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EXISTENCE OF TRAVELING WAVES BY MEANS OF FIXED POINT THEORY FOR AN EPIDEMIC MODEL WITH HATTAF-YOUSFI INCIDENCE RATE AND TEMPORARY IMMUNITY ACQUIRED BY VACCINATION

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Abstract. The main aim of this work is to investigate the existence of traveling waves of an epidemic model with temporary immunity acquired by vaccination. The incidence rate of the disease used in the epidemic model is of the form Hattaf-Yousfi that includes many types existing in the literature. By means of Schauder fixed point theorem and construction of a pair of upper and lower solutions, the existence of traveling wave solution that connects the disease-free equilibrium and the endemic equilibrium is obtained and characterized by two parameters that are the basic reproduction number and the minimal wave speed.

Keywords: traveling wave; Hattaf-Yousfi incidence rate; Schauder fixed point theorem.

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1. Introduction

In epidemiology, the existence of traveling wave which describes the transition of disease-free equilibrium to endemic equilibrium has been investigated by many authors. For instance,

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in 1995, Hosono and Ilyas [1] investigated the existence of traveling wave solution of diffusive epidemic model. In 2011, Yang et al. [2] interested to the existence of traveling waves to a SIR epidemic model with nonlinear incidence rate, spatial diffusion and time delay. In 2014, Xu [3] studied the traveling waves in a Kermack-Mckendrick epidemic model with diffusion and latent period. In [4], the authors studied the existence of solution of traveling waves of a delayed diffusive epidemic model with specific nonlinear incidence rate. In this study, we propose the following system with general incidence rate and temporary immunity acquired by vaccination:

$$\left\{\begin{aligned} \frac{\partial S(x,t)}{\partial t} &= D_S\Delta S + A - h(S(x,t),I(x,t-\tau))I(x,t-\tau) \\ &\quad -(\mu+\rho)S(x,t) + \rho\int_0^{\infty} S(x,t-u)g(u)e^{-\mu u}du, \\ \frac{\partial I(x,t)}{\partial t} &= D_I\Delta I + h(S(x,t),I(x,t-\tau))I(x,t-\tau) - (\mu+d+r)I(x,t), \\ \frac{\partial R(x,t)}{\partial t} &= D_R\Delta R + rI(x,t) - \mu R(x,t) - \rho\int_0^{\infty} S(x,t-u)g(u)e^{-\mu u}du + \rho S(x,t), \end{aligned}\right. \tag{1}$$

where $S(x,t)$, $I(x,t)$ and $R(x,t)$ represent respectively the densities of susceptible, infected and removed individuals at position x and time t. The constants D_S, D_I and D_R denote the corresponding diffusion coefficients for the susceptible, infected and removed populations. A is the recruitment rate of susceptible population, μ is the natural death rate of the population, d is the death rate due to disease, τ is the latent period, and r is the recovery rate of the infected population. ρ is the rate of vaccination, and $g(u)du$ is the probability of losing immunity between u and $u+du$, where $g(u)$ is the density of probability satisfying $\int_0^{+\infty} g(u)du = 1$. The incidence rate of infection is modeled by Hattaf-Yousfi function $h(S,I) = \frac{\beta S}{\alpha_0+\alpha_1 S+\alpha_2 I+\alpha_3 SI}$ [5], with $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \geq 0$ are constants and $\beta > 0$ is the infection process. This general incidence rate generalizes many type of incidence rate used in [6, 7, 8, 9].

Since the variable of removed individuals R does not appears in the first two equations of (1), then system (1) can be reduced to the following model:

$$\left\{\begin{aligned} \frac{\partial S(x,t)}{\partial t} &= d_S\Delta S + A - h(S(x,t),I(x,t-\tau))I(x,t-\tau) \\ &\quad -(\mu+\rho)S(x,t) + \rho\int_0^{\infty} S(x,t-u)g(u)e^{-\mu u}du, \\ \frac{\partial I(x,t)}{\partial t} &= d_I\Delta I + h(S(x,t),I(x,t-\tau))I(x,t-\tau) - (\mu+d+r)I(x,t), \end{aligned}\right. \tag{2}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial \nu}=\frac{\partial I}{\partial \nu}=0, \quad on \quad \partial\Omega\times(0,+\infty), \tag{3}$$

and initial conditions

$$S(x,\theta)=\phi_1(x,\theta), \quad I(x,\theta)=\phi_2(x,\theta), \quad x\in\bar{\Omega}, \quad \theta\in[-\tau,0], \tag{4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial\nu}$ denotes the outward normal derivative on $\partial\Omega$.

The rest of this paper is outlined as follows. Section 2 is devoted to the reproduction number and steady states. Section 3 deals with the existence of traveling waves by means of Schauder fixed point theorem. Section 4 treats the nonexistence of traveling waves. Finally, the paper ends with a conclusion.

2. Reproduction Number and Steady States

In this section, we study the existence of equilibria of system (2).

Let $\eta=\int_0^\infty g(u)e^{-\mu u}du$. It is obvious that $E^0\left(\frac{A}{\mu+\rho-\rho\eta},0\right)$ is the unique disease-free steady state. Hence, the basic reproduction number is as follows

$$R_0=\frac{\beta A}{(\mu\alpha_0+\rho\alpha_0(1-\eta)+\alpha_1 A)(\mu+d+r)}.$$

Theorem 2.1.

(i): *If* $R_0\leq 1$, *then system (2) has a disease-free equilibrium point given by* $E^0\left(\frac{A}{\mu+\rho-\rho\eta},0\right)$.

(ii): *If* $R_0>1$, *then the system (2) has a unique endemic equilibrium of the form* $E^*(S^*,I^*)$ *with* $S^*\in(0,\frac{\Lambda}{\mu+\rho-\rho\eta})$ *and* $I^*>0$.

Proof. Any uniform steady state of system (2) satisfies

$$A-(\mu+\rho)S-h(S,I)I+\rho\eta S=0, \tag{5}$$

$$h(S,I)I-(\mu+d+r)I=0. \tag{6}$$

From (6), we get $I=0$ or $h(S,I)=\mu+d+r$.

If $I=0$, then we obtain the disease-free equilibrium point $E^0\left(\frac{\Lambda}{\mu+\rho-\rho\eta},0\right)$.
If $I\neq 0$, then using (5) and (6) we get the following equation

$$h\left(S,\frac{A-(\mu+\rho(1-\eta))S}{\mu+d+r}\right)=\mu+d+r. \tag{7}$$

We have $I=\frac{A-(\mu+\rho(1-\eta))S}{\mu+d+r}\geq 0$ implies that $S\leq\frac{A}{\mu+\rho(1-\eta)}$. Hence, there is no positive equilibrium point if $S>\frac{\Lambda}{\mu+\rho(1-\eta)}$.
Now, we consider the following function L defined on the interval $\left[0,\frac{A}{\mu+\rho(1-\eta)}\right]$ as

$$L(S)=h\left(S,\frac{A-(\mu+\rho(1-\eta))S}{(\mu+d+r)}\right)-(\mu+d+r).$$

Since, $L(0)=-(\mu+d+r)<0$ and $L\left(\frac{A}{\mu+\rho(1-\eta)}\right)=(\mu+d+r)(R_0-1)>0$ for $R_0>1$. Further, $L'(S)=\frac{\partial h}{\partial S}-\frac{\mu+\rho(1-\eta)}{(\mu+d+r)}\frac{\partial h}{\partial I}>0$. Therefore, there exists a unique endemic equilibrium $E^*(S^*,I^*)$ with $S^*\in(0,\frac{\Lambda}{\mu+\rho(1-\eta)})$ and $I^*>0$. ■

3. Existence of Traveling Waves

In this section, we study the existence of traveling waves by means of Schauder fixed point theorem.

