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NEW RELATION-THEORETIC FIXED POINT RESULTS IN NEUTROSOPHIC *B*-METRIC SPACES

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Abstract. In this paper, we introduce the notion of neutrosophic $\Re - \psi -$contractive mappings in the setting of relational neutrosophic b-metric space. Our findings possibly open new way for another direction of relation-theoretic as well as neutrosophic fixed point theory. we prove some relevant results on the existence and uniqueness of fixed points for this type of mappings in the setting of neutrosophic b-metric space.

Keywords: relation-theoretic; fixed point; neutrosophic b-metric space; neutrosophic $\Re - \psi -$contractive mappings.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The concept of fuzzy sets was initially formulated by Zadeh [29] in 1965. This concept has been used extensively in topology as well as modern analysis by many researchers and plays a very important role in a large number of scientific and engineering applications. Atanassov [4]

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initiated Intuitionistic fuzzy sets for such cases. Neutrosophic set is a new version of the idea of the classical set which is defined by Smarandache [26]. Enter the previous concepts in different metric spaces has been the focus of many researchers' attention, as Kramosil and Michalek [11] introduced the notion of fuzzy metric space. Later, George and Veeramani [6] modified the notion of fuzzy metric spaces due to Kramosil and Michalek with view to have a Hausdorff topology. The approach of intuitionistic fuzzy metric spaces was introduced by Park in [19]. Kirisci and Simsek [10] generalized the approach of intuitionistic fuzzy metric space by presenting the approach of a neutrosophic metric space which deal with membership, nonmembership, and naturalness functions. Successively, several researches were carried out various contractive mappings in different spaces. For instance, Gregori and Sapena [7] defined fuzzy contractive mappings and proved a very natural extension of the well-known Banach contraction principle for such mappings in G-complete as well as M-complete fuzzy metric spaces. Mihet [15] extended the class of Gregori and Sapena's fuzzy contractive mappings [7] and proved a fuzzy Banach contraction result for complete non-Archimedean fuzzy metric spaces in the sense of Kramosil and Michalek. Sowndrarajan et al. [27] proved some fixed point results in the setting of neutrosophic metric spaces and others.

On the other hand, in 1993, Czerwik [5] introduced the notion of a b-metric with a view of generalizing the Banach contraction mapping theorem. After that, a lot of authors have worked in this directions and have presented some nice results related to the fixed point theory (see for example [8, 9, 16, 24] and others. As a generalization of b-metric space and neutrosophic metric space, Shakila and Jeyaraman [25] introduced the notion of neutrosophic b-metric space and proved neutrosophic b-metric versions of some conventional theorems fixed points via neutrosophic sets.

Relation-theoretic fixed point theory is a relatively new direction of fixed point theory. This direction was initiated by Turinici [28] and it becomes very active area after the existence of the well-known results due to Ran and Reurings [20] and Nieto and Lopez [17, 18] with their interesting applications to boundary value problems and matrix equations. Recently, there are several researchers working in this direction. See for example [2, 3, 22, 23], and others.

2. Preliminaries

In this section, some basic definitions are mentioned that are helpful to understand the main results.

Definition 2.1. [19] A binary operation $*: [0,1]\times[0,1]\to[0,1]$ is called a continuous triangular norm (t-norm) if it meets the following assertions:

(1) $*$ is associative and commutative;

(2) $*$ is continuous;

(3) $x*1=x, \forall x\in[0,1]$;

(4) If $x\leq z$ and $y\leq r$ with $x,y,z,r\in[0,1]$, then $x*y\leq z*r$.

Example 2.1. *The following three examples of basic (t-norm):*

(1) $x*y=\min(x,y)$.

(2) $x*y=xy$.

(3) $x*y=\max(x+y-1,0)$.

Definition 2.2. [19] A binary operation $\circ: [0,1]\times[0,1]\to[0,1]$ is called a continuous triangular conorm (t-conorm) if it meets the following assertions:

(1) $\circ$ is associative and commutative;

(2) $\circ$ is continuous;

(3) $x\circ 0=0, \forall x\in[0,1]$;

(4) If $x\leq z$ and $y\leq r$ with $x,y,z,r\in[0,1]$, then $x\circ y\leq z\circ r$.

Example 2.2. *The following three examples of basic (t-conorm):*

(1) $x\circ y=\min(x+y,1)$.

(2) $x\circ y=x+y-xy$.

(3) $x\circ y=\max(x,y)$.

