\pm} \frac{\|x+ty,\beta y\| - \|x,\beta y\|}{t} = 0.$$

□

(b) *For $x,y,z\in X$, limits $\rho'_+(x,y;z)$, $\rho'_-(x,y;z)$ exists.*

Proof. Suppose $f(t) = \frac{\|x+ty,z\|-\|x,z\|}{t}$. If $0<t_1<t_2$, then

$$\begin{aligned}
&f(t_1)-f(t_2)\\
=&\frac{\|x+t_1y,z\|-\|x,z\|}{t_1}-\frac{\|x+t_2y,z\|-\|x,z\|}{t_2}\\
=&\frac{\|t_2x+t_1t_2y,z\|-\|t_1x+t_1t_2y,z\|+(t_1-t_2)\|x,z\|}{t_1t_2}\\
\leq&\frac{\|(t_2-t_1)x,z\|+(t_1-t_2)\|x,z\|}{t_1t_2}\\
=&0.
\end{aligned}$$

So $f(t)$ is a monotonically increasing function with infimum. By the similar method, if $t_1 < t_2 < 0$, then $f(t)$ is a monotonically increasing function with supremum. Consequently the limit exists. □

Theorem 2.3. *Let $(X, \|\cdot,\cdot\|)$ be a real 2-normed space. Suppose $\rho'_+(x,y;z)$, $\rho'_-(x,y;z) : X\times X\times X \to \mathbb{R}$ are defined as above. Let $x,y,z \in X$, $\alpha,\beta\in\mathbb{R}$. Then*
(a) *There is always $\rho'_-(x,y;z)\leq \rho'_+(x,y;z)$.*
(b) $\rho'_\pm(\alpha x,y;z)=\alpha\rho'_\pm(x,y;z)=\rho'_\pm(x,\alpha y;z), \quad \alpha\geq 0.$
(b') $\rho'_\pm(\alpha x,y;z)=\alpha\rho'_\mp(x,y;z)=\rho'_\pm(x,\alpha y;z), \quad \alpha< 0.$
(c) $\rho'_\pm(x,\alpha x+\beta y;z)=\alpha\|x,z\|^2+\beta\rho'_\pm(x,y;z).$
(d) $|\ \rho'_\pm(x,y;z)\ |\leq \|x,z\|\|y,z\|.$
(e) *If $\|y_n,z\|\to\|y,z\|$, $y_n\in X(n=1,2\cdots)$, then $\rho'_\pm(x,y_n;z)\to\rho'_\pm(x,y;z)$.*

Proof. The proof is as follows.
(a) Suppose $f(t)=\frac{\|x+ty,z\|-\|x,z\|}{t}$. If $t_1<0<t_2$, by the similar method used in Remark 2.2(b), then we can get $f(t_1)-f(t_2)\leq 0$, $f(t_1)\leq f(t_2)$. Consequently we have $\rho'_-(x,y;z)\leq\rho'_+(x,y;z)$.
(b) Take $x,y,z\in X$, $\alpha\geq 0$. Then

$$\begin{aligned}
\rho'_\pm(\alpha x,y;z)=&\|\alpha x,z\|\lim_{t\to 0^\pm}\frac{\|\alpha x+ty,z\|-\|\alpha x,z\|}{t}\\
=&\alpha\|x,z\|\lim_{t\to 0^\pm}\frac{\|x+\frac{t}{\alpha}y,z\|-\|x,z\|}{\frac{t}{\alpha}}\\
=&\alpha\|x,z\|\lim_{s\to 0^\pm}\frac{\|x+sy,z\|-\|x,z\|}{s}\quad (s=\frac{t}{\alpha})
\end{aligned}$$

$$=\alpha\rho'_{\pm}(x,y;z).$$

Similarly

$$\begin{aligned}
\rho'_{\pm}(x,\alpha y;z) =&\|x,z\| \lim_{t\to 0^{\pm}} \frac{\|x+t\alpha y,z\|-\|x,z\|}{t}\\
=&\alpha\|x,z\| \lim_{t\to 0^{\pm}} \frac{\|x+t\alpha y,z\|-\|x,z\|}{t\alpha}\\
=&\alpha\|x,z\| \lim_{s\to 0^{\pm}} \frac{\|x+sy,z\|-\|x,z\|}{s} \quad (s=\alpha t)\\
=&\alpha\rho'_{\pm}(x,y;z).
\end{aligned}$$

So $\rho'_{\pm}(\alpha x,y;z)=\alpha\rho'_{\pm}(x,y;z)=\rho'_{\pm}(x,\alpha y;z)$.

