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SOME FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS VIA ISHIKAWA-TYPE ITERATIONS IN CAT(0) SPACES

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Abstract. In this manuscript, we establish strong convergence theorems for the Ishikawa iteration scheme involving quasi-nonexpansive multi-valued maps in the setting of CAT(0) spaces. Our results extend and improve some related results in the literature.

Keywords: quasi-nonexpansive multimap; multi-valued nonexpansive map; strong convergence; Ishikawa iterations; Banach space; CAT(0) space.

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1. INTRODUCTION

The fixed point theory for multi-valued mappings is one of the most important subject of set-valued analysis. In 1969, the stipulation of Banach in single-valued mappings was modified to multi-valued mappings by Nadler [21]. In Banach spaces, several well-known fixed point theorems of single-valued mappings such as Banach and Schauder have been extended to multi-valued mappings. Recently, many brilliant fixed point results of different multi-valued mappings have been studied in variety of settings (see e.g.[1, 9, 13, 23, 25, 28, 30, 33]). There

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are a lot of applications for multi-valued mappings such as optimal control theory, differential inclusions, game theory, and many branches in physics.

Let $X := (X, \|\,.\,\|)$ be a Banach space and $\mathscr{U}$ be a nonempty convex subset of X. The set $\mathscr{U}$ is called **proximinal** if for each $x \in X$, there exists $y \in \mathscr{U}$ such that $\|x-y\| = d(x,\mathscr{U})$, where $d(x,\mathscr{U}) = \inf\{\|x-z\| : z \in \mathscr{U}\}$. Throughout this paper, the simbols $\mathbb{CB}(\mathscr{U})$ and $\mathbb{P}(\mathscr{U})$ refer to the family of nonempty closed bounded subsets and nonempty proximinal bounded subsets of $\mathscr{U}$ respectively. For any $\mathscr{A},\mathscr{B} \in \mathbb{CB}(\mathscr{U})$, define the metric $\mathbb{H}_d : \mathbb{CB}(\mathscr{U}) \times \mathbb{CB}(\mathscr{U}) \longrightarrow \mathbb{R}^+$ by

$$\mathbb{H}_d(\mathscr{A},\mathscr{B}) = \max\Big\{\sup_{x\in\mathscr{A}} d(x,\mathscr{B}), \sup_{y\in\mathscr{B}} d(y,\mathscr{A})\Big\}.$$

We call such $\mathbb{H}_d$ the Hausdorff metric on $\mathbb{CB}(\mathscr{U})$. Here, let $\mathbb{R}^+ = [0,\infty)$.

Definition 1.1. Let $\mathscr{T} : \mathscr{U} \longrightarrow \mathscr{U}$ be single-valued mapping. Then $\mathscr{T}$ is called to be nonexpansive, if $\|\mathscr{T}(x) - \mathscr{T}(y)\| \leq \|x-y\|$ for $x,y \in \mathscr{U}$.

Definition 1.2. Let $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ be a multi-valued mapping. Then $\mathbb{T}$ is called to be nonexpansive, if $\mathbb{H}_d(\mathbb{T}(x),\mathbb{T}(y)) \leq \|x-y\|$ for all $x,y \in \mathscr{U}$.

Definition 1.3. [29] Let $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ be a multi-valued mapping. Then $\mathbb{T}$ is called to be quasi-nonexpansive, if $F(\mathbb{T}) \neq \emptyset$ and $\mathbb{H}_d(\mathbb{T}(x), p) \leq \|x-p\|$ for all $x \in \mathscr{U}$ and all $p \in F(\mathbb{T})$, where $F(\mathbb{T})$= the set of fixed point of multi-valued map $\mathbb{T}$.

Let $\mathscr{T} : \mathscr{U} \longrightarrow \mathscr{U}$ be a single-valued mapping, an element $p \in \mathscr{U}$ is called a fixed point of $\mathscr{T}$ if $p = \mathscr{T}(p)$. The set of fixed points of $\mathscr{T}$ is denoted by $F(\mathscr{T})$. An element $p \in \mathscr{U}$ is called a fixed poin the multi-valued mapping $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ if $p \in F(\mathbb{T})$ and $F(\mathbb{T}) \neq \emptyset$. It is clear that every nonexpansive multi-valued map $\mathbb{T}$ with $F(\mathbb{T}) \neq \emptyset$ is quasi-nonexpansive. But the converse is not true. The following example shows that there is a quasi-nonexpansive multimap which is not a nonexpansive multimap.

Example 1.4. (see [28]) Let $\mathscr{U} = [0,\infty)$ with the usual metric and $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ be defined by

$$\mathbb{T}x = \begin{cases} \{0\}, & \text{if } x \leq 1 \\ [x-\frac{3}{4}, x-\frac{1}{3}], & \text{if } x \geq 1 \end{cases}$$

Then $\mathbb{T}$ is a quasi-nonexpansive multi-valued map but not a nonexpansive multi-valued map.

The mapping $\mathbb{T}:\mathscr{U}\longrightarrow\mathbb{CB}(\mathscr{U})$ is called hemicompact if, for any sequence $\{x_n\}$ in $\mathscr{U}$ such that $d(x_n,\mathbb{T}x_n)\longrightarrow 0$ as $n\longrightarrow\infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\longrightarrow p\in\mathscr{U}$. We note that if $\mathscr{U}$ is compact, then every multi-valued mapping $\mathbb{T}:\mathscr{U}\longrightarrow\mathbb{CB}(\mathscr{U})$ is hemicompact.

The mapping $\mathbb{T}:\mathscr{U}\longrightarrow\mathbb{CB}(\mathscr{U})$ is said to satisfy Condition (I) if there is a nondecreasing function $h:[0,\infty)\longrightarrow[0,\infty)$ with $h(0)=0, h(r)>0$ for $r\in(0,\infty)$ such that

$$h(d(x,F(\mathbb{T})))\leq d(x,\mathbb{T}x) \text{ for all } x\in\mathscr{U}.$$

We recall the following definitions.

Definition 1.5. (Mann [20]) Let $\mathscr{T}:\mathscr{U}\longrightarrow\mathscr{U}$ be a single-valued mapping. The Mann iteration scheme, starting from $x_0\in\mathscr{U}$, is the sequence $\{x_n\}$ defined by

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n\mathscr{T}x_n, \alpha_n\in[0,1], n\geq 0,$$

where α_n satisfies certain conditions.