Using the following transformations in system (2)

$$\varsigma=\frac{A}{\mu+\rho(1-\eta)},\quad \widetilde{S}(x,t)=\frac{1}{\varsigma}S\left(x\sqrt{D_I},t\right),\quad \widetilde{I}(x,t)=\frac{1}{\varsigma}I\left(x\sqrt{D_I},t\right),$$
$$D=\frac{D_S}{D_I},\quad \widetilde{\beta}=\varsigma\beta,\quad \widetilde{\alpha}_0=\alpha_0,\quad \widetilde{\alpha}_1=\varsigma\alpha_1,\quad \widetilde{\alpha}_2=\varsigma\alpha_2,\quad \widetilde{\alpha}_3=\varsigma^2\alpha_3,$$

we get

$$\begin{cases}\dfrac{\partial S(x,t)}{\partial t}=D_S\dfrac{\partial^2 S(x,t)}{\partial x^2}+\dfrac{A}{\varsigma}-\widetilde{h}(S(x,t),I(x,t-\tau))I(x,t-\tau)\\ \qquad\qquad -(\mu+\rho)S(x,t)+\rho\displaystyle\int_0^{\infty}S(x,t-u)g(u)e^{-\mu u}du,\\ \dfrac{\partial I(x,t)}{\partial t}=\dfrac{\partial^2 I(x,t)}{\partial x^2}+\widetilde{h}(S(x,t),I(x,t-\tau))I(x,t-\tau)-(\mu+d+r)I(x,t),\end{cases} \tag{8}$$

where

$$\tilde{h}(S,I)=\frac{\widetilde{\beta}S}{\widetilde{\alpha}_0+\widetilde{\alpha}_1 S+\widetilde{\alpha}_2 I+\widetilde{\alpha}_3 SI}.$$

In this case, system (8) has always a disease-free equilibrium $\widetilde{E}^0(1,0)$ and an endemic equilibrium $\widetilde{E}^*\left(\frac{S^*}{\varsigma},\frac{I^*}{\varsigma}\right)$. The traveling wave solution of our new system connecting the disease-free equilibrium and the endemic equilibrium is a special solution of the form

$$(S(x,t),I(x,t)) = (\zeta(x+ct),\xi(x+ct)),$$

where $\zeta,\xi \in C^2(\mathbb{R},\mathbb{R})$ and $c>0$ is a constant representing the wave speed. By substituting $\zeta(x+ct)$ and $\xi(x+ct)$ into (8) and denoting $x+ct$ by y, we obtain two equations

$$\begin{aligned} c\zeta'(y) &= D\zeta''(y) + \frac{A}{\varsigma} - (\mu+\rho)\zeta(y) - \tilde{h}(\zeta(y),\xi(y-c\tau)\xi(y-c\tau) \\ &\quad + \rho\int_0^\infty \zeta(y-cu)g(u)e^{-\mu u}du, \end{aligned} \tag{9}$$

$$c\xi'(y) = \xi''(y) + \widetilde{h}(\zeta(y),\xi(y-c\tau))\xi(y-c\tau) - (\mu+d+r)\xi(y), \tag{10}$$

with the boundary conditions,

$$(\zeta,\xi)(-\infty) = (1,0) \text{ and } (\zeta,\xi)(+\infty) = \left(\frac{S^*}{\varsigma},\frac{I^*}{\varsigma}\right). \tag{11}$$

On the other hand, the characteristic equation of equation (10) at $\widetilde{E}^0$ satisfies

$$\Delta(\lambda,c) = \lambda^2 - c\lambda + \frac{\tilde{\beta}}{\widetilde{\alpha}_0+\widetilde{\alpha}_1}e^{-\lambda c\tau} - (\mu+d+r). \tag{12}$$

Hence, it is easy to prove the following result.

Lemma 3.1. *Suppose that $R_0>1$, then there exist $c^*>0$ and $\lambda^*>0$ such that $\frac{\partial\Delta(\lambda^*,c^*)}{\partial\lambda}=0$ and $\Delta(\lambda^*,c^*)=0$. Furthermore,*

- **(i):** *If $0<c<c^*$, then $\Delta(\lambda,c)>0$, for all $\lambda\geq 0$;*
- **(ii):** *If $c>c^*$, then $\Delta(\lambda,c)=0$ has two positive solutions $\lambda_1(c)$ and $\lambda_2(c)$ such that*
$$0<\lambda_1(c)<\lambda^*<\lambda_2(c) \text{ and } \Delta(\lambda,c)\begin{cases} >0, & \lambda\in[0,\lambda_1(c))\cup(\lambda_2(c),\infty), \\ <0, & \lambda\in(\lambda_1(c),\lambda_2(c)). \end{cases}$$

To establish the existence of traveling wave solution of equations (9)-(10), we construct a pair of upper and lower solutions. For the rest of this section, we assume that $R_0>1$ and $c>c^*$. Let $\lambda_i=\lambda_i(c), i=1,2$. Define the following functions

$$\bar{\zeta}(y) = 1, \tag{13}$$

$$\bar{\xi}(y)=\min\left\{e^{\lambda_1 y},\frac{\widetilde{\alpha}_0+\widetilde{\alpha}_1}{\widetilde{\alpha}_2+\widetilde{\alpha}_3}(R_0-1)\right\}. \tag{14}$$

Lemma 3.2. *The function* $(\bar{\zeta},\bar{\xi})$ *is an upper solution of equations (9)-(10).*

Proof. Let be the following function

$$\begin{aligned}\Sigma_1(\bar{\zeta}(y),\bar{\xi}(y)) &= D\bar{\zeta}''(y)-c\bar{\zeta}'(y)+\frac{A}{\varsigma}\\ &\quad -(\mu+\rho)\bar{\zeta}(y)-\tilde{h}(\bar{\zeta}(y),\bar{\xi}(y-c\tau))\bar{\xi}(y-c\tau)+\rho\eta\bar{\zeta}(y).\end{aligned}$$

We have

$$\begin{aligned}\Sigma_1(\bar{\zeta}(y),\bar{\xi}(y)) &= \frac{A}{\varsigma}-(\mu+\rho)-\tilde{h}(\bar{\zeta}(y),\bar{\xi}(y-c\tau))\bar{\xi}(y-c\tau)+\rho\eta\\ &\leq -\tilde{h}(\bar{\zeta}(y),\bar{\xi}(y-c\tau))\bar{\xi}(y-c\tau)\\ &\leq 0.\end{aligned}$$

Then, we conclude

$$c\bar{\zeta}'(y)\geq D\bar{\zeta}''(y)+\frac{A}{\varsigma}-(\mu+\rho)\bar{\zeta}(y)-\tilde{h}(\bar{\zeta}(y),\bar{\xi}(y-c\tau))\bar{\xi}(y-c\tau)+\rho\eta\bar{\zeta}(y). \tag{15}$$

It remains to prove that the function $\bar{\xi}$ satisfies

$$c\bar{\xi}'(y)\geq\bar{\xi}''(y)+\tilde{h}(\bar{\zeta}(y),\bar{\xi}(y-c\tau))\bar{\xi}(y-c\tau)-(\mu+d+r)\bar{\xi}(y), \tag{16}$$

for all $y\neq y_1:=\frac{1}{\lambda_1}\ln\left(\frac{\widetilde{\alpha}_0+\widetilde{\alpha}_1}{\widetilde{\alpha}_2+\widetilde{\alpha}_3}(R_0-1)\right)$.
If $y<y_1$, then $\bar{\xi}(y)=e^{\lambda_1 y}$. Let the following function

$$\Sigma_2(\bar{\zeta}(y),\bar{\xi}(y))=\bar{\xi}''(y)-c\bar{\xi}'(y)+\tilde{h}(\bar{\zeta}(y),\bar{\xi}(y-c\tau))\bar{\xi}(y-c\tau)-(\mu+d+r)\bar{\xi}(y).$$