Shakil and Jeyaraman [25] defined the notion of neutrosophic b-metric space as follows:

Definition 2.3. [25] A seven tuple $(\chi,\alpha,\beta,\gamma,*,\circ,b)$ is said to be a neutrosophic b-metric space if $\chi\neq\phi$, $b\geq 1$ is a given real number, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, α,β and γ neutrosophic sets on $\chi^2\times[0,\infty)$ meet the below circumstances for all $x,y,z\in\chi$,

(1) $\alpha(x,y,t)+\beta(x,y,t)+\gamma(x,y,t)\leq 3;$

(2) $0\leq\alpha(x,y,t)\leq 1;$

(3) $\alpha(x,y,0)=0;$

(4) $\alpha(x,y,t)=1$ for all $t>0$ iff $x=y;$

(5) $\alpha(x,y,t)=\alpha(y,x,t),$ for all $t>0;$

(6) $\alpha(x,z,b(t+s))\geq\alpha(x,y,t)*\alpha(y,z,s),$ for all $t,s>0;$

(7) $\alpha(x,y,.):[0,\infty)\to[0,1]$ is a left continuous and $\lim_{t\to+\infty}\alpha(x,y,t)=1;$

(8) $0\leq\beta(x,y,t)\leq 1;$

(9) $\beta(x,y,0)=1;$

(10) $\beta(x,y,t)=0$ for all $t>0$ iff $x=y;$

(11) $\beta(x,y,t)=\beta(y,x,t),$ for all $t>0;$

(12) $\beta(x,z,b(t+s))\leq\beta(x,y,t)\circ\beta(y,z,s),$ for all $t,s>0;$

(13) $\beta(x,y,.):[0,\infty)\to[0,1]$ is a left continuous and $\lim_{t\to+\infty}\beta(x,y,t)=0;$

(14) $0\leq\gamma(x,y,t)\leq 1;$

(15) $\gamma(x,y,0)=1;$

(16) $\gamma(x,y,t)=0$ for all $t>0$ iff $x=y;$

(17) $\gamma(x,y,t)=\gamma(y,x,t),$ for all $t>0;$

(18) $\gamma(x,z,b(t+s))\leq\gamma(x,y,t)\circ\gamma(y,z,s),$ for all $t,s>0;$

(19) $\gamma(x,y,.):[0,\infty)\to[0,1]$ is a left continuous and $\lim_{t\to+\infty}\gamma(x,y,t)=0.$

Example 2.3. *([25]) Let (χ, d, b) be a b-metric space and $x * y = \min(x,y), x \circ y = \max(x,y)$ for all $x,y \in [0,1]$ and let α, β and γ are fuzzy sets on $\chi^2 \times [0,\infty)$ defined as follows:*

$$\alpha(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)}, \ if\, t > 0; \\ 0, \ if\, t = 0. \end{cases}$$

$$\beta(x,y,t) = \begin{cases} \frac{d(x,y)}{t+d(x,y)}, \ if\, t > 0; \\ 1, \ if\, t = 0. \end{cases}$$

$$\gamma(x,y,t) = \begin{cases} \frac{d(x,y)}{t}, \ if\, t > 0; \\ 1, \ if\, t = 0. \end{cases}$$

Hence $(\chi, \alpha, \beta, \gamma, *, \circ, b)$ is a neutrosophic b- metric space.

Definition 2.4. [25] A sequence $\{x_n\}$ in a neutrosophic b-metric space $(\chi, \alpha, \beta, \gamma, *, \circ, b)$ is said to be convergent if there exists $x \in \chi$ such that

$$\lim_{n\to\infty} \alpha(x_n, x, t) = 1, \ \forall t > 0;$$

$$\lim_{n\to\infty} \beta(x_n, x, t) = 0, \ \forall t > 0;$$

and

$$\lim_{n\to\infty} \gamma(x_n, x, t) = 0, \ \forall t > 0.$$

In this case x is called the limit of the sequence $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 2.5. [25] A sequence $\{x_n\}$ in a neutrosophic b- metric space $(\chi, \alpha, \beta, \gamma, *, \circ, b)$ is said to be Cauchy sequence if for every $\varepsilon \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that

$$\alpha(x_n, x_m, t) > 1 - \varepsilon;$$

$$\beta(x_n, x_m, t) < \varepsilon;$$

and

$$\gamma(x_n, x_m, t) < \varepsilon;$$

for all $m, n \geq n_0, t > 0$.

Definition 2.6. [25] The space χ is said to be complete if and if only if every Cauchy sequence $\{x_n\}$ in χ converges to some $x \in \chi$.

Roldán-lópez-de-Hierro [21] defined a comparison $\psi : [0,1] \to [0,1]$ that satisfies the following conditions

(1) ψ is non-decreasing and left continuous;
(2) $\psi(t) < t$, for all $t \in (0,1)$;
(3) $\psi(0) = 0$.

Let Ψ denotes the family of all such functions ψ.
For example, $\psi(t) = t^2$ for all $t \in [0,1]$. Notice that, using the previous definition, the condition $\psi(1) = 1$ is not necessarily true.

