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EXISTENCE RESULTS FOR SINGLE AND MULTIVALUED MAPPINGS IN METRIC SPACES WITH APPLICATIONS

RAJENDRA PANT*, DEEPAK KHANTWAL

Department of Mathematics and Applied Mathematics, University of Johannesburg, Kingsway Campus,

Auckland Park 2006, South Africa

Abstract. We present some new existence results for single and multivalued mappings in metric spaces on very general settings. Some illustrative examples are presented to validate our theorems. Finally, we discuss an application to the Volterra-type integral inclusions.

Keywords: Fixed point; Proinov contraction; Suzuki contraction; Hausdorff distance.

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1. Introduction

The Banach contraction principle (BCP) is one of the most simple, elegant and classical tool in non-linear analysis with numerous extensions and generalizations. Some of the notable early generalizations and extensions of the BCP can be found in [3, 5, 6, 8, 12, 17, 19, 26]. For more details, one may refer to Rhoades [27]. Some recent extensions and generalizations of the BCP can be found in [10, 13, 22, 25, 29, 33]. In 1969, Boyd and Wong [3], presented an important generalization of the BCP by replacing the contraction constant with a real valued control function.

*Corresponding author

E-mail addresses: pant.rajendra@gmail.com, rpant@uj.ac.za

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Theorem 1.1. *Assume that (E,ρ) is a complete metric space and $f : E \to E$ a self-mapping satisfying*

$$\rho(f(x), f(y)) \leq \psi(\rho(x,y)) \text{ for all } x,y \in E,$$

where $\psi : [0,\infty) \to [0,+\infty)$ is upper semi-continuous from the right such that $\psi(t) < t$ for all $t > 0$. Then f admits a unique fixed point $z \in E$.

In 2006, Proinov [25] unified the Boyd-Wong [3], Jachymski [10], Matkowski [16] and Meir-Keeler [17] type contractions and proved the following interesting generalization of the BCP.

Theorem 1.2. [25]. *Let (E,ρ) be a complete metric space and $f : E \to E$ a continuous and asymptotically regular mapping such that*

(P1): $\rho(f(x), f(y)) \leq \psi(L(x,y))$ *for all $x,y \in E$;*

(P2): $\rho(f(x), f(y)) < L(x,y)$ *for all $x,y \in E$ whenever $L(x,y) \neq 0$;*

where $\mu \geq 0$, $L(x,y) = \rho(x,y) + \mu[\rho(x, f(x)) + \rho(y, f(y))]$ and $\psi : [0,\infty) \to [0,\infty)$ is such that for each $\xi > 0$ there exists a $\delta > \xi$ such that $\xi < t < \delta$ implies $\psi(t) \leq \xi$. Then f admits a unique fixed point in E.

Moreover, if $\mu = 1$ and ψ is continuous with $\psi(t) < t$ for all $t > 0$ then continuity of f is not needed.

Boyd and Wong [3] also pointed out that the requirement of upper semicontinuity of the control function ψ can be further weaken in Theorem 1.1. In 2016, Song-il Ri [29] used the Boyd-Wong's idea and obtained a generalization of the BCP and Theorem 1.1.

Lemma 1.3. [29]. *Let (E,ρ) be a metric space and $f : E \to E$ a self-mapping satisfying*

$$\rho(f(x), f(y)) \leq \varphi(\rho(x,y)) \text{ for all } x,y \in E, \tag{1.1}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is such that for all $t > 0$, $\varphi(t) < t$ and $\limsup_{s \to t^+} \varphi(s) < t$. Then the Picard sequence of iterates $(f^n x)$ at any point $x \in E$ is a Cauchy.

Theorem 1.4. [29]. *Suppose all the assumptions of Lemma 1.3 are true and (E,ρ) is complete. Then f admits a unique fixed point in E.*

In 2018, Bisht [2] claimed that the Lemma 1.3 is incorrect. He presented a counter example for his claim. He obtained the following theorem.