Definition 1.6. (Ishikawa [14]) Let $\mathscr{T}:\mathscr{U}\longrightarrow\mathscr{U}$ be a single-valued mapping. The Ishikawa iteration scheme, starting from $x_0\in\mathscr{U}$, is the sequence $\{x_n\}$ defined by

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n\mathscr{T}y_n, \alpha_n\in[0,1],$$
$$y_n=(1-\beta_n)x_n+\beta_n\mathscr{T}x_n, \beta_n\in[0,1], n\geq 0,$$

where α_n and β_n satisfy certain conditions.

Iterative techniques for approximating fixed points of nonexpansive single-valued mappings have been investigated by various authors (see e.g., [2, 15, 24, 27, 29, 32]) using the Mann iteration scheme or the Ishikawa iteration scheme. For details on the subject, we refer the reader to Berinde [3].

Sastry and Babu [25] defined the Mann and Ishikawa iteration schemes for multi-valued mappings as follows:

Definition 1.7. [25] Let $\mathbb{T}:\mathscr{U}\longrightarrow\mathbb{P}(\mathscr{U})$ a multi-valued mapping and $p\in F(\mathbb{T})$.

(i). The sequence of Mann iterates is defined by $x_0 \in \mathscr{U}$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \alpha_n \in [0,1], n \geq 0, \tag{1.1}$$

where $y_n \in \mathbb{T}x_n$ suchthat $\|y_n - p\| = d(p, \mathbb{T}x_n)$.

(ii). The sequence of Ishikawa iterates is defined by $x_0 \in \mathscr{U}$,

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n v_n, \alpha_n \in [0,1], \\ y_n &= (1-\beta_n)x_n + \beta_n w_n, \beta_n \in [0,1], n \geq 0, \end{aligned} \tag{1.2}$$

where $v_n \in \mathbb{T}y_n$ such that $\|v_n - p\| = d(p, \mathbb{T}y_n)$, and $w_n \in \mathbb{T}x_n$ such that $\|w_n - p\| = d(p, \mathbb{T}x_n)$.

Sastry and Babu [25] showed that the Mann and Ishikawa iteration schemes for a multi-valued mapping $\mathbb{T}$ with a fixed point p converge to a fixed point q of $\mathbb{T}$ under certain conditions. They claimed that the fixed point q may be different from p. They obtained a result for nonexpansive multi-valued map with compact domain. For more details, we refer readers to [25].

Recently, Panyanak [23] extended the result of Sastry and Babu [25] to uniformly convex Banach spaces and the domain of $\mathbb{T}$ remains compact. Panyanak [23] also modified the iteration schemes of Sastry and Babu [25] as follows:-

Definition 1.8. [23] Let $\mathbb{T}: \mathscr{U} \longrightarrow \mathbb{P}(\mathscr{U})$ be a multi-valued mapping and $F(\mathbb{T})$ be a nonempty proximinal subset of $\mathscr{U}$. The sequence of Mann iterates is defined by $x_0 \in \mathscr{U}$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \alpha_n \in [a,b], 0 < a < b < 1, n \geq 0, \tag{1.3}$$

where $y_n \in \mathbb{T}x_n$ such that $\|y_n - u_n\| = d(u_n, \mathbb{T}x_n)$, and $u_n \in F(\mathbb{T})$ such that $\|x_n - u_n\| = d(x_n, F(\mathbb{T}))$.

Definition 1.9. [23] Let $\mathbb{T}: \mathscr{U} \longrightarrow \mathbb{P}(\mathscr{U})$ be a multi-valued mapping and $F(\mathbb{T})$ be a nonempty proximinal subset of $\mathscr{U}$. The sequence of Ishikawa iterates is defined by $x_0 \in \mathscr{U}$,

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n v_n, \alpha_n \in [a,b], n \geq 0, \\ y_n &= (1-\beta_n)x_n + \beta_n w_n, \beta_n \in [a,b], 0 < a < b < 1, n \geq 0, \end{aligned} \tag{1.4}$$

where $v_n \in \mathbb{T}y_n$ such that $\|v_n - v_n'\| = d(v_n', \mathbb{T}y_n)$, and $v_n' \in F(\mathbb{T})$ such that $\|y_n - v_n'\| = d(y_n, F(\mathbb{T}))$, and $w_n \in \mathbb{T}x_n$ such that $\|w_n - v_n''\| = d(v_n'', \mathbb{T}x_n)$, and $v_n'' \in F(\mathbb{T})$ such that $\|x_n - v_n''\| = d(x_n, F(\mathbb{T}))$.

Moreover, Panyanak [23] proved the following result for for a nonexpansive multi-valued map.

Theorem 1.10. *([23], Theorem 3.8). Let X be a uniformly convex Banach space, $\mathscr{U}$ a nonempty closed bounded convex subset of X, and $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{P}(\mathscr{U})$ a nonexpansive multi-valued map that satisfies* **Condition (I)**. *Assume that (i) $0 \leq \alpha_n < 1$ and (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $F(\mathbb{T})$ is a nonempty proximinal subset of $\mathscr{U}$.Then the Mann iterates $\{x_n\}$ defined by (1.3) converges to a fixed point of $\mathbb{T}$.*

Remark 1.11. Panyanak [23] posted a question whether Theorem 1.10 is true or not for the **Ishikawa** iterates generated by the iterate (1.4).

Later, Song and Wang [31] found that there was a gap in the proofs of Theorem 3.1 in [23] and Theorem 5 in [25], because the iteration x_n depends on a fixed $p \in F(\mathbb{T})$ as well as $\mathbb{T}$. If $q \in F(\mathbb{T})$ and $q \neq p$, then the iteration x_n defined by q is different from the one defined by p. So the conclusion of Theorems 3.1 in [23] and Theorem 5 in [25], are ambiguous. They further solved the gap and also gave the affirmative answer to the above question using the following Ishikawa iteration scheme. They introduced the following definition.

Definition 1.12. [31] Let $\mathscr{U}$ be a nonempty convex subset of a Banach space X, $\alpha_n, \beta_n \in [0,1]$ and $\gamma_n \in (0, \infty)$ such that $\lim_{n \longrightarrow \infty} \gamma_n = 0$. The sequence $\{x_n\}$ by, starting from $x_0 \in \mathscr{U}$,

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n v_n, \\ y_n &= (1-\beta_n)x_n + \beta_n w_n, \end{aligned} \tag{1.5}$$

where $\|w_n - v_n\| \leq \mathbb{H}_d(\mathbb{T}x_n, \mathbb{T}y_n) + \gamma_n$ and $\|w_{n+1} - v_n\| \leq \mathbb{H}_d(\mathbb{T}x_{n+1}, \mathbb{T}y_n) + \gamma_n$ for $w_n \in \mathbb{T}x_n$ and $v_n \in \mathbb{T}y_n$.

Song and Wang [31] obtained the following results where the domain of $\mathbb{T}$ is compact, which is a strong condition.