We have

$$\begin{aligned}\Sigma_2(\bar{\zeta}(y),\bar{\xi}(y)) &\leq \lambda_1^2e^{\lambda_1 y}-c\lambda_1e^{\lambda_1 y}-(\mu+d+r)e^{\lambda_1 y}+\frac{\widetilde{\beta}}{\widetilde{\alpha}_0+\widetilde{\alpha}_1}e^{\lambda_1(y-c\tau)}\\ &= e^{\lambda_1 y}\Delta(\lambda_1,c).\end{aligned}$$

By Lemma 3.1, we have $\Delta(\lambda_1,c)=0$. Then we deduce (16).
If $y>y_1$, then $\bar{\xi}(y)=\frac{\widetilde{\alpha}_0+\widetilde{\alpha}_1}{\widetilde{\alpha}_2+\widetilde{\alpha}_3}(R_0-1)$ and

$$\widetilde{h}(\bar{\zeta}(y),\bar{\xi}(y-c\tau))\bar{\xi}(y-c\tau)-(\mu+d+r)\bar{\xi}(y)=0.$$

Therefore, we conclude inequality (16). ■

Lemma 3.3. *Let $\sigma \in (0,\lambda_1)$ sufficiently small. Then the function* $\underline{\zeta}(y) = \max\left\{1 - \frac{1}{\sigma}e^{\sigma y}, 0\right\}$. *satisfies*

$$c\underline{\zeta}'(y) \le D\underline{\zeta}''(y) + \frac{A}{\varsigma} - \tilde{h}(\underline{\zeta}(y), \bar{\xi}(y-c\tau))\bar{\xi}(y-c\tau) - (\mu+\rho)\underline{\zeta}(y) + \rho\int_0^\infty \underline{\zeta}(y-cu)g(u)e^{-\mu u}du, \tag{17}$$

for all $y \neq y_2 := \frac{1}{\sigma}\ln\sigma$.

Proof. If $y > y_2$, then we verify immediately the inequality (17) since $\underline{\zeta}(y) = 0$. On the other hand if $y < y_2$, then $\underline{\zeta}(y) = 1 - \frac{1}{\sigma}e^{\sigma y}$. Let the following function

$$\begin{aligned}\Sigma_3(\underline{\zeta}(y), \bar{\xi}(y)) &= D\underline{\zeta}''(y) - c\underline{\zeta}'(y) + \frac{A}{\varsigma} - \tilde{h}(\underline{\zeta}(y), \bar{\xi}(y-c\tau))\bar{\xi}(y-c\tau) \\ &\quad -(\mu+\rho)\underline{\zeta}(y) + \rho\int_0^\infty \underline{\zeta}(y-cu)g(u)e^{-\mu u}du.\end{aligned}$$

We have

$$\begin{aligned}\Sigma_3(\underline{\zeta}(y), \bar{\xi}(y)) &\ge -D\sigma e^{\sigma y} + ce^{\sigma y} + \mu + \rho(1-\eta) - \mu(1 - \frac{1}{\sigma}e^{\sigma y}) \\ &\quad -\frac{\widetilde{\beta}}{\widetilde{\alpha}_0}e^{\lambda_1(y-c\tau)}(1 - \frac{1}{\sigma}e^{\sigma y}) + \rho(\eta-1) \\ &\quad +\frac{\rho}{\sigma}e^{\sigma y}(1 - \int_0^\infty e^{-(\sigma c+\mu)u}g(u)du) \\ &\ge -D\sigma e^{\sigma y} + ce^{\sigma y} - \frac{\widetilde{\beta}}{\widetilde{\alpha}_0}e^{\lambda_1(y-c\tau)}(1 - \frac{1}{\sigma}e^{\sigma y}) \\ &\ge -D\sigma e^{\sigma y} + ce^{\sigma y} - \frac{\widetilde{\beta}}{\widetilde{\alpha}_0}e^{\lambda_1(y-c\tau)} \\ &\ge e^{\sigma y}\left(-D\sigma + c - \frac{\widetilde{\beta}}{\widetilde{\alpha}_0}e^{-c\tau\lambda_1}e^{(\lambda_1-\sigma)y}\right) \\ &\ge e^{\sigma y}\left(-D\sigma + c - \frac{\widetilde{\beta}}{\widetilde{\alpha}_0}e^{-\lambda_1 c\tau}\sigma^{\frac{\lambda_1-\sigma}{\sigma}}\right),\end{aligned}$$

By the choice of σ we get

$$-D\sigma + c - \frac{\widetilde{\beta}}{\widetilde{\alpha}_0}e^{-\lambda_1 c\tau}\sigma^{\frac{\lambda_1-\sigma}{\sigma}} \ge 0.$$

Then we conclude the inequality (17). ■

Lemma 3.4. *Let $\phi > 1$ sufficiently large, $0 < \omega < \min\{\sigma, \lambda_1, \lambda_2 - \lambda_1\}$. Then the function $\underline{\xi}(y) = \max\left\{e^{\lambda_1 y}(1-\phi e^{\omega y}), 0\right\}$ satisfies the following inequality*

$$c\underline{\xi}'(y) \leq \underline{\xi}''(y) + \widetilde{h}(\underline{\zeta}(y), \underline{\xi}(y-c\tau))\underline{\xi}(y-c\tau) - (\mu+d+r)\underline{\xi}(y), \tag{18}$$

for all $y \neq y_3 := \frac{1}{\omega}\ln\frac{1}{\phi}$.

Proof. If $y > y_3$, then we verify immediately the inequality (18), since $\underline{\xi}(y) = 0$.

On the other hand if $y < y_3$, then $\underline{\xi}(y) = e^{\lambda_1 y}(1-\phi e^{\omega y})$. Let $\phi_1 = e^{(1-y_2)\omega}$ and we choice ϕ

$$\phi > \max\left\{\frac{\left(\frac{\tilde{\alpha}_2+\tilde{\tilde{\alpha}}_3}{\tilde{\alpha}_0+\tilde{\alpha}_1} + 1/\sigma\right)\widetilde{\beta}}{-\Delta(\lambda_1+\omega, c)}, \phi_1, 1\right\}.$$

We have $\underline{\zeta}(y) = 1 - \frac{1}{\sigma}e^{\sigma y}$ since $y_3 < y_2$.