Theorem 1.5. [2]. *Suppose (E,ρ) is a metric space and $f : E \to E$ a mapping such that for some x_0 in E,*

$$\rho(f(x),f(y)) \leq \varphi_{x_0}(M(x,y)) \text{ for all } x,y \in \overline{O(x_0,f)} \text{ with } x \neq y, \tag{1.2}$$

where $M(x,y) = \max\{\rho(x,y), ad(x,f(x))+(1-a)\rho(y,f(y)), (1-a)\rho(x,f(x))+ad(y,f(y))\}$, $\varphi_{x_0} : (0,\infty) \to (0,\infty)$ with $\varphi_{x_0}(t) < t$ and $\limsup_{s\to t^+} \varphi_{x_0}(s) < t$ for all $t > 0$ and $0 < a < 1$. If E is f-orbitally complete then the Picard sequence of iterates $(f^n(x_0))$ is Cauchy in E and $\lim_{n\to\infty} f^n(x_0) = z$ for some $z \in E$. If f is orbitally continuous at z then z is a unique fixed point of f in $\overline{O(x_0,f)}$.

In [7], authors pointed out that Bisht's claim was incorrect (see [7, Example 3]).

In 2008, Suzuki [33] introduced a new class of contraction mappings where the contraction condition to be hold only on certain elements of the underlying space. He presented a remarkable generalization of the BCP which also characterises completeness of the metric space.

Theorem 1.6. [33]. *Suppose (E,ρ) is a complete metric space and $f : E \to E$ a self-mapping such that for all $x,y \in E$ and $h \in [0,1)$,*

$$\theta(h)d(x,f(x)) \leq \rho(x,y) \Longrightarrow d(f(x),f(y))) \leq h\rho(x,y),$$

where $\theta : [0,1) \to \left(\frac{1}{2},1\right]$ is a non-decreasing function such that

$$\theta(h) = \begin{cases} 1, & \text{if } 0 \leq h \leq \dfrac{\sqrt{5}-1}{2}, \\ \dfrac{(1-h)}{h^2}, & \dfrac{\sqrt{5}-1}{2} \leq h \leq \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{(1+h)}, & \text{if } \dfrac{1}{\sqrt{2}} \leq h < 1. \end{cases}$$

Then f has a unique fixed point in E.

Motivated by [3], [25], [29], [33] and others, we obtain some new fixed point results for single and multivalued mappings. The article is organized as follows. The section 1 is introductory. In section 2, we obtain a generalization of the BCP, Theorems 1.1, 1.4 and 1.5. In section 3, we present some results for multivalued mappings which extend Theorem 1.6, [14, Theorem 2] and generalize Theorem 1.1, 1.4, [23, Theorem 2.2], [31, Theorem 2.1] and [24, Theorem 5]. In section 4, we discuss an application to Volterra-type integral inclusion problems.

2. Fixed Point Results for Single-Valued Mappings

Now onwards, $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{R}$ the set of all real numbers, and $\varphi : [0,\infty) \to [0,\infty)$ is a function defined in Lemma 1.3. Let (E,ρ) be a metric space and $f : E \to E$ be a mapping. The orbit of f at some $u_0 \in E$ is defined as

$$O(u_0, f) := \{u_0, f(u_1), f(u_2), \dots\}.$$

The closure of $O(u_0, f)$ is denoted by $\overline{O(u_0, f)}$. If every Cauchy sequence in the orbit $O(u, f)$ for some $u \in E$ converges in E then E is said to be f-orbitally or orbitally complete [5].

The following lemma will be used to prove our theorems.