Theorem 1.13. *([31], Theorem 1). Let X be a uniformly convex Banach space, $\mathscr{U}$ a nonempty compact convex subset of X, and $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ a nonexpansive multi-valued mapping with $F(\mathbb{T}) \neq \emptyset$ satisfying $\mathbb{T}(p) = \{p\}$ for any $p \in F(\mathbb{T})$. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \longrightarrow 0$ and (iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$. Then the Ishikawa iterates $\{x_n\}$ generated by (1.5) converges to a fixed point of $\mathbb{T}$.*

Remark 1.14. For more other results by Song and Wang, we refer readers to [31].

Later, in 2009, Shahzad and Zegeye [28] extended and improved the results of Panyanak [23], Sastry and Babu [25] and Song and Wang [31] from nonexpansive multi-valued maps to quasi-nonexpansive multi-valued mappings in Banach spaces. Their results provided an affirmative answer to Panyanak's question [23] in a more general setting.

Shahzad and Zegeye [28] used the following lemma of Xu [33] in the proofs of their main results.

Lemma 1.15. [33] *Let $(X, \|\cdot\|)$ a Banach space and $c > 1$ be a fixed number. Then X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $h : [0,\infty) \longrightarrow [0,\infty)$ with $h(0) = 0$ such that*

$$\|ax + (1-a)y\|^2 \leq a\|x\|^2 + (1-a)\|y\|^2 - a(1-a)h(\|x-y\|)$$

for all $x, y \in B_c(0) = \{x \in X : \|x\| \leq c\}$, and $a \in [0,1]$.

More precisely, they obtained the following results.

Theorem 1.16. [28] *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, and $\mathscr{U}$ be a nonempty closed convex subset of X. Let $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ be a quasi-nonexpansive multi-valued mapping with $F(\mathbb{T}) \neq \emptyset$ and $\mathbb{T}(p) = \{p\}$ for each $p \in F(\mathbb{T})$. Let $\{x_n\}$ be the Ishikawa iterates defined by $x_0 \in \mathscr{U}$,*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n v_n, \, n \geq 0 \\ y_n &= (1-\beta_n)x_n + \beta_n w_n, \, n \geq 0 \end{aligned} \tag{1.6}$$

where $w_n \in \mathbb{T}x_n$ and $v_n \in \mathbb{T}y_n$. Assume that $\mathbb{T}$ satisfies Condition (I) and $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. Then $\{x_n\}$ converges strongly to a fixed point of $\mathbb{T}$.

Theorem 1.17. [28] *Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space,* $\mathscr{U}$ *a nonempty closed convex subset of* X *, and* $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ *a quasi-nonexpansive multi-valued map with* $F(\mathbb{T}) \neq \emptyset$ *and for which* $\mathbb{T}(p) = \{p\}$ *for each* $p \in F(\mathbb{T})$*. Let* $\{x_n\}$ *be the Ishikawa iterates generated by* (1.6)*. Assume that* $\mathbb{T}$ *is hemicompact and continuous, and (i)* $0 \leq \alpha_n, \beta_n < 1$*, (ii)* $\beta_n \longrightarrow 0$*, and (iii)* $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$*. Then the sequence* $\{x_n\}$ *converges strongly to a fixed point of* $\mathbb{T}$*.*

Theorem 1.18. [28] *Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space,* $\mathscr{U}$ *a nonempty closed convex subset of* X *, and* $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{P}(\mathscr{U})$ *a multi-valued map with* $F(\mathbb{T}) \neq \emptyset$*,* $P_{\mathbb{T}}(x) = \{y \in \mathbb{T}x : \|x - y\| = d(x, \mathbb{T}x)\}$ *and such that* $P_{\mathbb{T}}$ *is nonexpansive. Let* $\{x_n\}$ *be the Ishikawai iterates defined by* $x_0 \in \mathscr{U}$*,*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n v_n, \; n \geq 0 \tag{1.7}$$
$$y_n = (1 - \beta_n)x_n + \beta_n w_n, \; n \geq 0$$

where $w_n \in P_{\mathbb{T}}(x_n)$ *and* $v_n \in P_{\mathbb{T}}(y_n)$*. Assume that* $\mathbb{T}$ *satisfies condition (I) and* $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$*. Then* $\{x_n\}$ *converges strongly to a fixed point of* $\mathbb{T}$*.*

Remark 1.19. If $\mathbb{T} = \mathscr{T}$ is single-valued the above iteration scheme (1.7) reduces to the well-known Ishikawa iteration scheme (see [14]).

2. Preliminaries

In this section, we present some basic facts of CAT(0) spaces and hyperbolic spaces.

2.1. CAT(0) spaces. Let $(\mathbb{X}, d)$ be a metric space. A geodesic path joining $x \in \mathbb{X}$ to $y \in \mathbb{X}$ (or, more briefly, a geodesic from x to y) is a map $\omega : [0, l] \longrightarrow X, [0, l] \subset R$ such that $\omega(0) = x, \omega(l) = y$, and $d(\omega(t), \omega(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, ω is an isometry and $d(x, y) = l$. The image α of ω is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of $\mathbb{X}$ are joined by a geodesic, and $\mathbb{X}$ is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in \mathbb{X}$. A subset $\mathbb{Y} \subseteq \mathbb{X}$ is said to be convex if $\mathbb{Y}$ includes every geodesic segment joining any two of its points. A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space $(\mathbb{X}, d)$ consists of three points x_1, x_2, x_3 in $\mathbb{X}$ (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1,x_2,x_3)$ in $(\mathbb{X},d)$ is a triangle $\bar{\Delta}(x_1,x_2,x_3) := \Delta(\bar{x_1},\bar{x_2},\bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i},\bar{x_j}) = d(x_i,x_j)$ for $i,j \in 1,2,3$.

Definition 2.1. (CAT(0) space) Let $(\mathbb{X},d)$ be a geodesic space. It is a CAT(0) space if for any geodesic triangle $\Delta \subset \mathbb{X}$ and $x,y \in \Delta$ we have $d(x,y) \leq d(\bar{x},\bar{y})$ where $\bar{x},\bar{y} \in \bar{\Delta}$.

It is well-known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples of CAT(0) spaces include Pre-Hilbert spaces, R-trees (see [4]), Euclidean buildings (see [6]), the complex Hilbert ball with a hyperbolic metric (see [12]), and many others.

Definition 2.2. A geodesic triangle $\Delta(p,q,r)$ in $(\mathbb{X},d)$ is said to satisfy the CAT(0) *inequality* if for any $u,v \in \Delta(p,q,r)$ and for their comparison points $\bar{u},\bar{v} \in \bar{\Delta}(\bar{p},\bar{q},\bar{r})$, one has

$$d(u,v) \leq d_{\mathbb{R}^2}(\bar{u},\bar{v}).$$

For other equivalent definitions and basic properties of CAT(0) spaces, we refer the readers to standard texts such as [4]. It is well-known that every CAT(0) space is uniquely geodesic.