By simple calculus, we have

$$\begin{aligned}
\tilde{h}(\underline{\zeta}(y), \underline{\xi}(y-c\tau))\underline{\xi}(y-c\tau) &= \frac{\widetilde{\beta}\underline{\zeta}(y)\underline{\xi}(y-c\tau)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1\underline{\zeta}(y)+\widetilde{\alpha}_2\underline{\xi}(y-c\tau)+\widetilde{\alpha}_3\underline{\zeta}(y)\underline{\xi}(y-c\tau)}, \\
&= \frac{\widetilde{\beta}\underline{\xi}(y-c\tau)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1\underline{\zeta}(y)+\widetilde{\alpha}_2\underline{\xi}(y-c\tau)+\widetilde{\alpha}_3\underline{\zeta}(y)\underline{\xi}(y-c\tau)} \\
&\quad - \frac{\widetilde{\beta}\frac{1}{\sigma}e^{\sigma y}\underline{\xi}(y-c\tau)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1\underline{\zeta}(y)+\widetilde{\alpha}_2\underline{\xi}(y-c\tau)+\widetilde{\alpha}_3\underline{\zeta}(y)\underline{\xi}(y-c\tau)}, \\
&= \frac{\widetilde{\beta}\underline{\xi}(y-c\tau)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1\underline{\zeta}(\xi)+\widetilde{\alpha}_2\underline{\xi}(y-c\tau)+\widetilde{\alpha}_3\underline{\zeta}(y)\underline{\xi}(y-c\tau)} \\
&\quad - \frac{\widetilde{\beta}\xi(y-c\tau)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1} + \frac{\widetilde{\beta}\underline{\xi}(y-c\tau)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1} \\
&\quad - \frac{\widetilde{\beta}\frac{1}{\sigma}e^{\sigma y}\underline{\xi}(y-c\tau)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1\underline{\zeta}(y)+\widetilde{\alpha}_2\underline{\xi}(y-c\tau)+\widetilde{\alpha}_3\underline{\zeta}(y)\underline{\xi}(y-c\tau)}.
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{h}(\underline{\zeta}(y), \underline{\xi}(y-c\tau))\underline{\xi}(y-c\tau) &\geq \frac{\widetilde{\beta}\underline{\xi}(y-c\tau)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1} - \frac{\widetilde{\beta}(\widetilde{\alpha}_2+\widetilde{\alpha}_3)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1}\underline{\xi}^2(y-c\tau) - \widetilde{\beta}\frac{1}{\sigma}e^{(\sigma+\lambda_1)y} \\
&\geq \frac{\widetilde{\beta}\underline{\xi}(y-c\tau)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1} - \frac{\widetilde{\beta}(\widetilde{\alpha}_2+\widetilde{\alpha}_3)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1}e^{2\lambda_1 y} - \widetilde{\beta}\frac{1}{\sigma}e^{(\sigma+\lambda_1)y},
\end{aligned}$$

and we verify immediately that

$$\begin{aligned}\Sigma_4(\underline{\zeta}(y),\underline{\xi}(y)) &\geq \underline{\xi}''(y)-c\underline{\xi}'(y)+\frac{\widetilde{\beta}\underline{\xi}(y-c\tau)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1}-\frac{\widetilde{\beta}\,(\widetilde{\alpha}_2+\widetilde{\alpha}_3)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1}e^{2\lambda_1 y}-\widetilde{\beta}\frac{1}{\sigma}e^{(\sigma+\lambda_1)y}\\ &\quad -(\mu+d+r)\underline{\xi}(y),\\ &= -\Delta(\lambda_1+\omega,c)\,\phi e^{(\lambda_1+\omega)y}-\frac{\widetilde{\beta}\,(\widetilde{\alpha}_2+\widetilde{\alpha}_3)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1}e^{2\lambda_1 y}-\widetilde{\beta}\frac{1}{\sigma}e^{(\sigma+\lambda_1)y}\\ &\geq \left(-\Delta(\lambda_1+\omega,c)\,\phi-\frac{\widetilde{\beta}\,(\widetilde{\alpha}_2+\widetilde{\alpha}_3)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1}-\widetilde{\beta}\frac{1}{\sigma}\right)e^{(\lambda_1+\omega)y},\end{aligned}$$

where

$$\Sigma_4(\underline{\zeta}(y),\underline{\xi}(y))=\underline{\xi}''(y)-c\underline{\xi}'(y)+\widetilde{h}(\underline{\zeta}(y),\underline{\xi}(y-c\tau))\underline{\xi}(y-c\tau)-(\mu+d+r)\underline{\xi}(y).$$

By the choice of ϕ we have $-\Delta(\lambda_1+\omega,c)\,\phi-\frac{\widetilde{\beta}(\widetilde{\alpha}_2+\widetilde{\alpha}_3)}{\widetilde{\alpha}_0+\widetilde{\alpha}_1}-\frac{\widetilde{\beta}}{\sigma}>0$.

This completes the proof. ■

Next to verify Schauder fixed point theorem conditions, we use the upper and lower solutions $(\underline{\zeta},\underline{\xi})$ and $(\bar{\zeta},\bar{\xi})$ constructed above. Let the following functions

$$\begin{aligned}&H_1(\zeta,\xi)(y)=\gamma\zeta(y)+\frac{A}{\varsigma}-(\mu+\rho)\zeta(y)-\tilde{h}(\zeta(y),\xi(y-c\tau))\xi(y-c\tau)\\ &+\rho\int_0^\infty \zeta(y-cu)g(u)e^{-\mu u}du,\\ &H_2(\zeta,\xi)(y)=\gamma\xi(y)+\widetilde{h}(\zeta(y),\xi(y-c\tau))\xi(y-c\tau)-(\mu+d+r)\xi(y),\end{aligned}\tag{19}$$

where $\gamma>\max\{\mu+d+r,\mu+\widetilde{\beta}/2+\rho\}$.

Let Γ the set defined as follows

$$\Gamma=\left\{(\zeta,\xi)\in C\left(\mathbb{R},\mathbb{R}^2\right)/(\underline{\zeta},\underline{\xi})\leq(\zeta,\xi)\leq(\bar{\zeta},\bar{\xi})\right\}.$$

We can verify that Γ is nonempty, closed and convex in $C\left(\mathbb{R},\mathbb{R}^2\right)$. Consider the operator $F:\Gamma\longrightarrow C\left(\mathbb{R},\mathbb{R}^2\right)$ defined by

$$F(\zeta,\xi)(y)=(F_1(\zeta,\xi),F_2(\zeta,\xi))(y),$$

with

$$F_1(\zeta,\xi)(y)=\frac{1}{D(\lambda_{12}-\lambda_{11})}\left\{\int_{-\infty}^{y}e^{\lambda_{11}(y-x)}+\int_{y}^{+\infty}e^{\lambda_{12}(y-x)}\right\}H_1(\zeta,\xi)(x)dx,$$

$$F_2(\zeta,\xi)(y)=\frac{1}{(\lambda_{22}-\lambda_{21})}\left\{\int_{-\infty}^{y}e^{\lambda_{21}(y-x)}+\int_{y}^{+\infty}e^{\lambda_{22}(y-x)}\right\}H_2(\zeta,\xi)(x)dx,$$

where

$$\lambda_{11}=\frac{c-\sqrt{c^2+4D\gamma}}{2D},\quad \lambda_{12}=\frac{c+\sqrt{c^2+4D\gamma}}{2D},$$
$$\lambda_{21}=\frac{c-\sqrt{c^2+4\gamma}}{2},\quad \lambda_{22}=\frac{c+\sqrt{c^2+4\gamma}}{2}.$$

We verify easily that any fixed point of F is a solution of (19). Hence, the existence of solution of (9)-(10) is reduced to verify that the operator F satisfies the conditions of Schauder fixed point theorem. Here, we divide the proof into three lemmas.

Lemma 3.5. *The operator F maps Γ into Γ.*

Proof. We can verify easily by the choice of γ, that H_1 is monotone increasing in ζ and monotone decreasing in ξ, then we get for all $y\in\mathbb{R}$

$$F_1(\underline{\zeta},\bar{\xi})(y)\le F_1(\zeta,\xi)(y)\le F_1(\bar{\zeta},\underline{\xi})(y). \tag{20}$$

By (17), we get

$$F_1(\underline{\zeta},\bar{\xi})(y)\ge\frac{1}{D(\lambda_{12}-\lambda_{11})}\left\{\int_{-\infty}^{y}e^{\lambda_{11}(y-x)}+\int_{y}^{+\infty}e^{\lambda_{12}(y-x)}\right\}$$
$$\times\left[\gamma\underline{\zeta}(x)+c\underline{\zeta}'(x)-D\underline{\zeta}''(x)\right]dx.$$

for all $y\neq y_2$.