Lemma 2.1. *Assume that (E,ρ) be a metric space. Let (v_n) be a non-Cauchy sequence in E such that $\lim_{n\to\infty} \rho(v_n, v_{n+1}) = 0$. Then there exist an $\xi > 0$ and two sequences $(m(k))$ and $(n(k))$ of positive integers such that:*

- **(i):** $\rho(v_{n(k)}, v_{n(k)+1}) < \xi$ *for all* $n(k) \geq k \in \mathbb{N}$;
- **(ii):** $\rho(v_{n(k)}, v_{m(k)}) \geq \xi$ *and* $\rho(v_{m(k)}, v_{n(k)-1}) < \xi$ *for all* $n(k) > m(k) \geq k \in \mathbb{N}$;
- **(iii):** $\lim_{k\to\infty} \rho(v_{m(k)}, v_{n(k)}) = \lim_{k\to\infty} \rho(v_{m(k)}, v_{n(k)+1}) = \xi = \lim_{k\to\infty} \rho(v_{m(k)-1}, v_{n(k)})$
$= \lim_{k\to\infty} \rho(v_{m(k)-1}, v_{n(k)+1})$.

Proof. Proof is trivial. Therefore we omit it. □

Next, we present our main result of this section.

Theorem 2.2. *Suppose (E,ρ) is a metric space. Let $f : E \to E$ a mapping such that for some v_0 in E,*

$$\frac{1}{2}\rho(x, f(x)) \leq \rho(x,y) \implies \rho(f(x), f(y)) \leq \psi(N(x,y)) \text{ for all } x,y \in \overline{O(v_0, f)} \text{ with } x \neq y, \tag{2.1}$$

where $N(x,y) = \max\left\{\rho(x,y), \rho(x,f(x)), \rho(y,f(y)), \frac{\rho(y,f(x))+\rho(x,f(y))}{2}\right\}$. *If* E *is* f-*orbitally complete then the sequence of iterations* $(f^n(v_0))$ *is Cauchy in* E *and converges to the unique fixed point of* f *in* $\overline{O(v_0,f)}$.

Proof. Choose $v_0 \in E$. Define $v_n = f^n(v_0)$ for all $n \in \mathbb{N}$. Set $\rho_n := \rho(v_n, v_{n-1})$ then $\rho_n \geq 0$. If for any $j \in \mathbb{N}$, $\rho_j = 0$ then $v_j = v_{j+1}$. This implies v_j is a fixed point of f. Assume that $\rho_n > 0$ for all $n \in \mathbb{N}$. Since $\frac{1}{2}\rho(v_n, v_{n+1}) \leq \rho(v_n, v_{n+1})$ for all $n \in \mathbb{N}$, by (2.1),

$$\rho_{n+1} \leq \varphi\left(N(v_n, v_{n+1})\right) < N(v_n, v_{n+1}) = \rho_n. \tag{2.2}$$

Thus $\rho_{n+1} \leq \varphi(\rho_n) < \rho_n$. So, the sequences (ρ_n) and $(\varphi(\rho_n))$ are bounded below and monotone decreasing. This implies that $\lim_{n\to\infty} \rho_n$ and $\lim_{n\to\infty} \varphi(\rho_n)$ exist.

Suppose $\lim_{n\to\infty} \rho_n = \rho > 0$ and $\rho_n = \rho + \xi_n$ with $\xi_n > 0$. Since for all $t > 0$, $\limsup_{s\to t^+} \varphi(s) < t$ for (t_n) with $t_n \downarrow \rho^+$, we have $\limsup_{t_n\to\rho^+} \varphi(t_n) < \rho$. Hence, we get

$$\begin{aligned}
0 < \rho = \lim_{n\to+\infty} \rho_{n+1} &\leq \lim_{n\to+\infty} \varphi(\rho_n) \leq \lim_{n\to+\infty} \sup_{s\in(\rho,\rho_{n+1})} \varphi(s) \\
&= \lim_{\rho_{n+1}\to+0} \sup_{s\in(\rho,\rho+\xi_{n+1})} \varphi(s) \leq \lim_{\xi\to+0} \sup_{s\in(\rho,\rho+\xi)} \varphi(s) < \rho,
\end{aligned}$$

