Available online at http://scik.org
Adv. Fixed Point Theory, 2024, 14:47
https://doi.org/10.28919/afpt/7909
ISSN: 1927-6303

APPLICATION TO FRACTIONAL DIFFERENTIAL EQUATIONS VIA THE α-FIXED POINT THEOREM

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Abstract. This paper explores the application of α-Fixed point theorems given by [1] to perturbed fractional differential integral equations, we address the existence and uniqueness of solutions to these fractional equations under perturbations.

Keywords: α-fixed point; fractional derivative; fractional integral; perturbed fractional differential integral.

2020 AMS Subject Classification: 31A10, 47H10, 34K37.

1. INTRODUCTION

The fixed point theorem is a cornerstone in the field of mathematical analysis, with profound implications and applications in various branches of mathematics and science. Among its many applications, the theorem plays a critical role in the study of differential and integral equations. This paper focuses on the application of the α-fixed point theorem to perturbed fractional integral equations, an area of increasing interest due to its relevance in modeling complex systems in physics, engineering, and other applied sciences.[2, 9]

Fractional integral equations, which involve integrals of arbitrary (non-integer) order, generalize classical integral equations and provide a powerful framework for describing memory

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Received February 10, 2023

and hereditary properties of various materials and processes. Perturbations in these equations, arising from external influences or intrinsic system variations, present significant analytical challenges. Addressing these challenges requires robust mathematical tools, among which the Fixed Point Theorem stands out due to its ability to guarantee the existence and uniqueness of solutions under certain conditions.

A fractional derivative of order α is defined as the integral of order α of a function over a suitable interval. The concept of fractional derivatives was introduced by mathematicians such as Liouville, Riemann, and Caputo, and has found applications in various fields including physics, engineering, finance, and biology.

A fractional perturbed differential equation is an equation that contains fractional derivatives of an unknown function. It has the general form:

$$D_\alpha y(t) = f(t, y(t)) + g(t, y(t))$$

where D_α represents the fractional derivative of order $\alpha, y(t)$ is the unknown function, $f(t, y(t))$ and $g(t, y(t))$ are a given functions, and t is the independent variable.

Many researchers tried to put a definition of a fractional derivative. Most of them used an integral form for the fractional derivative.

Two of which are the most popular ones.

-Riemann-Liouville definition.

For $\alpha \in [n-1, n[$, the α derivative of f is:

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx \tag{1.1}$$

where Γ is the gamma function.

-Caputo definition definition.

For $\alpha \in [n-1, n[$, the α derivative of f is:

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx \tag{1.2}$$

In [5] is defined the conformable Jacobian matrix as: Let f and g be coordinate functions of a vector valued function $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ in variables $x_1 > a_1$ and $x_2 > a_2$, where $x = (x_1, x_2)$ and

$a = (a_1, a_2)$, such that their respective partial derivatives exist and are continuous. Then, the conformable Jacobian matrix is given by:

$$F_a^{\alpha(1)}(x) = \begin{pmatrix} \frac{\partial_{a_1}^{\alpha} f}{\partial x_1^{\alpha}} & \frac{\partial_{a_2}^{\alpha} f}{\partial x_2^{\alpha}} \\ \frac{\partial_{a_1} g}{\partial x_1^{\alpha}} & \frac{\partial_{a_2}^{\alpha} g}{\partial x_2^{\alpha}} \end{pmatrix} = \begin{pmatrix} (x_1 - a_1)^{1-\alpha} \frac{\partial f}{\partial x_1} & (x_2 - a_2)^{1-\alpha} \frac{\partial f}{\partial x_2} \\ (x_1 - a_1)^{1-\alpha} \frac{\partial g}{\partial x_1} & (x_2 - a_2)^{1-\alpha} \frac{\partial g}{\partial x_2} \end{pmatrix}$$

In [4], we find the properties of the Conformable fractional derivative of certain functions and the Rolle's and Mean Value Theorem for Conformable Fractional Differentiable Functions based on the following definition:

Given a function $f : [0, +\infty[\longrightarrow \mathbb{R}$, Then the conformable fractional derivative of f of order α is defined by:

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}$$

$\forall t > 0, \alpha \in]0,1[$. If f is α-differentiable in some $(0,a), a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$.