If $y>y_2$, then

$$\begin{aligned}F_1(\underline{\zeta},\bar{\xi})(y) &\ge \frac{1}{D(\lambda_{12}-\lambda_{11})}\int_{-\infty}^{y_2}e^{\lambda_{11}(y-x)}\left[\gamma\underline{\zeta}(x)+c\underline{\zeta}'(x)-D\underline{\zeta}''(x)\right]dx\\ &\quad+\frac{1}{D(\lambda_{12}-\lambda_{11})}\int_{y_2}^{y}e^{\lambda_{11}(y-x)}\left[\gamma\underline{\zeta}(x)+c\underline{\zeta}'(x)-D\underline{\zeta}''(x)\right]dx\\ &\quad+\frac{1}{D(\lambda_{12}-\lambda_{11})}\int_{y}^{+\infty}e^{\lambda_{12}(\zeta-x)}\left[\gamma\underline{\zeta}(x)+c\underline{\zeta}'(x)-D\underline{\zeta}''(x)\right]dx\\ &= \underline{\zeta}(y)-\frac{e^{\lambda_{11}(y-y_2)}\underline{\zeta}'(y_2-)}{\lambda_{12}-\lambda_{11}}\\ &\ge \underline{\zeta}(y).\end{aligned}$$

By the same technique, we get for all $y < y_2$, $F_1(\underline{\zeta}, \bar{\xi})(y) \geq \underline{\zeta}(y)$. From the continuity of $F_1(\underline{\zeta}, \bar{\xi})(y)$ and $\underline{\zeta}(y)$, we deduce

$$F_1(\underline{\zeta}, \bar{\xi})(y) \geq \underline{\zeta}(y), \quad \forall y \in \mathbb{R}. \tag{21}$$

On the other hand $H_1(\bar{\zeta}, \underline{\xi}) \leq \gamma \bar{\zeta}$, then

$$F_1(\bar{\zeta}, \underline{\xi})(y) \leq 1, \quad \forall y \in \mathbb{R}. \tag{22}$$

By inequalities (20)-(22), we deduce

$$\underline{\zeta}(y) \leq F_1(\zeta, \xi)(y) \leq \bar{\zeta}(y), \quad \forall y \in \mathbb{R}. \tag{23}$$

Similarly H_2 is monotone increasing in ζ and ξ, then we get for all $y \in \mathbb{R}$,

$$F_2(\underline{\zeta}, \underline{\xi})(y) \leq F_2(\zeta, \xi)(y) \leq F_2(\bar{\zeta}, \bar{\xi})(y). \tag{24}$$

By (16), we get for all $y \neq y_1$

$$\begin{aligned} F_2(\bar{\zeta}, \bar{\xi})(y) \leq \frac{1}{(\lambda_{22} - \lambda_{21})} & \left\{ \int_{-\infty}^{y} e^{\lambda_{21}(y-x)} + \int_{y}^{+\infty} e^{\lambda_{22}(y-x)} \right\} \\ & \times \left[\gamma \bar{\xi}(x) + c \bar{\xi}'(x) - \bar{\xi}''(x) \right] dx. \end{aligned}$$

If $y < y_1$, then

$$\begin{aligned} F_2(\bar{\zeta}, \bar{\xi})(y) & \leq \frac{1}{(\lambda_{22} - \lambda_{21})} \int_{-\infty}^{y} e^{\lambda_{21}(y-x)} \left[\gamma \bar{\xi}(x) + c \bar{\xi}'(x) - \bar{\xi}''(x) \right] dx \\ & \quad + \frac{1}{(\lambda_{22} - \lambda_{21})} \int_{\kappa}^{y_1} e^{\lambda_{22}(y-x)} \left[\gamma \bar{\xi}(x) + c \bar{\xi}'(x) - \bar{\xi}''(x) \right] dx \\ & \quad + \frac{1}{(\lambda_{22} - \lambda_{21})} \int_{y_1}^{+\infty} e^{\lambda_{22}(y-x)} \left[\gamma \bar{\xi}(x) + c \bar{\xi}'(x) - \bar{\xi}''(x) \right] dx \\ & \leq \bar{\xi}(y). \end{aligned}$$

If $y > y_1$, then

$$\begin{aligned} F_2(\bar{\zeta}, \bar{\xi})(y) & \leq \frac{1}{(\lambda_{22} - \lambda_{21})} \left\{ \int_{-\infty}^{y} e^{\lambda_{21}(y-x)} + \int_{y}^{+\infty} e^{\lambda_{22}(y-x)} \right\} \gamma \bar{\xi}(x) dx \\ & = \bar{\xi}(y). \end{aligned}$$

From the continuity of $F_2(\bar{\zeta}, \bar{\xi})(y)$ and $\bar{\xi}(y)$, we deduce

$$F_2(\bar{\zeta}, \bar{\xi})(y) \leq \bar{\xi}(y), \quad \forall y \in \mathbb{R}. \tag{25}$$

From (18), we have for all $y \neq y_3$

$$F_2(\underline{\zeta},\underline{\xi})(y) \geq \frac{1}{(\lambda_{22}-\lambda_{21})}\left\{\int_{-\infty}^{y} e^{\lambda_{21}(y-x)} + \int_{y}^{+\infty} e^{\lambda_{22}(y-x)}\right\} \times \left[\gamma\underline{\xi}(x) + c\underline{\xi}'(x) - \underline{\xi}''(x)\right] dx.$$

Similarly, then as well

$$F_2(\underline{\zeta},\underline{\xi})(y) \geq \underline{\xi}(y), \quad \forall y \in \mathbb{R}. \tag{26}$$

From (24)-(26), we deduce

$$\underline{\xi}(y) \leq F_2(\zeta,\xi)(y) \leq \bar{\xi}(y), \quad \forall y \in \mathbb{R}. \tag{27}$$

By (23) and (27), we conclude that F maps Γ to Γ. ■

Let $v > 0$ be a constant such that $v < \min\left\{-\lambda_{11}, -\lambda_{21}, \frac{\mu}{c}\right\}$, and

$$B_v\left(\mathbb{R},\mathbb{R}^2\right) = \left\{(\zeta,\xi) \in C : \sup_{y\in\mathbb{R}}\left\{|\zeta(y)|e^{-v|y|}\right\} < \infty, \quad \sup_{y\in\mathbb{R}}\left\{|\xi(y)|e^{-v|y|}\right\} < \infty\right\},$$

we can verify that $B_v\left(\mathbb{R},\mathbb{R}^2\right)$ is a Banach space with the norm $|.|_v$, defined by

$$|(\zeta,\xi)|_v = \max\left\{\sup_{y\in\mathbb{R}}|\zeta(y)|e^{-v|y|}, \sup_{y\in\mathbb{R}}|\xi(y)|e^{-v|y|}\right\}.$$

Lemma 3.6. *The operator F is continuous with respect to the norm $|\cdot|_v$ in $B_v\left(\mathbb{R},\mathbb{R}^2\right)$.*

Proof. Let $\Phi_1 = (\zeta_1,\xi_1) \in \Gamma$ and $\Phi_2 = (\zeta_2,\xi_2) \in \Gamma$, we verify that

$$\left|\widetilde{h}(\zeta_1(y),\xi_1(y-c\tau))\,\xi_1(y-c\tau) - \widetilde{h}(\zeta_2(y),\xi_2(y-c\tau))\,\xi_2(y-c\tau)\right| \leq \left(\frac{\widetilde{\beta}}{\widetilde{\alpha}_0\widetilde{\alpha}_2} + \frac{\widetilde{\beta}}{\widetilde{\alpha}_0}\right)|\zeta_1(y)-\zeta_2(y)| + \frac{\beta}{\widetilde{\alpha}_0^2}(1+\widetilde{\alpha}_1)\,|\xi_1(y-c\tau)-\xi_2(y-c\tau)|\,.$$