Note that if x,y_1,y_2 are points of CAT(0) space and if y_0 is the midpoint of the segment $[y_1,y_2]$ (we write $y_0 = \frac{1}{2}y_1 \bigoplus \frac{1}{2}y_2$), then the CAT$(0)$ inequality implies

$$d(x,y_0)^2 = d(x,\frac{1}{2}y_1 \bigoplus \frac{1}{2}y_2) \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2 \tag{2.1}$$

This inequality is known as the CN inequality of Bruhat and Tits [5]. Some brilliant known results in CAT(0) spaces can be found in [1, 9, 10] and references therein.

2.2. **Hyperbolic Spaces.**

In this section we recall some notions of the hyperbolic metric spaces. This class of spaces contains the class of CAT(0) spaces.

Definition 2.3. (See [19]) A hyperbolic space is a triple $(\mathbb{X},d,\mathbb{W})$ where $(\mathbb{X},d)$ is a metric space and $\mathbb{W} : \mathbb{X}\times\mathbb{X}\times[0,1] \longrightarrow \mathbb{X}$ is such that

W1. $d(z,\mathbb{W}(x,y,\alpha)) \leq (1-\alpha)d(z,x)+\alpha d(z,y)$,

W2. $d(\mathbb{W}(x,y,\alpha),\mathbb{W}(x,y,\beta)) = |\alpha-\beta|d(x,y)$,

W3. $\mathbb{W}(x,y,\alpha) = \mathbb{W}(y,x,(1-\alpha))$, W4. $d(\mathbb{W}(x,z,\alpha),\mathbb{W}(y,w,\alpha)) \leq (1-\alpha)d(x,y)+\alpha d(z,w)$ for all $x,y,z,w \in X, \alpha,\beta \in [0,1]$.

It follows from (W1.) that, for each $x,y \in \mathbb{X}$ and $\alpha \in [0,1]$,

$$d(x,\mathbb{W}(x,y,\alpha)) \leq \alpha d(x,y),\ d(y,\mathbb{W}(x,y,\alpha)) \leq (1-\alpha)d(x,y) \tag{2.2}$$

In fact, we have that (see [22])

$$d(x,\mathbb{W}(x,y,\alpha)) = \alpha d(x,y),\ d(y,\mathbb{W}(x,y,\alpha)) = (1-\alpha)d(x,y) \tag{2.3}$$

Comparing (2.3) to (2.2), we can also use the notation $(1-\alpha)x \bigoplus \alpha y$ for $\mathbb{W}(x,y,\alpha)$ in a hyperbolic space $(\mathbb{X},d,\mathbb{W})$.

An example of hyperbolic spaces is the family of Banach vector spaces or any normed vector spaces. Hadamard manifolds [6], the Hilbert open unit ball equipped with the hyperbolic metric [12], and the CAT(0) spaces [4, 10, 17, 18, 19] are examples of nonlinear structures which play a major role in recent research in metric fixed point theory. A subset $\mathscr{U}$ of a hyperbolic space $\mathbb{X}$ is said to be convex if $[x,y] \subset \mathscr{U}$, whenever $x,y \in \mathscr{U}$ (see also [30]).

Lemma 2.4. [19] *The hyperbolic space $(\mathbb{X},d,\mathbb{W})$ is called uniformly convex if for any $r>0$ and $\varepsilon \in (0,2]$ there exists a $\delta \in (0,1]$such that, for all $a,x,y \in \mathbb{X}$,*

$$\left.\begin{array}{l} d(x,a) \leq r \\ d(y,a) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \quad \Rightarrow \quad d(\frac{1}{2}x \bigoplus \frac{1}{2}y, a) \leq (1-\delta)r. \tag{2.4}$$

A mapping $\eta : (0,\infty)\times(0,2] \longrightarrow (0,1]$ providing such a $\delta := \eta(r,\varepsilon)$ for given $r>0$ and $\varepsilon \in (0,2]$ is called a modulus of uniform convexity.

Remark 2.5. Note that if $(\mathbb{X},d,\mathbb{W})$ is a hyperbolic space, then from (W4) we have

$$d\big(\alpha p \bigoplus (1-\alpha)x, \alpha q \bigoplus (1-\alpha)y\big) \leq \alpha d(p,q) + (1-\alpha)d(x,y) \tag{2.5}$$

for all $p,q,x,y \in \mathbb{X}$ and $\alpha \in [0,1]$ (See [11, 16]).

The following lemmas are useful.

Lemma 2.6. [10] *Let $(\mathbb{X},d)$ be a CAT(0) space.*

(i) For $x,y \in \mathbb{X}$ and $\alpha \in [0,1]$, there exists a unique point $z \in [x,y]$ such that

$$d(x,z) = \alpha d(x,y) \text{ and } d(y,z) = (1-\alpha)d(x,y). \tag{2.6}$$

We use the notation $(1-\alpha)x \bigoplus \alpha y$ for the unique point z satisfying (2.6).

(ii) For $x,y,z \in \mathbb{X}$ and $\alpha \in [0,1]$, we have

$$d\big((1-\alpha)x \bigoplus \alpha y, z\big) \leq (1-\alpha)d(x,z) + \alpha d(y,z).$$

(iii) For $x,y,z \in \mathbb{X}$ and $\alpha \in [0,1]$ we have

$$d\big((1-\alpha)x \bigoplus \alpha y, z\big)^2 \leq (1-\alpha)d(x,z)^2 + \alpha d(y,z)^2 - \alpha(1-\alpha)d(x,y)^2.$$

Lemma 2.7. [25] *Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences such that*

(i). $0 \leq \alpha_n, \beta_n < 1$,

(ii). $\beta_n \longrightarrow 0$ and $\sum \alpha_n \beta_n = \infty$.

Let $\{\gamma_n\}$ be nonnegative real sequence such that $\sum \alpha_n \beta_n (1-\beta_n)\gamma_n$ is bounded. Then $\{\gamma_n\}$ has a subsequence which converges to zero.

Lemma 2.8. [32] *Let $\{\lambda_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real numbers suchthat $\lambda_{n+1} \leq \lambda_n + \sigma_n$, $\forall\, n \geq 1$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then $\lim\limits_{n \longrightarrow \infty} \lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \longrightarrow 0$ as $j \longrightarrow \infty$, then $\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$.*

The purpose of this paper is to establish strong convergence theorems for the Ishikawa iteration-type scheme involving quasi-nonexpansive multi-valued mappings in the setting of CAT(0) spaces. Our results signifigantly extend and improve the results obtained by Shahzad and Zegeye [28], as well as the related results in the existing literature.