In [6], The Riemann-Liouville's (RL) fractional integral operator of order $\alpha \in [n-1, n[$ of a function $f \in L^1[a,b]$ is given as:

$$I^a w(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x w(\tau)(x-\tau)^{n-\alpha-1} d\tau \tag{1.3}$$

The object of this paper is to present a new application of the perturbed fractional differential equation, we consider a nonlinear fractional differential equation with an initial condition using this new formulation.

We show that this equation is equivalent to a perturbed integral equation and demonstrate the existence and uniqueness of solution to the nonlinear initial value problem using the definition of $\alpha-$Krasnoselski's fixed point theorem.

2. Preliminaries

We denote by $C[a,b]$ the space of continuous functions on the interval $[a,b]$, f on from $[a,b] \to \mathbb{R}$ with the norm:

$$\|f\|_C = \max_{t \in [a,b]} |f(t)|.$$

Definition 2.1. *Let I be an interval of $\mathbb{R}$, An element $c \in I$ is said to be a fixed point of a mapping $f : I \to I$ if $c = f(c)$.*

Definition 2.2. *Let f be an mapping defined on an interval I. We say that f is a strictly contractive if:*

$$\exists K \in [0,1[, \forall x,y \in I, |f(x)| - f(y)| \leq K|x-y|.$$

The following definition for the $\alpha-$fractional integral of a function f starting from $a \geq 0$.

Definition 2.3. [4]
$I_\alpha^a(f)(t) = I_1^a\left(t^{\alpha-1}f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx$, *where the integral is the usual Riemann improper imegral and* $\alpha \in]0,1[$.

With the above definition, it was shown that:

Theorem 2.4. [4, Theorem 3.1]
$T_\alpha I_\alpha^a(f)(t) = f(t)$, *for* $t \geq a$, *where* f *is any continuous function in the domain of* I_α.

The following results generalize the notion of the fixed point in $\mathbb{R}$, and the previous results is a particular case of which $\alpha = 1$.

Definition 2.5. [1, Definition 3.1]
An element $c \in [a,b]$ *is said to be a* α*-fixed point of a mapping* $f : [a,b] \to [a,b]$ *if* $c = f(c^{\frac{1}{\alpha}})$. *for some* $\alpha \in]0,1[$ *and* $c^{\frac{1}{\alpha}} \in [a,b]$.

Definition 2.6. [3]
Let f be an mapping defined on $\mathbb{R}$. We say that f is a strictly α-contractive if:

$$\exists K \in [0,1[, \forall x,y \in \mathbb{R}, |f(x) - f(y)| \leq K|x^\alpha - y^\alpha|.$$

Theorem 2.7. [3]
Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a given function that satisfies:

(1) *f is continuous on $[a,b]$.*
(2) *f is α-differentiable for some $\alpha \in]0,1[$.*
(3) $\max_{x\in[a,b]} \left|f^{(\alpha)}(x)\right| = K < \alpha$

Then, f is a strictly α-contractive withe $K' \in [0,1[$

Theorem 2.8. [3, Theorem 3.5]

Let $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be an mapping such that f is a strictly α-contractive withe $K \in [0,1[$ for some $\alpha \in]0,1[$.

Then, the mapping f has a unique α-fixed point $c \in \mathbb{R}_+^*$ and the sequence defined by its first term $(U_0)^{\frac{1}{\alpha}} \in \mathbb{R}_+^*$ and the recurrence relation: $U_{n+1} = f((U_n)^{\frac{1}{\alpha}})$ converges to c.

Corollary 2.9. [1]

Let $a > 0$ and f be an mapping defined on an closed interval $[a,b]$ such that:

(1) $\forall x \in [a,b], x^{\frac{1}{\alpha}} \in [a,b]$.
(2) *f is continuous on $[a,b]$.*
(3) *f is α-differentiable for some $\alpha \in]0,1[$.*
(4) $\max_{x\in[a,b]} |f^{\alpha}(x)| = K < \alpha$

Then, the mapping f has a unique α-fixed point $c \in [a,b]$ and the sequence defined by its first term $(U_0)^{\frac{1}{\alpha}} \in [a,b]$ and the recurrence relation: $U_{n+1} = f((U_n)^{\frac{1}{\alpha}})$ converges to c.