Then

$$|F_1(\zeta_1,\xi_1)(y) - F_1(\zeta_2,\xi_2)(y)| \leq \frac{\gamma-\mu-\rho}{D(\lambda_{12}-\lambda_{11})}\left\{\int_{-\infty}^{y} e^{\lambda_{11}(y-x)} + \int_{y}^{+\infty} e^{\lambda_{12}(y-x)}\right\}|\zeta_1(x)-\zeta_2(x)|\,dx$$

$$+\frac{\beta(1+\widetilde{\alpha}_2)}{D\widetilde{\alpha}_0\widetilde{\alpha}_2(\lambda_{12}-\lambda_{11})}\left\{\int_{-\infty}^{y}e^{\lambda_{11}(y-x)}+\int_{y}^{+\infty}e^{\lambda_{12}(y-x)}\right\}|\zeta_1(x)-\zeta_2(x)|\,dx$$
$$+\frac{\widetilde{\beta}\,(1+\widetilde{\alpha}_1)}{D\widetilde{\alpha}_0^2(\lambda_{12}-\lambda_{11})}\left\{\int_{-\infty}^{y}e^{\lambda_{11}(y-x)}+\int_{y}^{+\infty}e^{\lambda_{12}(y-x)}\right\}$$
$$\times|\xi_1(x-c\tau)-\xi_2(x-c\tau)|\,dx$$
$$+\frac{\rho}{D(\lambda_{12}-\lambda_{11})}\left\{\int_{-\infty}^{y}e^{\lambda_{11}(y-x)}+\int_{y}^{+\infty}e^{\lambda_{12}(y-x)}\right\}$$
$$\times\left\{\int_0^{\infty}|\zeta_1(x-cu)-\zeta_2(x-cu)|\,g(u)e^{-\mu u}du\right\}dx.$$

Therefore,

$$|F_1(\zeta_1,\xi_1)(y)-F_1(\zeta_2,\xi_2)(y)|\,e^{-\nu|y|}$$
$$\leq\frac{\gamma-\mu-\rho}{D(\lambda_{12}-\lambda_{11})}\left\{\int_{-\infty}^{y}e^{\lambda_{11}(y-x)}+\int_{y}^{+\infty}e^{\lambda_{12}(y-x)}\right\}$$
$$\times e^{-\nu|y|}e^{\nu|x|}|\zeta_1(x)-\zeta_2(x)|\,e^{-\nu|x|}dx$$
$$+\frac{\beta(1+\widetilde{\alpha}_2)}{D\widetilde{\alpha}_0\widetilde{\alpha}_2(\lambda_{12}-\lambda_{11})}\left\{\int_{-\infty}^{y}e^{\lambda_{11}(y-x)}+\int_{y}^{+\infty}e^{\lambda_{12}(y-x)}\right\}$$
$$\times^{-\nu|y|}e^{\nu|x|}|\zeta_1(x)-\zeta_2(x)|\,e^{-\nu|x|}dx$$
$$+\frac{\bar{\beta}\,(1+\widetilde{\alpha}_1)}{D\widetilde{\alpha}_0^2(\lambda_{12}-\lambda_{11})}\left\{\int_{-\infty}^{\xi}e^{\lambda_{11}(y-x)}+\int_{y}^{+\infty}e^{\lambda_{12}(y-x)}\right\}$$
$$\times e^{-\nu|y|}e^{\nu|x-c\tau|}|\xi_1(x-c\tau)-\xi_2(x-c\tau)|\,e^{-\nu|x-c\tau|}dx$$
$$+\frac{\rho}{D(\lambda_{12}-\lambda_{11})}\left\{\int_{-\infty}^{y}e^{\lambda_{11}(y-x)}+\int_{y}^{+\infty}e^{\lambda_{12}(y-x)}\right\}e^{-\nu|y|}$$
$$\times\left\{\int_0^{\infty}|\zeta_1(x-cu)-\zeta_2(x-cu)|\,e^{-\nu|x-cu|}g(u)e^{\nu|x-cu|}e^{-\mu u}du\right\}dx.$$

We have

$$\left\{\int_{-\infty}^{y}e^{\lambda_{11}(y-x)}+\int_{y}^{+\infty}e^{\lambda_{12}(y-x)}\right\}e^{-\nu|y|}e^{\nu|x|}dx\leq\frac{1}{\lambda_{12}-\nu}-\frac{1}{\lambda_{11}+\nu}.$$

Then

$$|F_1-F_2|_\nu\leq\frac{(\lambda_{11}-\lambda_{12}+2\nu)\left(\gamma-\mu+\frac{\widetilde{\beta}(1+\widetilde{\alpha}_0)}{\widetilde{\alpha}_0\widetilde{\alpha}_2}+\frac{\beta(1+\widetilde{\alpha}_1)}{\widetilde{\alpha}_0^2}e^{\nu c\tau}+\rho(\eta_2-1)\right)}{D(\lambda_{12}-\lambda_{11})(\lambda_{12}-\nu)(\lambda_{11}+\nu)}|\Phi_1-\Phi_2|_\nu,$$

where $\eta_2=\int_0^{+\infty}g(u)e^{-(\mu-c\nu)u}du$.

Hence , $F_1:\Gamma\longrightarrow\Gamma$ is continuous with respect to the norm $|\cdot|_\nu$. similarly, we show that

$F_2 : \Gamma \to \Gamma$ is continuous with respect to the norm $|\cdot|_v$. Therefore we conclude that, F is continuous with respect to the norm $|\cdot|_v$ in $B_v\left(\mathbb{R},\mathbb{R}^2\right)$. ∎

Lemma 3.7. *The operator F is compact with respect to the norm $|\cdot|_v$ in $B_v\left(\mathbb{R},\mathbb{R}^2\right)$.*

Proof. Let $(\zeta,\xi)\in\Gamma$, we have

$$\begin{aligned}\left|\frac{d}{dy}F_1(\zeta,\xi)(y)\right| &\leq \frac{(\gamma+\rho+\mu)\,|\lambda_{11}|}{D\,(\lambda_{12}-\lambda_{11})}\int_{-\infty}^{y} e^{\lambda_{11}(y-x)}dx\\ &+\frac{(\gamma+\rho+\mu)\lambda_{12}}{D\,(\lambda_{12}-\lambda_{11})}\int_{y}^{+\infty} e^{\lambda_{12}(y-x)}dx\\ &=\frac{2(\gamma+\rho+\mu)}{D\,(\lambda_{12}-\lambda_{11})}.\end{aligned}$$

On the other hand, $H_2(\zeta,\xi)(y)\leq(\gamma+\frac{\widetilde{\beta}}{\widetilde{\alpha}_0})\frac{\widetilde{\alpha}_0+\widetilde{\alpha}_1}{\widetilde{\alpha}_2+\widetilde{\alpha}_3}(R_0-1)$. Further, we get similarly

$$\begin{aligned}\left|\frac{d}{d\xi}F_2(\zeta,\psi)(y)\right| &\leq \frac{\left(\gamma+\frac{\widetilde{\beta}}{\widetilde{\alpha}_0}\right)(\widetilde{\alpha}_0+\widetilde{\alpha}_1)\,(R_0-1)\,|\lambda_{21}|}{(\lambda_{22}-\lambda_{21})\,(\widetilde{\alpha}_2+\widetilde{\alpha}_3)}\int_{-\infty}^{y} e^{\lambda_{21}(y-x)}dx\\ &+\frac{\left(\gamma+\frac{\widetilde{\beta}}{\widetilde{\alpha}_0}\right)(\widetilde{\alpha}_0+\widetilde{\alpha}_1)\,(R_0-1)\,\lambda_{22}}{(\lambda_{22}-\lambda_{21})\,(\widetilde{\alpha}_2+\widetilde{\alpha}_3)}\int_{y}^{+\infty} e^{\lambda_{22}(y-x)}dx\\ &=\frac{2\left(\gamma+\frac{\widetilde{\beta}}{\widetilde{\alpha}_0}\right)(\widetilde{\alpha}_0+\widetilde{\alpha}_1)\,(R_0-1)}{(\lambda_{22}-\lambda_{21})\,(\widetilde{\alpha}_2+\widetilde{\alpha}_3)}.\end{aligned}$$