Remark 2.1. [21] *Let* $\psi \in \Psi$

(1) $\psi(t) \leq t$, *for all* $t \in [0,1]$;
(2) If $\psi(t_0) = t_0$, *for some* $t_0 \in (0,1]$, *then* $t_0 = 1$;
(3) If $\{t_n\} \subset [0,1]$, *and* $\psi(t_n) \to 1$, *then* $t_n \to 1$.

In the following, we recall some relation-throretic notions as follows:

Definition 2.7. [12] A subset $\Re$ of χ^2 is called a binary relation on χ. If $(x,y) \in \Re$ (we may write $x\Re y$ instead of $(x,y) \in \Re$), then we say that "x is related to y under $\Re$". If either $x\Re y$ or $y\Re x$, then we write $[x,y] \in \Re$.

Observe that χ^2 is a binary relation on χ called the universal relation. In this presentation, χ is a non-empty set and $\Re$ refers for a non-empty binary relation on χ.

Definition 2.8. [13, 14] A binary relation $\Re$ on a non-empty set χ is said to be:

(1) reflexive if $x\Re x, \forall x \in \chi$;
(2) transitive if $x\Re y$ and $y\Re z$ imply $x\Re z, \forall x,y,z \in \chi$;
(3) antisymmetric if $x\Re y$ and $y\Re x$ imply $x = y, \forall x,y \in \chi$;
(4) partial order if it is reflexive, antisymmetric and transitive;
(5) complete if $[x,y] \in \Re, \forall x,y \in \chi$;

(6) f - closed if $(x,y) \in \Re \Rightarrow (fx,fy) \in \Re, \forall x,y \in \chi$ where $f : \chi \to \chi$ is a mapping.

Definition 2.9. [1] Let χ be a non-empty set and $\Re$ be a binary relation on χ. A sequence $\{x_n\} \subseteq \chi$ is said to be an $\Re$ - preserving sequence if $(x_n, x_{n+1}) \in \Re$ for all $n \in \mathbb{N}$.

In what follows, we provide relation-theoretic versions of the neutrosophic metrical notions: (α, β, γ)-self- closedness, convergence and completeness.

Definition 2.10. A binary relation $\Re$ on χ is said to be an (α, β, γ)- self- closed if given any convergent $\Re$ -preserving sequence $\{x_n\} \subseteq \chi$ which converges (in neutrosophic sense) to some $x \in \chi$, $\exists \{x_{n_k}\} \subseteq \{x_n\}$ with $(x_{n_k}, x) \in \Re$.

Definition 2.11. A sequence $\{x_n\} \subseteq \chi$ is called $\Re - (\alpha, \beta, \gamma)$- Cauchy if $x_n \Re x_{n+1}, \forall n \in \mathbb{N}_0$ and $\forall \varepsilon > 0$ and $t > 0$, $\exists n_0 \in \mathbb{N}$ satisfying

$$\begin{aligned} \alpha(x_n, x_{n+p}, t) &> 1 - \varepsilon, \forall n \geq N, p \in \mathbb{N}_0, \\ \beta(x_n, x_{n+p}, t) &< \varepsilon, \forall n \geq N, p \in \mathbb{N}_0, \\ \gamma(x_n, x_{n+p}, t) &< \varepsilon, \forall n \geq N, p \in \mathbb{N}_0. \end{aligned}$$

Remark 2.2. *Every Cauchy sequence is an $\Re - (\alpha, \beta, \gamma)$- Cauchy sequence, for any arbitrary binary relation $\Re$. $\Re - (\alpha, \beta, \gamma)$- Cauchyness coincides with Cauchyness if $\Re$ is taken to be the universal relation.*

Definition 2.12. A neutrosophic b- metric space $(\chi, \alpha, \beta, \gamma, *, \circ, b)$ which is endowed with a binary relation $\Re$ is said to be $\Re - (\alpha, \beta, \gamma)$- complete if every $\Re - (\alpha, \beta, \gamma)$-Cauchy sequence is convergent in χ.

Remark 2.3. *Every complete neutrosophic b- metric space is $\Re - (\alpha, \beta, \gamma)$-complete neutrosophic b- metric space, for any arbitrary binary relation $\Re$. $\Re - (\alpha, \beta, \gamma)$- -completeness coincides with completeness if $\Re$ is taken to be the universal relation.*

Very recently, Saleh et al. [22] introduced the notion of KM-fuzzy $\Re - \psi-$contractive mappings as follows:

Definition 2.13. Let $(\chi, M, *)$ be a non-Archimedean fuzzy metric space (in the sense of Kramosil and Michalek), $\Re$ a binary relation on χ and $f : \chi \to \chi$. We say that f is a KM-fuzzy $\Re - \psi$- contractive mapping if there exists $\psi \in \Psi$ such that (for all $x, y \in \chi$ and all $t > 0$ with $x\Re y$)

$$M(x,y,t) > 0 \Rightarrow \min\{M(x,y,t), \max\{M(fx,x,t), M(y,fy,t)\}\} \leq \psi(M(fx,fy,t)).$$