a contradiction. Thus $\lim_{n\to\infty} \rho_n = 0$. This shows that f is asymptotically regular at point v_0 of E. Using Lemma 2.1, it is easy to show the sequence (v_n) is Cauchy. Since E is f-orbitally complete therefore there exists $z \in E$ such that $v_n \to z$ as $n \to \infty$. Also, the sequence of iterations $(v_n) \in O(v_0,f)$ therefore its limit $z \in \overline{O(v_0,f)}$.
Now, for all $n \in \mathbb{N}$, we show that

$$\text{either } \frac{1}{2}\rho(v_n, v_{n+1}) \leq \rho(v_n, z) \text{ or } \frac{1}{2}\rho(v_{n+1}, v_{n+2}) \leq \rho(v_{n+1}, z). \tag{2.3}$$

Assuming by contradiction, we suppose that

$$\frac{1}{2}\rho(v_n, v_{n+1}) > \rho(v_n, z) \text{ and } \frac{1}{2}\rho(v_{n+1}, v_{n+2}) > \rho(v_{n+1}, z)$$

for all $n \in \mathbb{N}$. Then, by triangle inequality we have

$$\begin{aligned}\rho(v_n, v_{n+1}) &\leq \rho(v_n, z) + \rho(z, v_{n+1}) \\ &< \frac{1}{2}\rho(v_n, v_{n+1}) + \frac{1}{2}\rho(v_{n+1}, v_{n+2}) \\ &< \frac{1}{2}\rho(v_n, v_{n+1}) + \frac{1}{2}\rho(v_n, v_{n+1}) = \rho(v_n, v_{n+1}).\end{aligned}$$

This is a contradiction. Thus, the inequality (2.3) is true for all $n \in \mathbb{N}$.

In the first case, since $\frac{1}{2}\rho(v_n, v_{n+1}) = \frac{1}{2}\rho(v_n, f(v_n)) \leq \rho(v_n, z)$ by (2.1), we have

$$\rho(v_{n+1}, f(z)) \leq \varphi(N(v_n, z)),$$

where $N(v_n, z) = \max\left\{\rho(v_n, z), \rho(v_n, f(v_n)), \rho(z, f(z)), \frac{\rho(v_n, f(z)) + \rho(z, f(v_n))}{2}\right\}$.
Making $n \to \infty$, we get $\rho(z, f(z)) \leq \lim_{n\to\infty} \varphi(N(v_n, z))$. Also $\lim_{n\to\infty} N(v_n, z) = \rho(z, f(z))$. Let $\rho^* = \rho(z, f(z))$. Then by $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$, we obtain

$$\rho^* \leq \lim_{n\to\infty} \varphi(N(v_n, z)) \leq \lim_{\delta\to+0} \sup_{s\in(\rho^*, \rho^*+\delta)} \varphi(s) < \rho^*,$$

which is a contradiction unless $f(z) = z$. Similarly, in the other case, we can deduce that $f(z) = z$. Uniqueness of fixed point follows easily. $\square$

Example 2.3. Let $E = \mathbb{R}\times\mathbb{R}\backslash\{(1,1)\}$ and let ρ be a standard usual metric on E. Assume that

$$u_k = 12 - (-1)^k \left(\frac{1}{3}\right)^k,\ w_k = \frac{11}{4}\left\{2 - \left(\frac{1}{3}\right)^k\right\},\ k \in \mathbb{N}.$$

Define the mapping $f : E \to E$ by

$$f(u, w) = \begin{cases} (10, 0), & \text{if } (u, w) = (0, 0) \\ (10, 11), & \text{if } (u, w) = (0, 10) \\ (u_{k+1}, w_{k+1}), & \text{if } (u, w) = (u_k, w_k), \\ (12, 5.5), & \text{otherwise.} \end{cases}$$