For every $n\in N$, let F^n an operator defined by

$$F^n(\zeta,\xi)(y)=\begin{cases}F(\zeta,\xi)(y), & y\in[-n,n]\\ F(\zeta,\xi)(-n), & y\in(-\infty,-n)\\ F(\zeta,\xi)(n), & y\in(n,+\infty)\end{cases}$$

we can verify that F^n is uniformly bounded and equicontinuous for $(\zeta,\xi)\in\Gamma$. By applying Arzela-Ascoli theorem, we get that $F^n:\Gamma\to\Gamma$ is compact with respect to the super norm in $C\left(\mathbb{R},\mathbb{R}^2\right)$. Thus, $F^n:\Gamma\to\Gamma$ is compact with respect to the norm $|\cdot|_v$ in $B_v\left(\mathbb{R},\mathbb{R}^2\right)$.

Hence, $\{F^n\}_0^{+\infty}$ is a compact series, In addition, we have

$$
\begin{aligned}
|\, F^n(\zeta,\xi)(y) - F(\zeta,\xi)(y)|_v &= \sup_{y\in\mathbb{R}} |F^n(\zeta,\xi)(y) - F(\zeta,\xi)(y)|\, e^{-v|y|} \\
&= \sup_{y\in(-\infty,-n]\cup[n,+\infty)} |F^n(\zeta,\xi)(y) - F(\zeta,\xi)(y)|\, e^{-v|y|} \\
&\leq \sup_{y\in(-\infty,-n]\cup[n,+\infty)} \left[1 + \frac{\widetilde{\alpha}_0+\widetilde{\alpha}_1}{\widetilde{\alpha}_2+\widetilde{\alpha}_3}(R_0-1)\right] e^{-v|y|} \\
&\leq \left[1 + \frac{\widetilde{\alpha}_0+\widetilde{\alpha}_1}{\widetilde{\alpha}_2+\widetilde{\alpha}_3}(R_0-1)\right] e^{-vn} \to 0 \quad \text{as} \quad n\to+\infty.
\end{aligned}
$$

Thus, $\{F^n\}_0^{+\infty}$ converges to F in Γ with respect to the norm $|\cdot|_v$. From Proposition 2.1 in [10], we deduce that $F:\Gamma\to\Gamma$ is also compact with respect to the norm $|\cdot|v$. ■

Theorem 3.8. *For $R_0 > 1$ and $c > c^*$, system (8) admits a traveling wave solution $(\zeta(x+ct),\xi(x+ct))$ connecting the disease-free equilibrium $\widetilde{E}^0(1,0)$ and the endemic equilibrium $\widetilde{E}^*\left(\frac{S^*}{\varsigma},\frac{I^*}{\varsigma}\right)$. Moreover, $\lim_{y\to-\infty}\xi(y)e^{-\lambda_1 y}=1$.*

Proof. From Lemma 3.6 and 3.7, we conclude that the operator F satisfies the conditions of Schauder fixed point theorem. Then F admits a fixed point $(\zeta,\xi)\in\Gamma$. We verify immediately that this fixed point is a solution of (9) and (10).

On the other hand, we verify that the fixed point (ζ,ξ) satisfies the boundary conditions (11). Since $(\zeta,\xi)\in\Gamma$, we have $1-\frac{1}{\sigma}e^{\sigma y}\leq\zeta(y)\leq 1$ and $0\leq\xi(y)\leq e^{\lambda_1 y}$. Then $\zeta(-\infty)=1$, $\xi(-\infty)=0$.

Is is not hard to prove that the endemic equilibrium is globally asymptotically stable when $R_0>1$. Then $\zeta(+\infty)=\frac{S^*}{\varsigma}$ and $\xi(+\infty)=\frac{I^*}{\varsigma}$.

Since $\xi\in\Gamma$, we have $1-\phi e^{\omega y}\leq\xi(y)e^{-\lambda_1 y}\leq 1$, and we get, $\lim_{y\to-\infty}\xi(y)e^{-\lambda_1 y}=1$. ■

Corollary 3.9. *For $R_0 > 1$ and $c > c^*$, system (1) admits a traveling wave solution connecting the disease-free equilibrium $Q_f(\frac{A}{\mu+\rho-\rho\eta},0,\frac{A\rho(1-\eta)}{\mu(\mu+\rho-\rho\eta)})$ and the endemic equilibrium $Q^*(S^*,I^*,R^*)$.*

Proof. We have to prove that $R(-\infty)=\frac{A\rho(1-\eta)}{\mu(\mu+\rho-\rho\eta)}$ and $R(+\infty)=R^*$.

By the wave equation of $R(y)$, we get

$$
cR'(y) = D_R R''(y) + rI(y) - \mu R(y) - \rho\int_0^\infty S(y-cu)g(u)e^{-\mu u}du + \rho S(y).
$$

Let $\omega_1 < 0 < \omega_2$ be the solutions of the characteristic equation $D_R\omega^2 - c\omega - \mu = 0$.

Then

$$\begin{aligned} R(y) &= \frac{r}{D_R(\omega_2-\omega_1)}\left(\int_{-\infty}^{y} e^{\omega_1(y-x)}I(x)dx + \int_{y}^{+\infty} e^{\omega_2(y-x)}I(x)dx\right) \\ &\quad -\frac{\rho}{D_R(\omega_2-\omega_1)}\left(\int_{-\infty}^{y} e^{\omega_1(y-x)}J(x)dx + \int_{y}^{+\infty} e^{\omega_2(y-x)}J(x)dx\right) \\ &\quad +\frac{\rho}{D_R(\omega_2-\omega_1)}\left(\int_{-\infty}^{y} e^{\omega_1(y-x)}S(x)dx + \int_{y}^{+\infty} e^{\omega_2(y-x)}S(x)dx\right), \end{aligned}$$

with $J(x) = \int_0^\infty S(x-cu)g(u)e^{-\mu u}du$

From Hopital rule, we obtain

$$\frac{r}{D_R(\omega_2-\omega_1)}\lim_{y\to+\infty}\left(\frac{\int_{-\infty}^{y} e^{-\omega_1 x}I(x)dx}{e^{-\omega_1 y}} + \frac{\int_{y}^{+\infty} e^{-\omega_2 x}I(x)dx}{e^{-\omega_2 \xi}}\right) = \frac{rI^*}{\mu},$$

$$\frac{\rho}{D_R(\omega_2-\omega_1)}\lim_{y\to+\infty}\left(\frac{\int_{-\infty}^{y} e^{-\omega_1 x}S(x)dx}{e^{-\omega_1 y}} + \frac{\int_{y}^{+\infty} e^{-\omega_2 x}S(x)dx}{e^{-\omega_2 y}}\right) = \frac{\rho S^*}{\mu},$$

and

$$\frac{\rho}{D_R(\omega_2-\omega_1)}\lim_{y\to+\infty}\left(\frac{\int_{-\infty}^{y} e^{-\omega_1 x}J(x)dx}{e^{-\omega_1 y}} + \frac{\int_{y}^{+\infty} e^{-\omega_2 x}J(x)dx}{e^{-\omega_2 y}}\right) = \frac{\rho\eta S^*}{\mu},$$

Hence, $R(+\infty) = R^* = \frac{rI^*+\rho(1-\eta)S^*}{\mu}$.