3. Main Results

First, we will define the concept of neutrosophic $\Re - \psi-$ contractive as follows:

Definition 3.1. Let $(\chi, \alpha, \beta, \gamma, *, \circ, b)$ a neutrosophic b- metric space, $\Re$ a binary relation on χ and $f : \chi \to \chi$. We called That f is called neutrosophic $\Re - \psi$ contraction if there exists $\psi \in \Psi$ and $k \in (0, \frac{1}{b})$ such that for all $x, y \in \chi, t > 0$ with $x\Re y$,

$$\begin{aligned} \psi(\alpha(fx,fy,kt)) &\geq \min\{\alpha(x,y,t), \max\{\alpha(fx,x,t), \alpha(y,fy,t)\}\}, \\ \beta(fx,fy,kt) &\leq \psi(\max\{\beta(x,y,t), \min\{\beta(fx,x,t), \beta(y,fy,t)\}\}), \\ \gamma(fx,fy,kt) &\leq \psi(\max\{\gamma(x,y,t), \min\{\gamma(fx,x,t), \gamma(y,fy,t)\}\}). \end{aligned} \tag{3.1}$$

Now, we will state and prove the first main results:

Theorem 3.1. *Let $(\chi, \alpha, \beta, \gamma, *, \circ, b)$ a neutrosophic b-metric space equipped with a binary relation $\Re$. Assume that $f : \chi \to \chi$ is a neutrosophic $\Re - \psi$- contractive mapping defined in (3.1) and χ is a $\Re - (\alpha, \beta, \delta)-$ complete such that*
(i) there exists x_0 in χ such that $x_0 \Re f x_0$;
(ii) $\Re$ is transitive and $f-$ closed;
(iii) one of the following holds:
(a) f is continuous or
(b) $\Re$ is (α, β, γ)-self closed.
then f has a fixed point $\xi \in \chi$.

Proof. For arbitrary $x_0 \in \chi$, define a sequence $\{x_n\}$ in χ by $x_n = fx_{n-1}, \ \forall n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some $N \in \mathbb{N}_0$, then x_n is a fixed point of f. Assume that $x_n \neq x_{n+1}, \ \forall n \in \mathbb{N}_0$.
By (i), there exists $x_0 \in \chi$ such that $x_0 \Re f x_0$ and $x_n = fx_{n-1}$, then $x_0 \Re x_1$. As $\Re$ is $f-$closed and

$x_0\Re x_1$, we have that $x_1\Re x_2$. continuing this process, we get $x_n\Re x_{n+1}$, $\forall n\in\mathbb{N}_0$.
Using a neutrosophic $\Re-\psi-$ contractive (3.1). For $t>0$ and $n\in\mathbb{N}_0$, we have

$$\begin{aligned}
\alpha(x_{n+1},x_n,t) &\geq \alpha(x_{n+1},x_n,kt)\geq\psi(\alpha(x_{n+1},x_n,kt))=\psi(\alpha(fx_n,fx_{n-1},kt))\\
&\geq \min\{\alpha(x_n,x_{n-1},t),\max\{\alpha(fx_n,x_n,t),\alpha(x_{n-1},fx_{n-1},t)\}\},\\
&= \min\{\alpha(x_n,x_{n-1},t),\max\{\alpha(x_{n+1},x_n,t),\alpha(x_{n-1},x_n,t)\}\},\\
&= \alpha(x_n,x_{n-1},t)\\
&\geq \psi(\alpha(x_n,x_{n-1},t))\\
&= \psi(\alpha(fx_{n-1},fx_{n-2},t))\\
&\geq \min\{\alpha(x_{n-1},x_{n-2},\frac{t}{k}),\max\{\alpha(fx_{n-1},x_{n-1},\frac{t}{k}),\alpha(x_{n-2},fx_{n-2},\frac{t}{k})\}\}\\
&= \min\{\alpha(x_{n-1},x_{n-2},\frac{t}{k}),\max\{\alpha(x_n,x_{n-1},\frac{t}{k}),\alpha(x_{n-2},x_{n-1},\frac{t}{k})\}\}\\
&= \alpha(x_{n-1},x_{n-2},\frac{t}{k})\\
&\geq \psi(\alpha(x_{n-1},x_{n-2},\frac{t}{k}))\\
&= \psi(\alpha(fx_{n-2},fx_{n-3},\frac{t}{k}))\\
&\geq \min\{\alpha(x_{n-2},x_{n-3},\frac{t}{k^2}),\max\{\alpha(fx_{n-2},x_{n-2},\frac{t}{k^2}),\alpha(x_{n-3},fx_{n-3},\frac{t}{k^2})\}\}\\
&= \alpha(x_{n-2},x_{n-3},\frac{t}{k^2})\geq\psi(\alpha(fx_{n-3},fx_{n-4},\frac{t}{k^2}))\\
&\geq \ldots\geq\alpha(x_1,x_0,\frac{t}{k^{n-1}}),
\end{aligned}\tag{3.2}$$