Corollary 2.10. [1]

Let $a > 0$ and f be an mapping defined on an closed interval $[a,b]$ such that:

(1) *the interval $[a,b]$ is stable by $I_{\alpha}^{a}(f)(t)$.*
(2) *f is continuous on $[a,b]$.*
(3) $\max_{x\in[a,b]} |f(x)| = M < \alpha$ *for some* $\alpha \in]0,1[$.

Then, the mapping F defined by:

$$F(t) = I_{\alpha}^{a}(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx \qquad \forall t \geq a$$

has a unique α-fixed point $c \in [a,b]$.

Definition 2.11. *An element $f \in C[a,b]$ is said to be a α-fixed point of a mapping $T : C[a,b] \to C[a,b]$ if $f = T(f^{\frac{1}{\alpha}})$.*
for some $\alpha \in]0,1[$ and $f^{\frac{1}{\alpha}} \in C[a,b]$.

Definition 2.12. *Let $T : C[a,b] \to C[a,b]$,*

We say that T is a strictly α-contractive for some $\alpha \in]0,1[$ if:

$$\exists K \in]0,1[, \forall f,g \in C[a,b], \|Tf - Tg\|_C \leq K\|f^{\alpha} - g^{\alpha}\|_C$$

Theorem 2.13. *Let $T : C[a,b] \to C[a,b]$ be an mapping such that T is a strictly α-contractive withe $K \in]0,1[$ for some $\alpha \in]0,1[$.*
Then, the mapping T has a unique α-fixed point $f \in C[a,b]$.
and the sequence defined by its first term $(U_0)^{\frac{1}{\alpha}} \in C[a,b]$ and the recurrence relation: $U_{n+1} = T((U_n)^{\frac{1}{\alpha}})$ converges to f.

The following result is Krasnoselski's fixed point theorem.

Corollary 2.14. [2] *Let $S,T : C[a,b] \to C[a,b]$ be an mapping such that T is a strictly α-contractive and S is compact.*
Then, the mapping $S+T$ has a α-fixed point $f \in C[a,b]$.

3. Main Results

We consider the following perturbed fractional differential equation:

$$y^{(\alpha)}(t) = f(t, y^{\frac{1}{\alpha}}(t)) + g(t, y^{\frac{1}{\alpha}}(t)), \quad y(a) = y_0, \quad y_0 \in \mathbb{R}. \tag{3.1}$$

Where $f,g : [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ are a given functions, $0 < \alpha < 1, (a > 0)$.

Lemma 3.1. *The equation (3.1) is equivalent to the integral equation:*

$$y(t) = y_0 + I_\alpha^a(f(s, y^{\frac{1}{\alpha}}(s) + g(s, y^{\frac{1}{\alpha}}(s))(t) \tag{3.2}$$

Proof. If y is a solution to the The equation (4.1), y is continuous because it is α-differentiable ([4, Theorem 2.1]) and and we will have:

$$\begin{aligned} I_\alpha^a(f(s, y^{\frac{1}{\alpha}}(s) + g(s, y^{\frac{1}{\alpha}}(s))(t) &= \int_a^t s^{\alpha-1}(f(s, y^{\frac{1}{\alpha}}(s)) + g(s, y^{\frac{1}{\alpha}}(s))ds \\ &= \int_a^t s^{\alpha-1} y^{(\alpha)}(s)ds \\ &= \int_a^t s^{\alpha-1} s^{1-\alpha} y'(s)ds \\ &= \int_a^t y'(s)ds \\ &= y(t) - y(a) \\ &= y(t) - y_0 \end{aligned}$$

□

Theorem 3.2. *Assume the following hypotheses:*

(1) f and g are continuous.

(2) $f+g$ is bounded.

(3) g is completely continuous.