(b') When $\alpha<0$, the proof method is the same as (b).

(c) Take $x,y,z\in X$, $\alpha,\beta\in\mathbb{R}$. Suppose t is small enough such that $1+t\alpha>0$. Then

(i) If $\beta=0$, then

$$\begin{aligned}
\rho'_{\pm}(x,\alpha x;z) =&\|x,z\| \lim_{t\to 0^{\pm}} \frac{\|x+t\alpha x,z\|-\|x,z\|}{t}\\
=&\|x,z\| \lim_{t\to 0^{\pm}} \frac{t\alpha\|x,z\|}{t}\\
=&\alpha\|x,z\|^2.
\end{aligned}$$

(ii) If $\beta\neq 0$, then

$$\begin{aligned}
\rho'_{\pm}(x,\alpha x+\beta y;z) =&\|x,z\| \lim_{t\to 0^{\pm}} \frac{\|x+t(\alpha x+\beta y),z\|-\|x,z\|}{t}\\
=&\|x,z\| \lim_{t\to 0^{\pm}} \frac{(1+t\alpha)(\|x+\frac{t\beta}{1+t\alpha}y,z\|-\|x,z\|)+t\alpha\|x,z\|}{t}\\
=&\alpha\|x,z\|^2+\|x,z\| \lim_{t\to 0^{\pm}} \frac{\|x+\frac{t\beta}{1+t\alpha}y,z\|-\|x,z\|}{\frac{t\beta}{1+t\alpha}}\beta\\
=&\alpha\|x,z\|^2+\|x,z\| \lim_{s\to 0^{\pm}} \frac{\|x+sy,z\|-\|x,z\|}{s}\beta \quad (s=\frac{t\beta}{1+t\alpha})\\
=&\alpha\|x,z\|^2+\beta\rho'_{\pm}(x,y;z).
\end{aligned}$$

(d) Take $x,y,z \in X$. Then

$$\begin{aligned}
| \rho'_{\pm}(x,y;z) | =& \|x,z\| \lim_{t\to 0^{\pm}} | \frac{\|x+ty,z\| - \|x,z\|}{t} | \\
\leq& \|x,z\| \lim_{t\to 0^{\pm}} | \frac{\|x,z\| + | t | \|y,z\| - \|x,z\|}{t} | \\
\leq& \|x,z\| \|y,z\|.
\end{aligned}$$

(e) Take $x,z \in X$. Suppose $\|y_n,z\| \to \|y,z\|$, then

$$\begin{aligned}
&\rho'_{\pm}(x,y_n;z) - \rho'_{\pm}(x,y;z) \\
=& \|x,z\| \lim_{t\to 0^{\pm}} \frac{\|x+ty_n,z\| - \|x,z\|}{t} - \|x,z\| \lim_{t\to 0^{\pm}} \frac{\|x+ty,z\| - \|x,z\|}{t} \\
=& \|x,z\| \lim_{t\to 0^{\pm}} \frac{\|x+ty_n,z\| - \|x+ty,z\|}{t} \\
\leq& \|x,z\| \lim_{t\to 0^{\pm}} \frac{\|ty_n - ty,z\|}{t} \\
=& \|x,z\| \|y_n - y,z\| \to 0.
\end{aligned}$$

□

Proposition 2.4. *If $(X,(\cdot,\cdot|\cdot))$ is a 2-inner product space with 2-norm defined by $\|x,z\| = (x,x|z)^{\frac{1}{2}}$, then we can get*

$$\rho'_{+}(x,y;z) = (x,y|z) = \rho'_{-}(x,y;z).$$

Proof. For the integrity of the content, we will give its proof process

$$\begin{aligned}
\rho'_{\pm}(x,y;z) =& \lim_{t\to 0^{\pm}} \frac{\|x+ty,z\|^2 - \|x,z\|^2}{2t} \\
=& \lim_{t\to 0^{\pm}} \frac{(x+ty,x+ty|z) - (x,x|z)}{2t} \\
=& \lim_{t\to 0^{\pm}} \frac{(x,x|z) + 2t(x,y|z) + t^2(y,y|z) - (x,x|z)}{2t} \\
=& \lim_{t\to 0^{\pm}} \frac{2t(x,y|z) + t^2(y,y|z)}{2t} \\
=& \lim_{t\to 0^{\pm}} \frac{2(x,y|z) + t(y,y|z)}{2} \\
=& (x,y|z).
\end{aligned}$$