$$\begin{aligned}
\beta(x_{n+1},x_n,t) &\leq \beta(x_{n+1},x_n,kt)=\beta(fx_n,fx_{n-1},kt)\\
&\leq \psi(\max\{\beta(x_n,x_{n-1},t),\min\{\beta(fx_n,x_n,t),\beta(x_{n-1},fx_{n-1},t)\}\}),\\
&= \psi(\max\{\beta(x_n,x_{n-1},t),\min\{\beta(x_{n+1},x_n,t),\beta(x_{n-1},x_n,t)\}\}),\\
&= \psi(\beta(x_n,x_{n-1},t))\\
&\leq \beta(x_n,x_{n-1},t)\\
&= \beta(fx_{n-1},fx_{n-2},t)
\end{aligned}$$

$$\begin{aligned}
&\le \psi(\max\{\beta(x_{n-1},x_{n-2},\frac{t}{k}),\min\{\beta(fx_{n-1},x_{n-1},\frac{t}{k}),\beta(x_{n-2},fx_{n-2},\frac{t}{k})\}\})\\
&= \psi(\max\{\beta(x_{n-1},x_{n-2},\frac{t}{k}),\min\{\beta(x_{n},x_{n-1},\frac{t}{k}),\beta(x_{n-2},x_{n-1},\frac{t}{k})\}\})\\
&= \psi(\beta(x_{n-1},x_{n-2},\frac{t}{k}))\\
&\le \beta(x_{n-1},x_{n-2},\frac{t}{k})\\
&= \beta(fx_{n-2},fx_{n-3},\frac{t}{k})\\
&\le \psi(\max\{\beta(x_{n-2},x_{n-3},\frac{t}{k^2}),\min\{\beta(fx_{n-2},x_{n-2},\frac{t}{k^2}),\beta(x_{n-3},fx_{n-3},\frac{t}{k^2})\}\})\\
&= \psi(\beta(x_{n-2},x_{n-3},\frac{t}{k^2}))\le \beta(fx_{n-3},fx_{n-4},\frac{t}{k^2})\\
&\le \dots \le \beta(x_1,x_0,\frac{t}{k^{n-1}}), \qquad (3.3)
\end{aligned}$$

$$\begin{aligned}
\gamma(x_{n+1},x_n,t) &\le \gamma(x_{n+1},x_n,kt)=\gamma(fx_n,fx_{n-1},kt)\\
&\le \psi(\max\{\gamma(x_n,x_{n-1},t),\min\{\gamma(fx_n,x_n,t),\gamma(x_{n-1},fx_{n-1},t)\}\}),\\
&= \psi(\max\{\gamma(x_n,x_{n-1},t),\min\{\gamma(x_{n+1},x_n,t),\gamma(x_{n-1},x_n,t)\}\}),\\
&= \psi(\gamma(x_n,x_{n-1},t))\\
&\le \gamma(x_n,x_{n-1},t)\\
&= \gamma(fx_{n-1},fx_{n-2},t)\\
&\le \psi(\max\{\alpha(x_{n-1},x_{n-2},\frac{t}{k}),\min\{\gamma(fx_{n-1},x_{n-1},\frac{t}{k}),\gamma(x_{n-2},fx_{n-2},\frac{t}{k})\}\})\\
&= \psi(\max\{\alpha(x_{n-1},x_{n-2},\frac{t}{k}),\min\{\gamma(x_{n},x_{n-1},\frac{t}{k}),\gamma(x_{n-2},x_{n-1},\frac{t}{k})\}\})\\
&= \psi(\gamma(x_{n-1},x_{n-2},\frac{t}{k}))\\
&\le \gamma(x_{n-1},x_{n-2},\frac{t}{k})\\
&= \gamma(fx_{n-2},fx_{n-3},\frac{t}{k})\\
&\le \psi(\max\{\gamma(x_{n-2},x_{n-3},\frac{t}{k^2}),\min\{\gamma(fx_{n-2},x_{n-2},\frac{t}{k^2}),\gamma(x_{n-3},fx_{n-3},\frac{t}{k^2})\}\})\\
&= \psi(\gamma(x_{n-2},x_{n-3},\frac{t}{k^2}))\le \gamma(fx_{n-3},fx_{n-4},\frac{t}{k^2})\\
&\le \dots \le \gamma(x_1,x_0,\frac{t}{k^{n-1}}). \qquad (3.4)
\end{aligned}$$