Similarly, we get $R(-\infty) = \frac{A\rho(1-\eta)}{\mu(\mu+\rho-\rho\eta)}$. ∎

4. Nonexistence of Traveling Waves

In this section, we study the nonexistence of nontrivial traveling wave solutions of system (1), connecting the disease-free equilibrium and the endemic equilibrium.

Theorem 4.1. *If $R_0 > 1$ and $c \in (0, c^*)$, or $R_0 < 1$, then system (9)-(10) with the boundary conditions (11) has no nontrivial positive solution.*

Proof. First, for $R_0 > 1$ and $c \in (0, c^*)$, we suppose that problem (9)-(10) admits a positive solution (ζ, ξ). By the result of Lemma 3.1, there exists a $\varepsilon > 0$ sufficiently small such that the equation

$$\lambda^2 - \bar{c}\lambda + \frac{\bar{\beta}}{\bar{\alpha}_0+\bar{\alpha}_1}(1-2\varepsilon)e^{-\lambda\bar{c}r} - (\mu+d+r) = 0,$$

has no real solution for $\bar{c} \in \left(0, \frac{c+2c^*}{3}\right)$.

Since $\zeta(-\infty) = 1$, then there exists $N_\varepsilon > 0$ such that $1-\varepsilon \leq \zeta(y) \leq 1$, for any $y < -N_\varepsilon$. Therefore, for $y < -N_\varepsilon$, we get

$$\begin{aligned} c\xi'(y) &\geq \xi''(y) + \frac{\widetilde{\beta}(1-\varepsilon)\xi(y-c\tau)}{\widetilde{\alpha}_0 + \widetilde{\alpha}_1 + (\widetilde{\alpha}_2 + \widetilde{\alpha}_3)\,\xi(y-c\tau)} - (\mu+d+r)\xi(y) \\ &\geq \xi''(y) + \frac{\tilde{\beta}(1-\varepsilon)\xi(y-c\tau)}{[\widetilde{\alpha}_0 + \widetilde{\alpha}_1 + (\widetilde{\alpha}_2 + \widetilde{\alpha}_3)\,\xi(y-c\tau)]^{h+1}} - (\mu+d+r)\xi(y), \quad \forall h > 0. \end{aligned}$$

Hence,

$$c\xi'(y) \geq \xi''(y) + f(\xi(y-cr)) - (\mu+d+r)\xi(y),$$

with

$$f(u) = \inf_{v \in \left(u, \frac{\widetilde{\alpha}_0+\widetilde{\alpha}_1}{\widetilde{\alpha}_2+\widetilde{\alpha}_3}(R_0-1)\right)} \frac{\widetilde{\beta}(1-\varepsilon)v}{[\widetilde{\alpha}_0 + \widetilde{\alpha}_1 + (\widetilde{\alpha}_2 + \widetilde{\alpha}_3)\,v]^{h+1}}.$$

Hence, the positive function $u(x,t) = \xi(y+ct)$ satisfies

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} & \geq \frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t-\tau)) - (\mu+d+r)u(x,t), x \in \mathbb{R}, t > 0, \\ u(x,s) & = \psi(x+cs) > 0, x \in \mathbb{R}, s \in [-\tau, 0]. \end{cases}$$

By the comparison principal theorem [11], $u(x,t)$ is an upper solution of the following initial value problem

$$\begin{cases} \frac{\partial w(x,t)}{\partial t} = \frac{\partial^2 w(x,t)}{\partial x^2} + f(w(x,t-\tau)) - (\mu+d+r)w(x,t), x \in \mathbb{R}, t > 0, \\ w(x,s) = \xi(x+cs) > 0, x \in \mathbb{R}, s \in [-\tau, 0]. \end{cases}$$

By the theory of asymptotic spreading [12], we get

$$\liminf_{t \to +\infty} w(x,t) > 0, \quad |x| \leq \frac{c+c^*}{2}t. \tag{28}$$

Hence,

$$\liminf_{t \to +\infty} u(x,t) \geq \liminf_{t \to +\infty} w(x,t) > 0, |x| \leq \frac{c+c^*}{2}t.$$

If $-x = \frac{c+c^*}{2}t$, then $x+ct \to -\infty$ as $t \to +\infty$, and

$$\lim_{t \to +\infty} u(x,t) = 0, -x = \frac{c+c^*}{2}t,$$

which contradicts (28). Then, system (9)-(10) has no nontrivial positive solution satisfying (11) when $R_0 > 1$ and $c \in (0, c^*)$.

On the other hand, for any speed $c > 0$, let $R_0 < 1$. The wave equation of $\xi(y)$ is given by

$$c\xi'(y) = \xi''(y) + \tilde{h}(\zeta(y), \xi(y - c\tau))\xi(y - c\tau) - (\mu + d + r)\xi(y)$$

Let $\lambda'_{21} < 0 < \lambda'_{22}$ such that $\lambda^2 - c\lambda - (\mu + d + r) = 0$. Hence, we have

$$\begin{aligned}
\xi(y) = &\frac{1}{(\lambda'_{22} - \lambda'_{21})}\left(\int_{-\infty}^{y} e^{\lambda'_{21}(y-x)}\widetilde{h}(\zeta(x), \xi(x - c\tau))\xi(x - c\tau)dx\right. \\
&\left. + \int_{y}^{+\infty} e^{\lambda'_{22}(y-x)}\widetilde{h}(\zeta(x), \xi(x - c\tau))\xi(x - c\tau)dx\right) \\
= &\frac{1}{(\lambda'_{22} - \lambda'_{21})}\left(\int_{0}^{+\infty} e^{\lambda'_{21}x}\widetilde{h}(\zeta(y - x), \xi(y - x - c\tau))\xi(y - x - c\tau)dx\right. \\
&\left. + \int_{-\infty}^{0} e^{\lambda'_{22}x}\widetilde{h}(\zeta(y - x), \xi(y - x - c\tau))\xi(y - x - c\tau)dx\right).
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} \xi(y)dy &= \frac{1}{\mu + d + r}\int_{-\infty}^{+\infty} \tilde{h}(\zeta(x), \xi(x - c\tau))\xi(x - c\tau)dy \\
&\leq \frac{\widetilde{\beta}}{(\widetilde{\alpha}_0 + \widetilde{\alpha}_1)(\mu + d + r)}\int_{-\infty}^{+\infty} \xi(y - c\tau)dy \\
&= R_0\int_{-\infty}^{+\infty} \xi(y)dy \\
&< \int_{-\infty}^{+\infty} \xi(y)dy.
\end{aligned}$$

This is a contradiction. Hence, we conclude that there is no nontrivial traveling wave when $R_0 > 1$ and $c \in (0, c^*)$, or $R_0 < 1$. ■

According to the above theorem, we deduce immediately the following result.

Corollary 4.2. *If $R_0 > 1$ and $c \in (0, c^*)$, or $R_0 < 1$, then system (1) has no nontrivial positive traveling wave solution connecting the disease-free equilibrium $Q_f(\frac{A}{\mu+\rho-\rho\eta}, 0, \frac{A\rho(1-\eta)}{\mu(\mu+\rho-\rho\eta)})$ and the endemic equilibrium $Q^*(S^*, I^*, R^*)$.*

5. Conclusion

In this work, we have proposed an epidemic model with diffusion, Hattaf-Yousfi incidence rate and temporary immunity acquired by vaccination. The proposed model contains two delays one is discrete representing the latent period and the other is infinite distributed delay modeling immunity period. We first determined the basic reproduction number and steady states of the proposed model. The existence of traveling waves describing the transition of disease-free equilibrium to endemic equilibrium has been established by means of Schauder fixed point theory. In addition, we have studied the nonexistence of nontrivial traveling wave solution of the proposed model.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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