3. Main Results

Let $(\mathbb{X}, d)$ be a CAT(0) space and $\mathscr{U}$ be a nonempty convex subset of $\mathbb{X}$. Similarly, we introduce the following definitions.

Definition 3.1. A multi-valued mapping $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{P}(\mathscr{U})$ is said to be:

(a) nonexpansive if $\mathbb{H}_d(\mathbb{T}x, \mathbb{T}y) \leq d(x,y)$ for all $x, y \in \mathscr{U}$,

(b) quasi-nonexpansive if $\mathbb{H}_d(\mathbb{T}x, \mathbb{T}p) \leq d(x,p)$ for all $x \in \mathscr{U}$ and $p \in F(\mathbb{T})$.

Following [28], we give the following definition.

Definition 3.2. Let $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ be a multi-valued map. Suppose $\alpha_n, \beta_n \in [0,1]$, the sequence of Ishikawa iterates is generated by $x_0 \in \mathscr{U}$,

$$
\begin{aligned}
x_{n+1} &= (1-\alpha_n)x_n \bigoplus \alpha_n v_n, \, n \geq 0 \\
y_n &= (1-\beta_n)x_n \bigoplus \beta_n w_n, \, n \geq 0
\end{aligned} \tag{3.1}
$$

where $w_n \in \mathbb{T}x_n$ and $v_n \in \mathbb{T}y_n$.

Note that we significantly make use of Lemma 2.6 (see [10]) in the proof of our main results.

Lemma 3.3. *Let $(\mathbb{X}, d)$ be a CAT(0) space and $\mathscr{U}$ be a nonempty closed convex subset of $\mathbb{X}$. Let $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ be a quasi-nonexpansive multi-valued mapping with $F(\mathbb{T}) \neq \emptyset$ and for which $\mathbb{T}(p) = \{p\}$ for each $p \in F(\mathbb{T})$. Let $\{x_n\}$ be the Ishikawa iterates generated by* (3.1)*. Then $\lim\limits_{n \longrightarrow \infty} d(x_n, p)$ exists for each $p \in F(\mathbb{T})$.*

Proof. Let $p \in F(\mathbb{T})$. Then, from (3.1) and using Lemma 2.6(ii), we have

$$
\begin{aligned}
d(y_n, p) &= d\big((1-\beta_n)x_n \bigoplus \beta_n w_n, p\big) \\
&\leq (1-\beta_n)d(x_n, p) + \beta_n d(w_n, p) \\
&\leq (1-\beta_n)d(x_n, p) + \beta_n \mathbb{H}_d(\mathbb{T}x_n, \mathbb{T}p) \\
&\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\
&= d(x_n, p)
\end{aligned} \tag{3.2}
$$

and

$$
\begin{aligned}
d(x_{n+1}, p) &= d\big((1-\alpha_n)x_n \bigoplus \alpha_n v_n, p\big) \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(v_n, p) \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n \mathbb{H}_d(\mathbb{T}y_n, \mathbb{T}p) \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(y_n, p) \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\
&= d(x_n, p).
\end{aligned} \tag{3.3}
$$

Consequently, the sequence $\{d(x_n,p)\}$ is decreasing and bounded below and thus $\lim\limits_{n\longrightarrow\infty} d(x_n,p)$ exists for each $p \in F(\mathbb{T})$. Also $\{x_n\}$ is bounded. □

Theorem 3.4. *Let $(\mathbb{X},d)$ be a CAT(0) space and $\mathscr{U}$ be a nonempty closed convex subset of $\mathbb{X}$. Let $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ be a quasi-nonexpansive multi-valued mapping with $F(\mathbb{T}) \neq \emptyset$ and $\mathbb{T}(p) = \{p\}$ for each $p \in F(\mathbb{T})$. Let $\{x_n\}$ be the Ishikawa iterates generated by* (3.1). *Assume that T satisfies Condition (I) and $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. Then $\{x_n\}$ converges strongly to a fixed point of $\mathbb{T}$.*

Proof. Let $p \in F(\mathbb{T})$. Then, as in the proof of Lemma 3.3, $\{x_n\}$ is bounded and so $\{y_n\}$ is bounded. Therefore, there exists $R > 0$ such that $x_n - p, y_n - p \in B_R(0)$ for all $n \geq 0$. Applying Lemma 2.6(iii), wehave

$$\begin{aligned} d(x_{n+1},p)^2 &= d\big((1-\alpha_n)x_n \bigoplus \alpha_n v_n, p\big)^2 \\ &\leq (1-\alpha_n)d(x_n,p)^2 + \alpha_n d(v_n,p)^2 - \alpha_n(1-\alpha_n)d(x_n,v_n)^2 \\ &\leq (1-\alpha_n)d(x_n,p)^2 + \alpha_n \mathbb{H}_d^2(\mathbb{T}y_n,\mathbb{T}p) - \alpha_n(1-\alpha_n)d(x_n,v_n)^2 \\ &\leq (1-\alpha_n)d(x_n,p)^2 + \alpha_n d(y_n,p)^2 - \alpha_n(1-\alpha_n)d(x_n,v_n)^2 \\ &\leq (1-\alpha_n)d(x_n,p)^2 + \alpha_n d(y_n,p)^2 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} d(y_n,p)^2 &= d\big((1-\beta_n)x_n \bigoplus \beta_n w_n, p\big)^2 \\ &\leq (1-\beta_n)d(x_n,p)^2 + \beta_n d(w_n,p)^2 - \beta_n(1-\beta_n)d(x_n,w_n)^2 \\ &\leq (1-\beta_n)d(x_n,p)^2 + \beta_n \mathbb{H}_d^2(\mathbb{T}x_n,\mathbb{T}p) - \beta_n(1-\beta_n)d(w_n,x_n)^2 \\ &\leq (1-\beta_n)d(x_n,p)^2 + \beta_n d(x_n,p)^2 - \beta_n(1-\beta_n)d(w_n,x_n)^2 \\ &= d(x_n,p)^2 - \beta_n(1-\beta_n)d(w_n,x_n)^2. \end{aligned} \tag{3.5}$$

Substitute (3.5) in (3.4), we obtain

$$\begin{aligned} d(x_{n+1},p)^2 &\leq (1-\alpha_n)d(x_n,p)^2 + \alpha_n d(x_n,p)^2 - \alpha_n\beta_n(1-\beta_n)d(w_n,x_n)^2 \\ &= d(x_n,p)^2 - \alpha_n\beta_n(1-\beta_n)d(w_n,x_n)^2. \end{aligned}$$