Also , for all $t > 0$, $p \geq 1$, we have

$$\begin{aligned}
\alpha(x_n, x_{n+p}, t) &\geq \alpha(x_n, x_{n+1}, \frac{t}{2b}) * \alpha(x_{n+1}, x_{n+p}, \frac{t}{2b}), \\
\beta(x_n, x_{n+p}, t) &\leq \beta(x_n, x_{n+1}, \frac{t}{2b}) * \alpha(x_{n+1}, x_{n+p}, \frac{t}{2b}), \\
\gamma(x_n, x_{n+p}, t) &\leq \gamma(x_n, x_{n+1}, \frac{t}{2b}) * \alpha(x_{n+1}, x_{n+p}, \frac{t}{2b}).
\end{aligned}$$

Continuing in this way we get

$$\begin{aligned}
\alpha(x_n, x_{n+p}, t) &\geq \alpha(x_n, x_{n+1}, \frac{t}{2b}) * \alpha(x_{n+1}, x_{n+2}, \frac{t}{(2b)^2}) * \ldots * \alpha(x_{n+p-1}, x_{n+p}, \frac{t}{(2b)^{p-1}}), \\
\beta(x_n, x_{n+p}, t) &\leq \beta(x_n, x_{n+1}, \frac{t}{2b}) * \beta(x_{n+1}, x_{n+2}, \frac{t}{(2b)^2}) * \ldots * \beta(x_{n+p-1}, x_{n+p}, \frac{t}{(2b)^{p-1}}), \\
\gamma(x_n, x_{n+p}, t) &\leq \gamma(x_n, x_{n+1}, \frac{t}{2b}) * \gamma(x_{n+1}, x_{n+2}, \frac{t}{(2b)^2}) * \ldots * \gamma(x_{n+p-1}, x_{n+p}, \frac{t}{(2b)^{p-1}}).
\end{aligned}$$

Using (3.2)-(3.4) in the above inequalities, we conclude

$$\begin{aligned}
\alpha(x_n, x_{n+p}, t) &\geq \alpha(x_0, x_1, \frac{t}{2bk^{n-1}}) * \alpha(x_0, x_1, \frac{t}{(2b)^2 k^n}) * \ldots * \alpha(x_0, x_1, \frac{t}{(2b)^{p-1} k^{n+p-3}}), \\
\beta(x_n, x_{n+p}, t) &\leq \beta(x_0, x_1, \frac{t}{2bk^{n-1}}) * \beta(x_0, x_1, \frac{t}{(2b)^2 k^n}) * \ldots * \beta(x_0, x_1, \frac{t}{(2b)^{p-1} k^{n+p-3}}), \\
\gamma(x_n, x_{n+p}, t) &\leq \gamma(x_0, x_1, \frac{t}{2bk^{n-1}}) * \gamma(x_0, x_1, \frac{t}{(2b)^2 k^n}) * \ldots * \gamma(x_0, x_1, \frac{t}{(2b)^{p-1} k^{n+p-3}}).
\end{aligned}$$

Using the properties of the definition 2.3, in the above inequalities we get

$$\begin{aligned}
&\lim_{n\to\infty} \alpha(x_n, x_{n+p}, t) = 1 * 1 * \ldots * 1 = 1, \\
&\lim_{n\to\infty} \beta(x_n, x_{n+p}, t) = 0 * 0 * \ldots * 0 = 0, \\
&\lim_{n\to\infty} \gamma(x_n, x_{n+p}, t) = 0 * 0 * \ldots * 0 = 0. \qquad (3.5)
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in χ. Thus $\{x_n\}$ is $\mathfrak{R} - (\alpha, \beta, \gamma)$- Cauchy sequence, and since $(\chi, \alpha, \beta, \gamma, *, \circ, b)$ is $\mathfrak{R} - (\alpha, \beta, \gamma)-$ complete, there exists $x \in \chi$ such that $x_n \to x$.
Now, If f is continuous, then taking the limit as $n \to \infty$ on the both sides of the $x_{n+1} = fx_n, n \in \mathbb{N}_0$, we get $x = fx$.
Otherwise, If $\mathfrak{R}$ is (α, β, γ)- self closed, then there exists $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \mathfrak{R} x, \forall k \in \mathbb{N}_0$.