Obviously, if $z=\alpha x$ or $z=\beta y$, the result also true. Therefore $\rho_+^{'}(x,y;z)=(x,y|z)=\rho_-^{'}(x,y;z)$. The proposition is proved. □

Definition 2.5. ([7]) *Let* $[\cdot,\cdot|\cdot]$ *be a real valued function on* $X\times X\times X$ *which satisfies the following conditions:*

(a) $[x,x|z]\geq 0$, $[x,x|z]=0$ *if and only if* x *and* z *are linearly dependent;*

(b) $[\alpha x,y|z]=\alpha[x,y|z]$ *for* $\alpha\in\mathbb{R}$*;*

(c) $[x+x^{'},y|z]=[x,y|z]+[x^{'},y|z]$*;*

(d) $|\ [x,y|z]\ |\leq [x,x|z]^{\frac{1}{2}}[y,y|z]^{\frac{1}{2}}$ *for every* $x,x^{'},y,z\in X$.

$[\cdot,\cdot|\cdot]$ *is a 2-semi-inner product and* $(X,[\cdot,\cdot|\cdot])$ *is called a 2-semi-inner product space. A 2-semi-inner-product space is a 2-normed space with the 2-norm* $\|x,z\|=[x,x|z]^{\frac{1}{2}}$ *provided* $[x,x|z]=[z,z|x]$ [18].

Proposition 2.6. *If* $(X,[\cdot,\cdot|\cdot])$ *is a 2-semi-inner product space with* $[x,x|z]=[z,z|x]$, *then we can get*

$$\rho_{\pm}^{'}(x,y;z)=\lim_{t\to 0^{\pm}}[y,x+ty|z].$$

Proof. According to the Theorem 2.3(d) and $\|x,z\|=[x,x|z]^{\frac{1}{2}}$, we can get

$$\begin{aligned}
\rho_{\pm}^{'}(x,y;z)=&\|x,z\|\lim_{t\to 0^{\pm}}\frac{\|x+ty,z\|-\|x,z\|}{t}\\
&=\lim_{t\to 0^{\pm}}\frac{\|x+ty,z\|-\|x,z\|}{t}\|x+ty,z\|\\
&\leq\lim_{t\to 0^{\pm}}\frac{[x+ty,x+ty|z]-[x,x+ty|z]}{t}\\
&=\lim_{t\to 0^{\pm}}[y,x+ty|z]\\
&=\lim_{t\to 0^{\pm}}\frac{[x+2ty,x+ty|z]-[x+ty,x+ty|z]}{t}\\
&\leq\lim_{t\to 0^{\pm}}\frac{\|x+2ty,z\|\|x+ty,z\|-\|x+ty,z\|^2}{t}\\
&=\lim_{t\to 0^{\pm}}\frac{\|x+2ty,z\|-\|x+ty,z\|}{t}\|x,z\|\\
&=\lim_{t\to 0^{\pm}}[2\frac{\|x+2ty,z\|-\|x,z\|}{2t}-\frac{\|x+ty,z\|-\|x,z\|}{t}]\|x,z\|
\end{aligned}$$

$$=\lim_{t\to 0^{\pm}} \frac{\|x+ty,z\|-\|x,z\|}{t}\|x,z\|$$
$$=\rho_{\pm}^{'}(x,y;z).$$

(i) If $z=\alpha x$, then

$$\rho_{\pm}^{'}(x,y;\alpha x)=\lim_{t\to 0^{\pm}} \frac{\|x+ty,\alpha x\|^2-\|x,\alpha x\|^2}{2t}=0,$$

$$\begin{aligned}0\leq&\lim_{t\to 0^{\pm}}[y,x+ty|\alpha x]\\ \leq&\lim_{t\to 0^{\pm}}[y,y|\alpha x]^{\frac{1}{2}}[x+ty,x+ty|\alpha x]^{\frac{1}{2}}\\ =&\lim_{t\to 0^{\pm}}\|y,\alpha x\|\|x+ty,\alpha x\|\\ =&\lim_{t\to 0^{\pm}}\alpha^2 t\|x,y\|\\ =&0.\end{aligned}$$