Take $v_0 = (u_1, w_1)$ and $\varphi(t) = \frac{t}{3}$. Then $\overline{O(v_0, f)} = \{(u_1, w_1), (u_2, w_2), (u_3, w_3), \ldots, (12, 5.5)\}$. For $x = (u_k, w_k)$ and $y = (u_m, w_m)$ with $m > k \in \mathbb{N}$, we get $\frac{1}{2}\rho(x, f(x)) \leq \rho(x, y)$,

$$\begin{aligned}
\rho(f(x), f(y)) = \rho\left((u_{k+1}, w_{k+1}), (u_{m+1}, w_{m+1})\right) &= \sqrt{(u_{k+1} - u_{m+1})^2 + (w_{k+1} - w_{m+1})^2} \\
&= \sqrt{(u_{k+1} - u_{m+1})^2 + 7.5625(u_{k+1} - u_{m+1})^2} = 2.75\left|u_{k+1} - u_{m+1}\right| \\
&= \frac{2.75}{3}\left|u_k - u_m\right|
\end{aligned}$$

and

$$\begin{aligned}
\rho(x, y) = \rho\left((u_k, w_k), (u_m, w_m)\right) &= \sqrt{(u_k - u_m)^2 + (w_k - w_m)^2} \\
&= \sqrt{(u_k - u_m)^2 + 7.5625(u_k - u_m)^2} = 2.75\left|u_k - u_m\right|.
\end{aligned}$$

Hence $\rho(f(x), f(y)) = \frac{1}{3}\rho(x, y) \leq \varphi(N(x, y)$ and f satisfies condition (2.1) on $\overline{O(v_0, f)}$. Thus all the hypothesis of Theorem 2.2 are verified and f has a fixed point at $(12, 5.5)$.
However for $x = (0,0)$ and $y = (0,10)$, $\rho(f(x), f(y)) \geq \max\{\rho(x, y), \rho(x, f(x)), \rho(y, f(y))\}$. Thus, we can not apply Theorem 1.1, 1.4 , [15, Theorem 1.2] and [23, Theorem 2.2] here. Since E is not complete therefore Theorem 1.2 is also not applicable.

Now we present several consequences of Theorem 2.2. If we replace $N(x, y)$ with $m(x, y) = \max\{\rho(x, y), \rho(x, f(x)), \rho(y, f(y))\}$ in Theorem 2.2 then we get a weaker version of [23, Theorem 2.2].

Corollary 2.4. *Suppose (E, ρ) is a metric space. Let $f : E \to E$ a mapping such that for some v_0 in E,*

$$\frac{1}{2}\rho(x, f(x)) \leq \rho(x, y) \implies \rho(f(x), f(y)) \leq \psi(m(x, y)) \text{ for all } x, y \in \overline{O(v_0, f)} \text{ with } x \neq y.$$

If E is f-orbitally complete then the sequence of iterations $(f^n(v_0))$ is Cauchy in E and converges to the unique fixed point of f in $\overline{O(v_0, f)}$.

If we replace $N(x, y)$ with $M(x, y)$ in Theorem 2.2, then we get the following result.

Corollary 2.5. *Suppose (E,ρ) is a metric space. Let $f : E \to E$ a mapping such that for some v_0 in E,*

$$\frac{1}{2}\rho(x,f(x)) \le \rho(x,y) \implies \rho(f(x),f(y)) \le \psi(M(x,y)) \text{ for all } x,y \in \overline{O(v_0,f)} \text{ with } x \ne y.$$

If E is f-orbitally complete then the sequence of iterations $(f^n(v_0))$ is Cauchy in E and converges to the unique fixed point of f in $\overline{O(v_0,f)}$.

Proof. It is obvious that the value of $ad(x,f(x))+(1-a)\rho(y,f(y))$, where $0<a<1$, is strictly less than $\max\{\rho(x,f(x)),\rho(y,f(y))\}$ for all $x,y \in E$. Hence $M(x,y) < N(x,y)$. □

The following result is a consequence of Corollary 2.5 which generalizes the Theorem 1.5 without using the assumption of orbitally continuity on f.