Thus

$$\alpha_n\beta_n(1-\beta_n)d(w_n,x_n)^2 \leq d(x_n,p)^2 - d(x_{n+1},p)^2.$$

This implies

$$\sum_{n=1}^{m}\alpha_n\beta_n(1-\beta_n)d(w_n,x_n)^2 \leq d(x_1,p)^2 - d(x_{m+1},p)^2 < \infty.$$

for all $m \geq 1$. Thererfore $\sum_{n=1}^{\infty}\alpha_n\beta_n(1-\beta_n)d(w_n,x_n)^2 < \infty$. Thus, $\lim\limits_{n\longrightarrow\infty} d(w_n,x_n)^2 = 0$. Since d is continuous, we have $\lim\limits_{n\longrightarrow\infty} d(w_n,x_n) = 0$. Also $d(x_n,\mathbb{T}x_n) \leq d(x_n,w_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Since $\mathbb{T}$ satisfies Condition (I), we have

$$h(d(x_n,F(\mathbb{T}))) \leq d(x_n,\mathbb{T}x_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Thus $\lim\limits_{n\longrightarrow\infty} d(x_n,F(\mathbb{T})) = 0$. Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k},p_k) < \frac{1}{2^k}$ for some $\{p_k\} \subset F(\mathbb{T})$ for all k. Note that in the proof of Lemma 3.3 we obtain

$$d(x_{n_k+1},p_k) \leq d(x_{n_k},p_k) < \frac{1}{2^k}.$$

We will show that $\{p_k\}$ is a Cauchy sequence in $\mathscr{U}$. Notice that

$$\begin{aligned} d(p_{k+1},p_k) &\leq d(p_{k+1},x_{n_k+1}) + d(x_{n_k+1},p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence in $\mathscr{U}$ and thus converges to $q \in \mathscr{U}$. Since

$$\begin{aligned} d(p_k,Tq) &\leq \mathbb{H}_d(\mathbb{T}p_k,\mathbb{T}q) \\ &\leq d(p_k,q) \end{aligned}$$

and $p_k \longrightarrow q$ as $k \longrightarrow \infty$, it follows that $d(q,Tq) = 0$ and thus $q \in F(\mathbb{T})$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim\limits_{n\longrightarrow\infty} d(x_n,q)$ exists, it follows that $\{x_n\}$ converges strongly to a fixed point q of $\mathbb{T}$. This completes our proof. □

Example 3.5. The existence of a fixed point of $\mathbb{T}$ is very interesting. Let $\mathbb{T}$ be a quasi-nonexpansive multi-valued map with $F(\mathbb{T}) \neq \emptyset$, then a fixed point of $\mathbb{T}$ exists under the assumptions of Theorem 3.4. Indeed, fix $x_0 \in \mathscr{U}$, define

$$\mathbb{T}_n x = \frac{1}{n} x_0 \bigoplus (1 - \frac{1}{n}) \mathbb{T} x$$

foe $x \in \mathscr{U}$ and $n \geq 1$. Then, using (2.5) for $x^* \in F(\mathbb{T})$ we have

$$\begin{aligned} d(\mathbb{T}_n x, \mathbb{T}_n x^*) &\leq d(\frac{1}{n} x_0 \bigoplus (1 - \frac{1}{n}) \mathbb{T} x, \frac{1}{n} x_0 \bigoplus (1 - \frac{1}{n}) \mathbb{T} x^*) \\ &\leq (1 - \frac{1}{n}) \mathbb{H}_d(\mathbb{T} x, \mathbb{T} x^*) \\ &\leq (1 - \frac{1}{n}) d(x, x^*) \end{aligned}$$

That is, $\mathbb{T}_n$ is a contraction. By the Banach contraction principle, $\mathbb{T}_n$ has a unique fixed point x_n in $\mathscr{U}$. Since the closure of $\mathbb{T}(\mathscr{U})$ is compact, there exists a subsequence $\{\mathbb{T} x_{n_i}\}$ of $\{\mathbb{T} x_n\}$ such that $\mathbb{T} x_{n_i} \longrightarrow u$. Since $\mathbb{T}(\mathscr{U})$ is bounded and

$$d(x_n, \mathbb{T} x_n) = d(\mathbb{T}_n x_n, \mathbb{T} x_n) = d(\frac{1}{n} x_0 \bigoplus (1 - \frac{1}{n}) \mathbb{T} x_n, \mathbb{T} x_n) \leq \frac{1}{n} d(x_0, \mathbb{T} x_n),$$

we have $d(x_n, \mathbb{T} x_n) \longrightarrow 0$. In particular, we have $x_{n_i} \longrightarrow u$. Continuity of $\mathbb{T}$ implies $\mathbb{T} u = u$. Moreover, since $\mathbb{T}$ satisfies Condition (I), so we have

$$h(d(x_n, F(\mathbb{T}))) \leq d(x_n, \mathbb{T} x_n) \longrightarrow 0.$$

Thus $\lim_{n \longrightarrow \infty} d(x_n, F(\mathbb{T})) = 0$. This implies that $\{x_n\}$ converges strongly to a fixed point of $\mathbb{T}$.

Corollary 3.6. *Let $(\mathbb{X}, d)$ be a CAT(0) space and $\mathscr{U}$ be a nonempty closed convex subset of $\mathbb{X}$. Let $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ be a nonexpansive multi-valued mapping with $F(\mathbb{T}) \neq \emptyset$ and $\mathbb{T}(p) = \{p\}$ for each $p \in F(\mathbb{T})$. Let $\{x_n\}$ be the Ishikawa iterates defined by* (3.1). *Assume that $\mathbb{T}$ satisfies Condition (I) and $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ converges strongly to a fixed point of $\mathbb{T}$.*

Theorem 3.7. *Let $(\mathbb{X}, d)$ be a CAT(0) space and $\mathscr{U}$ be a nonempty closed convex subset of $\mathbb{X}$. Let $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ be a quasi-nonexpansive multi-valued mapping with $F(\mathbb{T}) \neq \emptyset$ and $\mathbb{T}(p) = \{p\}$ for each $p \in F(\mathbb{T})$. Let $\{x_n\}$ be the Ishikawa iterates generated by* (3.1). *Assume that $\mathbb{T}$ is hemicompact and*

(i). $0 \leq \alpha_n, \beta_n < 1$;

(ii). $\beta_n \longrightarrow 0$;

(iii). $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then $\{x_n\}$ *converges strongly to a fixed point of* $\mathbb{T}$.