(ii) If $z=\beta y$, then

$$\rho_{\pm}^{'}(x,y;\beta y)=\lim_{t\to 0^{\pm}} \frac{\|x+ty,\beta y\|^2-\|x,\beta y\|^2}{2t}=0,$$

$$\begin{aligned}0\leq&\lim_{t\to 0^{\pm}}[y,x+ty|\beta y]\\ \leq&\lim_{t\to 0^{\pm}}[y,y|\beta y]^{\frac{1}{2}}[x+ty,x+ty|\beta y]^{\frac{1}{2}}\\ =&0.\end{aligned}$$

□

For any arbitrary non-zero elements $x,y\in X$, let $V(x,y)$ denote the subspace of X generated by x,y.

Compared with the definition of ρ-orthogonality in normed space, we give the definition of ρ-orthogonality in 2-normed space.

Definition 2.7. *Let $(X,\|\cdot,\cdot\|)$ be a real 2-normed space. Let $x,y,z\in X$ and $z\notin V(x,y)$. We define ρ_+-orthogonality, ρ_--orthogonality, ρ-orthogonality as follows.*

(a) *We call x is ρ_+-orthogonality to y denoted by $x\perp_{\rho_+} y$, if $\rho_{+}^{'}(x,y;z)=0$ for each $z\notin V(x,y)$.*

(b) *We call x is ρ_--orthogonality to y denoted by $x \perp_{\rho_-} y$, if $\rho_-^{'}(x,y;z) = 0$ for each $z \notin V(x,y)$.*

(c) *We call x is ρ-orthogonality to y denoted by $x \perp_\rho y$, if $\rho_+^{'}(x,y;z) + \rho_-^{'}(x,y;z) = 0$ for each $z \notin V(x,y)$.*

The above case is that $z \notin V(x,y)$. If $z \in V(x,y)$, we need to pay attention to the following two cases

Remark 2.8. *Let $(X, \|\cdot,\cdot\|)$ be a real 2-normed space. Take $x,y,z \in X$ and $z \in V(x,y)$.*

(a) *Take $z = \alpha x + \beta y$, $\alpha\beta \neq 0$. If $\rho_\pm^{'}(x,y;z) = 0$, then $y = sx$.*

(b) *Take $z = \beta y$ or $z = \alpha x$, $\alpha \neq 0$, $\beta \neq 0$. Then we can get $\rho_\pm^{'}(x,y;z) = 0$.*

Proof. (a) Suppose $z = \alpha x + \beta y$, $\alpha\beta \neq 0$. Then

$$\begin{aligned}
\rho_\pm^{'}(x,y;z) &= \lim_{t\to 0^\pm} \frac{\|x+ty, \alpha x + \beta y\|^2 - \|x, \alpha x + \beta y\|^2}{2t} \\
&= \lim_{t\to 0^\pm} \frac{|\beta - \alpha t|^2 \|x,y\|^2 - \beta^2 \|x,y\|^2}{2t} \\
&= \lim_{t\to 0^\pm} \frac{(-2\alpha\beta t + \alpha^2 t^2)\|x,y\|^2}{2t} \\
&= -2\alpha\beta \|x,y\|^2.
\end{aligned}$$

So if $\rho_\pm^{'}(x,y;z) = 0$, that is $\|x,y\| = 0$, then $y = sx$.

(b) The conclusion has been given in Remark 2.2. □

Theorem 2.9. *Let $(X, \|\cdot,\cdot\|)$ be a real 2-normed space. Then the following conditions are equivalent:*

(a) $\rho_+^{'} = \rho_-^{'}$;

(b) $\perp_{\rho_+} \subset \perp_{\rho_-}$;

(c) $\perp_{\rho_-} \subset \perp_{\rho_+}$;

(d) $\perp_{\rho_-} = \perp_{\rho_+}$;

(e) $\perp_{\rho_+} \subset \perp_\rho$;

(f) $\perp_\rho \subset \perp_{\rho_+}$;

(g) $\perp_\rho = \perp_{\rho_+}$;

(h) $\perp_{\rho_-} \subset \perp_\rho$;

(i) $\perp_{\rho} \subset \perp_{\rho_-}$;

(j) $\perp_{\rho} = \perp_{\rho_-}$.