Corollary 2.6. *Suppose (E,ρ) be metric space and $f : E \to E$ be a mapping such that for some $v_0 \in E$,*

$$\rho(f(x),f(y)) \le \psi(M(x,y)) \text{for all } x,y \in \overline{O(v_0,f)} \text{ with } x \ne y.$$

If E is f-orbitally complete then the sequence of iterations $(f^n(v_0))$ is Cauchy in E and converges to the unique fixed point of f in $\overline{O(v_0,f)}$.

If we take $N(x,y) = \rho(x,y)$ in Theorem 2.2, then we get following extension of Theorem 1.4 and 1.6.

Corollary 2.7. *Suppose (E,ρ) is a metric space. Let $f : E \to E$ a mapping such that for some v_0 in E,*

$$\frac{1}{2}\rho(x,f(x)) \le \rho(x,y) \text{ implies } \rho(f(x),f(y)) \le \varphi(\rho(x,y)) \text{ for all } x,y \in \overline{O(v_0,f)} \text{ with } x \ne y.$$

If E is f-orbitally complete then the sequence of iterations $(f^n(v_0))$ is Cauchy in E and converges to the unique fixed point of f in $\overline{O(v_0,f)}$.

Next, we get a generalized version of Matkowski's fixed point result [15].

Corollary 2.8. *Assume that $\gamma : [0,\infty) \to [0,\infty)$ an increasing such that $\lim_{n\to\infty} \gamma^n(t) = 0$ for every $t > 0$. Let (E,ρ) be a metric space and let $f : E \to E$ be a mapping such that for some $v_0 \in E$,*

$$\frac{1}{2}\rho(x,f(x)) \le \rho(x,y) \text{ implies } \rho(f(x),f(y)) \le \gamma(N(x,y)) \text{ for all } x,y \in \overline{O(v_0,f)} \text{ with } x \ne y.$$

If E is f-orbitally complete then the sequence of iterations $(f^n(v_0))$ is Cauchy in E and converges to the unique fixed point of f in $\overline{O(v_0,f)}$.

Similarly, we get a generalized version of Boyd and Wong's [3] result as follows:

Corollary 2.9. *Let (E,ρ) be a metric space and let $f: E \to E$ be a mapping such that for some $v_0 \in E$,*

$$\frac{1}{2}\rho(x,f(x)) \leq \rho(x,y) \text{ implies } \rho(f(x),f(y)) \leq \psi(N(x,y)) \text{ for all } x,y \in \overline{O(v_0,f)} \text{ with } x \neq y,$$

where ψ is defined in Theorem 1.1. *If E is f-orbitally complete then the sequence of iterations $(f^n(v_0))$ is Cauchy in E and converges to the unique fixed point of f in $\overline{O(v_0,f)}$.*

3. Fixed Point Results for Multivalued Mappings

In this section, we discuss fixed point results for multivalued mappings on very general settings. First, we recall some notations, definitions and results from [5], [19] and [20]. Let (E,ρ) be a metric space, $CB(E)$ the family of nonempty closed and bounded subsets of E and $C(E)$ the family of all nonempty compact subsets of E. For $A,B \in CB(E)$,

$$\rho(a,B) = \inf\{\rho(a,b) \ : \ b \in B\},$$

$$H(A,B) = \max\left\{\sup_{a\in A}\rho(a,B),\ \sup_{b\in B}\rho(A,b)\right\},$$

and

$$\delta(A,B) = \sup\{\rho(a,b) \ : \ a \in A,\ b \in B\}.$$

Lemma 3.1. [32, 34]. *Suppose (E,ρ) is a metric space and $A,B \in CB(E)$. If there exists $\vartheta > 0$ such that*

(1): *for every $a \in A$ there is $b \in B$ so that $\rho(a,b) \leq \vartheta$,*

(2): *for every $b \in B$ there is $a \in A$ so that $\rho(b,a) \leq \vartheta$.*

Then $H(A,B) \leq \vartheta$.

Definition 3.2. [28, 30]. A mapping $F: E \to CB(E)$ is called asymptotically regular at $v_0 \in E$, if for each sequence $(v_n) \in E$ such that $v_n \in F(v_{n-1})$, we have

$$\lim_{n\to\infty}\rho(v_n,v_{n+1}) = 0.$$

If F is asymptotically regular at each point v of E then we called F is asymptotically regular on E.