Proof. Let $p \in F(\mathbb{T})$. As in the proof of Theorem 3.4, by using Lemma 2.6(iii) we obtain

$$\sum_{n=1}^{m} \alpha_n \beta_n (1-\beta_n) d(w_n, x_n)^2 \leq d(x_1, p)^2 - d(x_{m+1}, p)^2 < \infty$$

and therefore

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) d(w_n, x_n)^2 < \infty.$$

Thus, $\lim_{n \longrightarrow \infty} d(x_n, w_n) = 0$. Since $d(x_n, \mathbb{T}x_n) \leq d(x_n, w_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\mathbb{T}$ is hemicompact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow q$ for some $q \in \mathscr{U}$. Since $\mathbb{T}$ is continuous, $d(x_{n_k}, \mathbb{T}x_{n_k}) \longrightarrow d(q, Tq)$. As a result, we have $d(q, \mathbb{T}q) = 0$ and so $q \in F(\mathbb{T})$. By Lemma 3.3, we have $\lim_{n \longrightarrow \infty} d(x_n, p)$ exists for each $p \in F(\mathbb{T})$, it follows that $\{x_n\}$ converges strongly to q. This completes our proof. □

Corollary 3.8. *Let* $(\mathbb{X}, d)$ *be a CAT(0) space and* $\mathscr{U}$ *be a nonempty closed convex subset of* $\mathbb{X}$. *Let* $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{CB}(\mathscr{U})$ *be a nonexpansive multi-valued mapping with* $F(\mathbb{T}) \neq \emptyset$ *and* $\mathbb{T}(p) = \{p\}$ *for each* $p \in F(\mathbb{T})$. *Let* $\{x_n\}$ *be the Ishikawa iterates defined by* (3.1). *Assume that* $\mathbb{T}$ *is hemicompact and*

(i). $0 \leq \alpha_n, \beta_n < 1$;

(ii). $\beta_n \longrightarrow 0$;

(iii). $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then $\{x_n\}$ *converges strongly to a fixed point of* $\mathbb{T}$.

Definition 3.9. Let $(\mathbb{X}, d)$ be a CAT(0) space and $\mathscr{U}$ be a nonempty closed convex subset of $\mathbb{X}$. Let $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{P}(\mathscr{U})$ be a multi-valued map . Let $P_{\mathbb{T}}(x) = \{y \in \mathbb{T}x : d(x,y) = d(x, \mathbb{T}x)\}$.The sequence of Ishikawa iterates is defined by $x_0 \in \mathscr{U}$,

$$\begin{aligned} x_{n+1} &= (1-\alpha_n) x_n \bigoplus \alpha_n v_n, \\ y_n &= (1-\beta_n) x_n \bigoplus \beta_n w_n,\ n \geq 0, \end{aligned} \tag{3.6}$$

where $w_n \in P_{\mathbb{T}}(x_n)$ and $v_n \in P_{\mathbb{T}}(y_n)$ and $\alpha_n, \beta_n \in [0,1]$.

Note that, if we remove the restriction on $\mathbb{T}$ i.e.$\mathbb{T}(p) = \{p\}$ for all $p \in F(\mathbb{T})$ in Theorem 3.4, we obtain the following theorems.

Theorem 3.10. *Let $(\mathbb{X}, d)$ be a CAT(0) space and $\mathscr{U}$ be a nonempty closed convex subset of $\mathbb{X}$. Let $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{P}(\mathscr{U})$ a multi-valued mapping with $F(\mathbb{T}) \neq \emptyset$ and such that $P_{\mathbb{T}}$ is quasi-nonexpansive. Let $\{x_n\}$ be the Ishikawa iterates generated by* (3.6). *Assume that $\mathbb{T}$ satisfies Condition (I) and $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. Then $\{x_n\}$ converges strongly to a fixed point of $\mathbb{T}$.*

Proof. Let $p \in F(\mathbb{T})$. Then $p \in P_{\mathbb{T}}(p) = \{p\}$. By usinng Lemma 2.6(ii), we get

$$\begin{aligned} d(y_n, p) &= d\big((1-\beta_n)x_n \bigoplus \beta_n w_n, p\big) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(w_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n \mathbb{H}_d(P_{\mathbb{T}}(x_n), P_{\mathbb{T}}(p)) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &\leq d(x_n, p) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d\big((1-\alpha_n)x_n \bigoplus \alpha_n v_n, p\big) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(v_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n \mathbb{H}_d(P_{\mathbb{T}}(y_n), P_{\mathbb{T}}(p)) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(y_n, p). \end{aligned} \tag{3.8}$$

Substitute (3.7) in (3.8), we obtain

$$d(x_{n+1}, p) \leq d(x_n, p). \tag{3.9}$$

Consequently, the sequence $\{d(x_n, p)\}$ is decreasing and bounded below and thus $\lim_{n \longrightarrow \infty} d(x_n, p)$ exists for any $p \in F(\mathbb{T})$.

Applying Lemma 2.6(iii), we get

$$\begin{aligned} d(x_{n+1}, p)^2 &= d\big((1-\alpha_n)x_n \bigoplus \alpha_n v_n, p\big)^2 \\ &\leq (1-\alpha_n)d(x_n, p)^2 + \alpha_n d(v_n, p)^2 - \beta_n(1-\beta_n)d(x_n, v_n)^2 \\ &\leq (1-\alpha_n)d(x_n, p)^2 + \alpha_n \mathbb{H}_d^2(P_{\mathbb{T}}(y_n), P_{\mathbb{T}}(p)) - \beta_n(1-\beta_n)d(x_n, v_n)^2 \\ &\leq (1-\alpha_n)d(x_n, p)^2 + \alpha_n d(y_n, p)^2 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} d(y_n,p)^2 &= d\big((1-\beta_n)x_n \bigoplus \beta_n w_n, p\big)^2 \\ &\leq (1-\beta_n)d(x_n,p)^2 + \beta_n d(w_n,p)^2 - \beta_n(1-\beta_n)d(x_n,w_n)^2 \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n \mathbb{H}_d^2(P_{\mathbb{T}}(x_n),P_{\mathbb{T}}(p)) - \beta_n(1-\beta_n)d(x_n,w_n)^2 \quad (3.11) \\ &\leq (1-\beta_n)d(x_n,p)^2 + \beta_n d(x_n,p)^2 - \beta_n(1-\beta_n)d(x_n,w_n)^2 \\ &= d(x_n,p)^2 - \beta_n(1-\beta_n)d(x_n,w_n)^2. \end{aligned}$$

From (3.11) and (3.10), we obtain

$$d(x_{n+1},p)^2 \leq d(x_n,p)^2 - \alpha_n\beta_n(1-\beta_n)d(x_n,w_n)^2.$$

This implies

$$\sum_{n=1}^{m} \alpha_n\beta_n(1-\beta_n)d(x_n,w_n)^2 \leq d(x_1,p)^2 - d(x_{m+1},p)^2 < \infty, \quad (3.12)$$

for all $m \geq 1$. Therefore $\sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n)d(x_n,w_n)^2 < \infty$. Consequently, we have $\lim_{n\longrightarrow\infty} d(x_n,w_n) = 0$. So $d(x_n,\mathbb{T}x_n) \leq d(x_n,w_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Since $\mathbb{T}$ satisfies condition (I), we have $\lim_{n\longrightarrow\infty} d(x_n,F(\mathbb{T})) = 0$. Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k},p_k) < \frac{1}{2^k}$ for some $\{p_k\} \subset F(\mathbb{T})$ and for all k. As in the proof of Theorem 3.4, $\{p_k\}$ is a Cauchy sequence in $\mathscr{U}$ and thus converges to $q \in \mathscr{U}$.