(4) f is α^2-contractive with respect to the second variable, that is,there exists $L>0$ for almost all $t\in[a,b]$ and every $x,y\in C[a,b]$, and for some $\alpha\in]0,1[$, we have:

$$|f(t,x(t))-f(t,y(t))|\leq L|(x(t))^{(\alpha^2)}-(y(t))^{(\alpha^2)}|$$

If $\frac{L}{\alpha}|b^\alpha-a^\alpha|<1$, then, there exists a unique solution y for the problem (4.1) in the in the interval $[a,b]$.

Proof. According to Lemma 3.1, we just have to prove that there exists a unique solution for the integral equation (3.1).

This equation can be written as:

$$(S+T)(y^{\frac{1}{\alpha}})(t)=y(t)$$

where

$$T(y^{\frac{1}{\alpha}})(t)=y_0+I_a^\alpha(f(s,y^{\frac{1}{\alpha}}(s))(t) \text{ and } S(y^{\frac{1}{\alpha}})(t)=I_a^\alpha(g(s,y^{\frac{1}{\alpha}}(s))(t)$$

with $S+T:C[a,b]\longrightarrow C[a,b]$ which is well defined because $\forall y\in C[a,b]$ $(S+T)y$ is continuous on $[a,b]$.

T is a strictly α-contractive.

Indeed, Then for ary $x,y\in C[a,b],\alpha\in]0,1[$

$$\begin{aligned}
|Ty(t)-Tx(t)| &= \left|I_a^\alpha(f(s,y^{\frac{1}{\alpha}}(s))(t)-I_a^\alpha(f(s,x^{\frac{1}{\alpha}}(s))(t)\right| \\
&= \left|\int_a^t\left(s^{\alpha-1}f(s,y^{\frac{1}{\alpha}}(s))-s^{\alpha-1}f(s,x^{\frac{1}{\alpha}}(s))\right)ds\right| \\
&= \left|\int_a^t s^{\alpha-1}\left(f(s,y^{\frac{1}{\alpha}}(s))-f(s,x^{\frac{1}{\alpha}}(s))\right)ds\right| \\
&\leq \int_a^t s^{\alpha-1}\left|f(s,y^{\frac{1}{\alpha}}(s))-f(s,x^{\frac{1}{\alpha}}(s))\right|ds \\
&\leq L\int_a^t s^{\alpha-1}\left|(y^{\frac{1}{\alpha}})^{\alpha^2}(s)-(x^{\frac{1}{\alpha}})^{\alpha^2}(s)\right|
\end{aligned}$$

$$= \quad L\int_a^t s^{\alpha-1}\left|y^{\alpha}(s)-x^{\alpha}(s)\right|ds$$

So,

$$\begin{aligned}
\|Ty-Tx\|_C &= \max_{t\in[a,b]}|Ty(t)-Tx(t)| \\
&\leq \max_{t\in[a,b]}\left(L\int_a^t s^{\alpha-1}\left|y^{\alpha}(s)-x^{\alpha}(s)\right|ds\right) \\
&\leq L\max_{t\in[a,b]}\left|y^{\alpha}(s)-x^{\alpha}(s)\right|\int_a^t s^{\alpha-1}ds \\
&= L\|y^{\alpha}-x^{\alpha}\|_C\int_a^t s^{\alpha-1}ds \\
&\leq L\|y^{\alpha}-x^{\alpha}\|_C\left|\frac{b^{\alpha}}{\alpha}-\frac{a^{\alpha}}{\alpha}\right| \\
&= \frac{L}{\alpha}\left|b^{\alpha}-a^{\alpha}\right|\|y^{\alpha}-x^{\alpha}\|_C
\end{aligned}$$

Since $\frac{L}{\alpha}|b^{\alpha}-a^{\alpha}|<1$, then T is a strictly α-contractive withe $K=\frac{L}{\alpha}|b^{\alpha}-a^{\alpha}|<1$.

The Ascoli-Arzela theorem and the condition (1), (2) allows us to conclude that S is compact.

By the corolllary 2.14, The mapping $S+T$ has a α-fixed point $y\in C[a,b]$.

Then, there exists a solution for the problem (3.1) in the in the interval $[a,b]$. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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