Example 3.3. Let $E = \left\{ \frac{1}{n} : \ n \in \mathbb{N} \right\} \cup \{0\}$ and $F : E \to CB(E)$ is defined by

$$F(x) = \begin{cases} \left\{ \frac{1}{n+1}, \frac{1}{n+2} \right\}, & \text{when } x = \frac{1}{n}, \\ \{0\}, & \text{when } x = 0. \end{cases}$$

Then it is easily seen that for each $u \in E$ and $(u_n) \in E$ such that $u_n \in F(u_{n-1}),\ \rho(u_n, u_{n+1}) \to 0$. Thus F is asymptotic regular on E.

In the above definition, if we take $F = f$ as a single-valued self mapping on E then we get the following definition of asymptotic regularity [4].

Definition 3.4. A mapping $f : E \to E$ is called asymptotically regular at some $v_0 \in E$ if

$$\lim_{n\to\infty} \rho(f^n(v_0), f^{n+1}(v_0)) = 0 \tag{3.1}$$

and if equality (3.1) is true for all $u \in E$, then the mapping f is called asymptotically regular on E.

A mapping f is asymptotically regular at its fixed points but the converse is not always true. The following example illustrates this fact.

Example 3.5. Let $E = [1, \infty)$ equipped with the usual metric ρ and let $f : E \to E$ be a mapping such that

$$f(x) = \log[e^x + 1] \text{ for all } x \in E.$$

Then for any $u \in E$, we have $f^n(u) = \log[e^u + n]$ and

$$\lim_{n\to\infty} \rho(f^n(u), f^{n+1}(u)) = 0.$$

However, the mapping f is fixed point free.

Definition 3.6. [5]. Let $F : E \to CB(E)$ be a mapping and $v_0 \in E$. If, there exists a sequence (v_n) in E such that $v_n \in F(v_{n-1})$ for all $n \in \mathbb{N}$ then $O(v_0, F) = \{v_0, v_1, v_2, \dots\}$ is called an orbit of F at point v_0.

Definition 3.7. A mapping $g : E \to \mathbb{R}$ is called lower semi-continuous on E, if for any $v \in E$ and $(v_n) \in E$ such that $v_n \to v$ implies $g(v) \leq \liminf_{n\to\infty} g(v_n)$.

Definition 3.8. [1]. A multivalued mapping $F : E \to CB(E)$ is said to be lower semi-continuous at v_0, if for any $y \in F(v_0)$ and for any sequence $v_n \in E$ converges to v_0, there exists a sequence $y_n \in F(v_n)$ converges to y. In other words, the mapping F is lower semi continuous at v_0 if for any $y_0 \in F(v_0)$ and any neighbourhood $V(y_0)$ of y_0, there exists an open neighbourhood U of v_0 such that $v \in U$, we have $F(v) \cap V(y_0) \neq \emptyset$.

Hicks and Rhoades [9] introduced the following notion of orbital lower semi-continuity.

Definition 3.9. Let (E, ρ) be a metric space and $f : E \to E$ be a mapping. A mapping $g : E \to \mathbb{R}$ is said to be a f-orbitally lower semi-continuous at a point $v_0 \in E$, if (v_n) is a sequence in $O(v, f)$ for some $v \in E$ such that $v_n \to v_0$ implies $g(v_0) \leq \liminf_{n\to\infty} g(v_n)$ (see also [11]).

Definition 3.10. [21]. A mapping $g : E \to \mathbb{R}$ is said to be F-orbitally lower semi-continuous, if (v_n) is a sequence in $O(v_0, F)$ and $v_n \to z$ implies $g(z) \leq \liminf_{n\to\infty} g(v_n)$.

Remark 3.11. The condition of F-orbitally continuity is more general than the condition of orbital continuity and k-continuity for $k \geq 1$ (see [21, Example 1]).