Consider

$$\begin{aligned} d(p_k,\mathbb{T}q) &\leq d(p_k,P_{\mathbb{T}}(q)) \\ &\leq \mathbb{H}_d(P_{\mathbb{T}}(p_k),P_{\mathbb{T}}(q)) \quad (3.13) \\ &\leq d(p_k,q) \end{aligned}$$

and $p_k \longrightarrow q$ as $k \longrightarrow \infty$, it follows that $d(q,\mathbb{T}q) = 0$ and so $q \in F(\mathbb{T})$ and $\{x_n\}$ converges strongly to q. Our proof is finished. □

Theorem 3.11. *Let $(\mathbb{X},d)$ be a CAT(0) space and $\mathscr{U}$ be a nonempty closed convex subset of $\mathbb{X}$. Let $\mathbb{T} : \mathscr{U} \longrightarrow \mathbb{P}(\mathscr{U})$ a multi-valued mapping with $F(\mathbb{T}) \neq \emptyset$, $P_{\mathbb{T}}$ is quasi-nonexpansive. Let $\{x_n\}$ be the Ishikawa iterates generated by* (3.6). *Assume that $\mathbb{T}$ is hemicompact and*

(i). $0 \leq \alpha_n, \beta_n < 1$;

(ii). $\beta_n \longrightarrow 0$;

(iii). $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then $\{x_n\}$ *converges strongly to a fixed point of* $\mathbb{T}$ *.*

Proof. Let $p \in F(\mathbb{T})$. Then, as in the proof of Theorem 3.10, $\lim_{n\longrightarrow\infty} d(x_n, p)$ exists for any $p \in F(\mathbb{T})$. And as in the proof of Theorem 3.10, it can be shown that $d(x_n, w_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Since $\mathbb{T}$ is hemicompact, we may assume that $x_{n_k} \longrightarrow q$ for some $q \in \mathscr{U}$. Notice that

$$\begin{aligned} d(q, Tq) &\leq d(q, x_{n_k}) + d(x_{n_k}, w_{n_k}) + d(w_{n_k}, \mathbb{T}q) \\ &\leq d(q, x_{n_k}) + d(x_{n_k}, w_{n_k}) + \mathbb{H}_d(P_{\mathbb{T}}(x_{n_k}), P_{\mathbb{T}}(q)) \qquad (3.14) \\ &\leq 2d(x_{n_k}, q) + d(x_{n_k}, w_{n_k}) \longrightarrow 0. \end{aligned}$$

This implies that $d(q, \mathbb{T}q) = 0$ and thus $q \in F(\mathbb{T})$. Since $\lim_{n\longrightarrow\infty} d(x_n, p)$ exists for any $p \in F(\mathbb{T})$, it follows that $x_n \longrightarrow q$ as $n \longrightarrow \infty$. This completes our proof. □

Remark 3.12. Some examples of a multi-valued map $\mathbb{T}$ for which $P_{\mathbb{T}}$ is quasi-nonexpansive can be found in [13, 34].

4. CONCLUSIONS

In this manuscript, we prove some strong convergence theorems in CAT(0) spaces. Let $\mathscr{U}$ be a nonempty closed covex subset of a CAT(0) space $\mathbb{X}$.

We obtained the following results:-

1). **Lemma** 3.3: An extension of Lemma 2.2 in [28].

2). **Theorem** 3.4: If $\mathbb{T} : \mathscr{U} \longrightarrow CB(\mathscr{U})$ is a quasi-nonexpansive multi-valued mapping satisfying Condition (I) and $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ defined by (3.1) converges strongly to a fixed point of $\mathbb{T}$.

This theorem extends and improves Theorem 1.16 (see [28], Theorem 2.3).

3). **Theorem** 3.7: If $\mathbb{T} : \mathscr{U} \longrightarrow CB(\mathscr{U})$ be a quasi-nonexpansive multi-valued mapping with $F(\mathbb{T}) \neq \emptyset$ and $\mathbb{T}(p) = \{p\}$ for each $p \in F(\mathbb{T})$. Suppose $\mathbb{T}$ is hemicompact and

(i). $0 \leq \alpha_n, \beta_n < 1$;

(ii). $\beta_n \longrightarrow 0$;

(iii). $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then $\{x_n\}$ defined by (3.1) converges strongly to a fixed point of $\mathbb{T}$.

This theorem extends and improves Theorem 1.17 (see [28], Theorem 2.5).

4). **Theorem** 3.10: If $\mathbb{T}: \mathscr{U} \longrightarrow \mathbb{P}(\mathscr{U})$ a multi-valued mapping with $F(\mathbb{T}) \neq \emptyset$, T satisfies Condition (I), $P_{\mathbb{T}}$ is quasi-nonexpansive, and $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. Then the sequence $\{x_n\}$ defined by (3.6) converges strongly to a fixed point of $\mathbb{T}$.

This theorem extends and improves Theorem 1.18 (see [28], Theorem 2.7).

4). **Theorem** 3.11: If $\mathbb{T}: \mathscr{U} \longrightarrow \mathbb{P}(\mathscr{U})$ a multi-valued mapping with $F(\mathbb{T}) \neq \emptyset$, $P_{\mathbb{T}}$ is quasi-nonexpansive, $\mathbb{T}$ is hemicompact and

(i). $0 \leq \alpha_n, \beta_n < 1$;

(ii). $\beta_n \longrightarrow 0$;

(iii). $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ defined by (3.6) converges strongly to a fixed point of $\mathbb{T}$.

This theorem extends and improves Theorem 2.8 in [28].

As consequence, we obtain Corollaries 3.6 and Corollaries 3.8 accordingly. Our results extend and improve some other related results in the literature.

AUTHORS' CONTRIBUTIONS

The authors wrote, read and approved the final manuscript.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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