Proof. We first prove that $(a) \Leftrightarrow (b) \Leftrightarrow (d)$. We know that $(a) \Rightarrow (d) \Rightarrow (b)$ is obvious. Next we prove that $(b) \Rightarrow (a)$. Suppose that (b) holds. Let $x, y, z \in X$ and $z \notin V(x, y)$(We may assume $x \neq 0$, otherwise (a) holds trivially). We define $\alpha := \frac{\rho'_+(x,y;z)}{\|x,z\|^2}$, $w := -\alpha x + y$. From Theorem 2.3(c), we have

$$\begin{aligned}\rho'_+(x, w; z) =&\rho'_+(x, -\alpha x + y; z)\\ =&-\alpha\|x, z\|^2 + \rho'_+(x, y; z)\\ =&0,\end{aligned}$$

Therefore, $x \perp_{\rho_+} w$. According to the hypothesis, we can get, $x \perp_{\rho_-} w$,

$$\rho'_-(x, -\alpha x + y; z) = 0.$$

According to Theorem 2.3(c), we can get

$$\rho'_-(x, -\alpha x + y; z) = -\alpha\|x, z\|^2 + \rho'_-(x, y; z) = 0.$$

Therefore, we can get $\rho'_+(x, y; z) = \rho'_-(x, y; z)$, which proves (a).

We also know that $(a) \Rightarrow (d) \Rightarrow (c)$ and the proof of $(c) \Rightarrow (a)$ can also be obtained. So we can prove that $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$.

With the above proof, we can also prove that $(a) \Leftrightarrow (e) \Leftrightarrow (f) \Leftrightarrow (g)$ and $(a) \Leftrightarrow (h) \Leftrightarrow (i) \Leftrightarrow (j)$. □

Definition 2.10. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. Let $x, y, z \in X$. We call x is b-orthogonality to y denoted by $x \perp^b y$ if $\|x + ty, z\| \geq \|x, z\|$ for every real number t and each element $z \notin V(x, y)$.*

Theorem 2.11. *Let $(X, \|\cdot,\cdot\|)$ be a real 2-normed space. Let $x, y, z \in X$ and $z \notin V(x, y)$. Then the following statements are equivalent:*

(a) $\rho'_-(x, y; z) \leq 0 \leq \rho'_+(x, y; z)$;

(b) $x \perp^b y$.

Proof. (a) $\Rightarrow$ (b). Suppose that $\rho'_{-}(x,y;z) \leq 0 \leq \rho'_{+}(x,y;z)$.
If $t \geq 0$, it follows from Theorem 2.3(d) that

$$\rho'_{+}(x,x+ty;z) \leq \|x,z\|\|x+ty,z\|.$$

In addition, according to Theorem 2.3(c),

$$\rho'_{+}(x,x+ty;z) = t\rho'_{+}(x,y;z) + \|x,z\|^2,$$

which implies

$$t\rho'_{+}(x,y;z) \leq (\|x+ty,z\| - \|x,z\|)\|x,z\|.$$

Since $\rho'_{+}(x,y;z) \geq 0$, $t \geq 0$, we have

$$\|x+ty,z\| - \|x,z\| \geq 0.$$

Since $\rho'_{-}(x,y;z) \leq 0$, from Theorem 2.3(b) we get $-\rho'_{-}(x,y;z) = \rho'_{+}(x,-y;z) \geq 0$ which implies that $\|x-ty,z\| - \|x,z\| \geq 0$ for all $t \geq 0$. Consequently, we get $x \perp^b y$. (a) $\Rightarrow$ (b) is proved.
(b) $\Rightarrow$ (a). Supppose that $x \perp^b y$. According to the definition of b-orthogonality, we know $\|x+ty,z\| \geq \|x,z\|$. Therefore, we can get $\rho'_{-}(x,y;z) \leq 0 \leq \rho'_{+}(x,y;z)$. (b) $\Rightarrow$ (a) is proved. $\square$

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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