Since $\lim_{n\to\infty} x_{n_k} = x$, we have for $t > 0$,

$$\begin{aligned}
\lim_{n\to\infty} \alpha(x_{n_k}, x, t) &= 1,\\
\lim_{n\to\infty} \beta(x_{n_k}, x, t) &= 0,\\
\lim_{n\to\infty} \gamma(x_{n_k}, x, t) &= 0. \qquad (3.6)
\end{aligned}$$

As $x_{n_k} \Re x, \forall k \in \mathbb{N}_0$, by condition 3.1, we get

$$\begin{aligned}
\psi(\alpha(fx_{n_k}, fx, kt)) &\geq \min\{\alpha(x_{n_k}, x, t), \max\{\alpha(fx_{n_k}, x_{n_k}, t), \alpha(x, fx, t)\}\},\\
\beta(fx_{n_k}, fx, kt) &\leq \psi(\max\{\beta(x_{n_k}, x, t), \min\{\beta(fx_{n_k}, x_{n_k}, t), \beta(x, fx, t)\}\}),\\
\gamma(fx_{n_k}, fx, kt) &\leq \psi(\max\{\gamma(x_{n_k}, x, t), \min\{\gamma(fx_{n_k}, x_{n_k}, t), \gamma(x, fx, t)\}\}).
\end{aligned}$$

Hence

$$\begin{aligned}
\psi(\alpha(x_{n_k+1}, fx, kt)) &\geq \min\{\alpha(x_{n_k}, x, t), \max\{\alpha(x_{n_k+1}, x_{n_k}, t), \alpha(x, fx, t)\}\},\\
\beta(x_{n_k+1}, fx, kt) &\leq \psi(\max\{\beta(x_{n_k}, x, t), \min\{\beta(x_{n_k+1}, x_{n_k}, t), \beta(x, fx, t)\}\}),\\
\gamma(x_{n_k+1}, fx, kt) &\leq \psi(\max\{\gamma(x_{n_k}, x, t), \min\{\gamma(x_{n_k+1}, x_{n_k}, t), \gamma(x, fx, t)\}\}).
\end{aligned}$$

Letting $n \to \infty$, we find

$$\begin{aligned}
\lim_{n\to\infty} \psi(\alpha(x_{n_k+1}, fx, kt)) &\geq \min\{1, \max\{1, \lim_{n\to\infty} \alpha(x_{n_k+1}, x_{n_k}, t), \alpha(x, fx, t)\}\} = 1,\\
\lim_{n\to\infty} \beta(x_{n_k+1}, fx, kt) &\leq \psi(\max\{0, \min\{0, \lim_{n\to\infty} \beta(x_{n_k+1}, x_{n_k}, t), \beta(x, fx, t)\}\}) = \psi(0) = 0,\\
\lim_{n\to\infty} \gamma(x_{n_k+1}, fx, kt) &\leq \psi(\max\{0, \min\{0, \lim_{n\to\infty} \gamma(x_{n_k+1}, x_{n_k}, t), \gamma(x, fx, t)\}\}) = \psi(0) = 0.
\end{aligned}$$

Thus, using (3) of the remark 2.1, we get

$$\begin{aligned}
\lim_{n\to\infty} \alpha(x_{n_k+1}, fx, t) \geq \lim_{n\to\infty} \alpha(x_{n_k+1}, fx, kt) &= 1,\\
\lim_{n\to\infty} \beta(x_{n_k+1}, fx, t) \leq \lim_{n\to\infty} \beta(x_{n_k+1}, fx, kt) &= 0,\\
\lim_{n\to\infty} \gamma(x_{n_k+1}, fx, t) \leq \lim_{n\to\infty} \gamma(x_{n_k+1}, fx, kt) &= 0.
\end{aligned}$$

Thus $x = fx$ that is x a fixed point of f. □

For the uniqueness of the fixed point, we add suitable condition to the theorem 3.1 as follows:

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1, the following condition holds: (iv) for all $x,y \in Fix(f)$ there exists $z \in \chi$ such that $x\Re z$ and $y\Re z$. Then the fixed point is unique.*

Proof. Theorem 3.1 decides that $Fix(f) \neq \phi$. Let $x,y \in Fix(f)$, by condition (iv), there exists $z \in \chi$ such that $x\Re z$ and $y\Re z$. Define $z_0 = z$ and $z_{n+1} = fz_n, \forall n \geq 0$, we claim that $x = y$. As $x\Re z_0$, we have for all $t > 0, k \in (0, \frac{1}{b})$ that,

$$\begin{aligned}
\alpha(fx, fz_0, t) &\geq \alpha(fx, fz_0, kt) \\
&\geq \psi(\alpha(fx, fz_0, kt)) \geq \min\{\alpha(x, z_0, t), \max\{\alpha(fx, x, t), \alpha(z_0, fz_0, t)\}\}, \\
\Rightarrow \alpha(x, z_1, t) &\geq \psi(\alpha(fx, fz_0, kt)) \geq \min\{\alpha(x, z_0, t), \max\{\alpha(x, x, t), \alpha(z_0, z_1, t)\}\}, \\
\Rightarrow \alpha(x, z_1, t) &\geq \psi(\alpha(fx, fz_0, kt)) \geq \min\{\alpha(x, z_0, t), \max\{1, \alpha(z_0, z_1, t)\}\}, \\
\Rightarrow \alpha(x, z_1, t) &\geq \psi(\alpha(fx, fz_0, kt)) \geq \min\{\alpha(x, z_0, t), 1\}, \\
\Rightarrow \alpha(x, z_1, t) &\geq \psi(\alpha(fx, fz_0, kt)) \geq \alpha(x, z_0, t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\beta(fx, fz_0, t) \leq \beta(fx, fz_0, kt) &\leq \psi(\max\{\beta(x, z_0, t), \min\{\beta(fx, x, t), \beta(z_0, fz_0, t))\}\}), \\
\Rightarrow \beta(x, z_1, t) &\leq \psi(\max\{\beta(x, z_0, t), \min\{0, \beta(z_0, z_1, t))\}\}), \\
\Rightarrow \beta(x, z_1, t) &\leq \psi(\max\{\beta(x, z_0, t), 0\}), \\
\Rightarrow \beta(x, z_1, t) &\leq \psi(\beta(x, z_0, t)) \leq \beta(x, z_0, t),
\end{aligned}$$

and

$$\begin{aligned}
\gamma(fx, fz_0, t) \leq \gamma(fx, fz_0, kt) &\leq \psi(\max\{\gamma(x, z_0, t), \min\{\gamma(fx, x, t), \gamma(z_0, fz_0, t))\}\}), \\
\Rightarrow \gamma(x, z_1, t) &\leq \psi(\max\{\gamma(x, z_0, t), \min\{0, \gamma(z_0, z_1, t))\}\}), \\
\Rightarrow \gamma(x, z_1, t) &\leq \psi(\max\{\gamma(x, z_0, t), 0\}), \\
\Rightarrow \gamma(x, z_1, t) &\leq \psi(\gamma(x, z_0, t)) \leq \gamma(x, z_0, t).
\end{aligned}$$

Since $\Re$ is f- closed, we conclude by induction that $x\Re z_n, \forall n \in \mathbb{N}_0$, using the same way above, we can find

$$\alpha(x,z_{n+1},t) \geq \psi(\alpha(x,z_{n+1},t) \geq \alpha(x,z_n,t);$$

$$\beta(x,z_{n+1},t) \leq \psi(\beta(x,z_n,t)) \leq \beta(x,z_n,t),$$

$$\gamma(x,z_{n+1},t) \leq \psi(\gamma(x,z_n,t)) \leq \gamma(x,z_n,t). \tag{3.7}$$

Thus, $\{\alpha(x,z_n,t)\}$ is non-decreasing and bounded above, $\{\beta(x,z_n,t)\},\{\gamma(x,z_n,t)\}$ are non-increasing and bounded below. Hence there are $0 < \alpha(t) \leq 1, 0 \leq \beta(t) < 1$ and $0 \leq \gamma(t) < 1$ such that for $t > 0$ we have $\lim_{n\to\infty} \alpha(x,z_n,t) = \alpha(t), \lim_{n\to\infty} \beta(x,z_n,t) = \beta(t)$ and $\lim_{n\to\infty} \gamma(x,z_n,t) = \gamma(t)$. Thus

$$\lim_{n\to\infty} \alpha(x,z_n,t) = 1,\ \lim_{n\to\infty} \beta(x,z_n,t) = 0,\ \lim_{n\to\infty} \gamma(x,z_n,t) = 0,\ \forall t > 0.$$

By the same way, we can prove that

$$\lim_{n\to\infty} \alpha(y,z_n,t) = 1,\ \lim_{n\to\infty} \beta(y,z_n,t) = 0,\ \lim_{n\to\infty} \gamma(y,z_n,t) = 0,\ \forall t > 0.$$

Since

$$\begin{aligned} \alpha(x,y,t) &\geq \alpha(x,z_n,\frac{t}{2b}) * \alpha(z_n,y,\frac{t}{2b}), \\ \alpha(x,y,t) &\leq \beta(x,z_n,\frac{t}{2b}) \circ \beta(z_n,y,\frac{t}{2b}), \\ \alpha(x,y,t) &\leq \gamma(x,z_n,\frac{t}{2b}) \circ \gamma(z_n,y,\frac{t}{2b}). \end{aligned}$$

Letting $n \to \infty$, by the continuity of $*, \circ$, we find

$$\alpha(x,y,t) \geq 1 * 1 = 1 \quad \Rightarrow \quad \alpha(x,y,t) = 1,$$

$$\beta(x,y,t) \leq 0 \circ 0 = 0 \quad \Rightarrow \quad \beta(x,y,t) = 0,$$

$$\gamma(x,y,t) \leq 0 \circ 0 = 0 \quad \Rightarrow \quad \gamma(x,y,t) = 0.$$

Hence $x = y$. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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