Now, we present a result of multivalued mappings on complete metric spaces which generalizes Theorem 1.2, 1.4, [23, Theorem 2.2] and [31, Theorem 2.1].

Theorem 3.12. *Suppose (E, ρ) is a complete metric space and $F : E \to C(E)$ a multivalued mapping. If F is an asymptotically regular at some point $v_0 \in E$ such that*

$$\frac{1}{2}\rho(x, F(x)) \leq \rho(x, y) \implies H(F(x), F(y)) \leq \varphi(D(x, y)), \tag{3.2}$$

for all $x, y \in E$, where $D(x, y) = \rho(x, y) + \mu[\rho(x, F(x)) + \rho(y, F(y))]$ and $\mu \geq 0$. Then, there exists a sequence of iterations $(v_n) \in E$ starting from v_0, converges to a point $z \in E$. If $g : E \to \mathbb{R}$ defined by $g(x) = \rho(x, F(x))$ for all $x \in E$, is lower semi-continuous at the point z then z is a fixed point of F.

Proof. Pick $v_0 \in E$ and $v_1 \in F(v_0)$. We choose a point $v_2 \in F(v_1)$ such that

$$\rho(v_1, v_2) \leq H(F(v_0), F(v_1)).$$

Such a choice is permissible because $F(v_1)$ is compact. Again since $F(v_2)$ is also compact, we can choose $v_3 \in F(v_2)$ such that

$$\rho(v_2, v_3) \leq H(F(v_1), F(v_2)).$$

Continuing in the same manner, we get a sequence $(v_n) \in E$ such that

$$v_{n+1} \in F(v_n) \text{ and } \rho(v_n, v_{n+1}) \leq H(F(v_{n-1}), F(v_n)) \text{ for all } n \in \mathbb{N}.$$

Since F is asymptotically regular at v_0 then

$$\lim_{n\to\infty} \rho(v_n, v_{n+1}) = 0 = \lim_{n\to\infty} \rho(v_n, F(v_n)).$$

Then by Lemma 2.1, there exist $\xi > 0$ and two positive sequences $n(k)$, $m(k)$ with $k \leq m(k) < n(k)$ such that

$$\rho(v_{m(k)}, v_{n(k)}) \geq \xi \tag{3.3}$$

and

$$\rho(v_{n(k)}, v_{n(k)+1}) < \xi \text{ for all } n \geq k \in \mathbb{N}. \tag{3.4}$$

By (3.3) and (3.4), we have

$$\frac{1}{2}\rho(v_{n(k)}, F(v)_{n(k)}) \leq \frac{1}{2}\rho(v_{n(k)}, v_{n(k)+1}) \leq \rho(v_{n(k)}, v_{m(k)}).$$

Then, by triangle inequality and contraction (3.2), we have

$$\begin{aligned}
\rho(v_{m(k)}, v_{n(k)}) &\leq \rho(v_{m(k)}, v_{m(k)+1}) + \rho(v_{m(k)+1}, v_{n(k)+1}) + \rho(v_{n(k)}, v_{n(k)+1}) \\
&\leq \rho(v_{m(k)}, v_{m(k)+1}) + H(F(v_{n(k)}), F(v_{m(k)})) + \rho(v_{n(k)}, v_{n(k)+1}) \\
&\leq \rho(v_{m(k)}, v_{m(k)+1}) + \varphi(D(v_{m(k)}, v_{n(k)})) + \rho(v_{n(k)}, v_{n(k)+1})).
\end{aligned}$$

Making $k \to +\infty$ and using Lemma 2.1, we have

$$\xi \leq \lim_{k\to+\infty} \varphi(D(v_{m(k)}, v_{n(k)})).$$

We also note that $\lim_{k\to+\infty} D(v_{m(k)}, v_{n(k)}) = \xi$. Then by $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$, we get

$$\xi \leq \lim_{k\to+\infty} \varphi(D(v_{m(k)}, v_{n(k)}) \leq \lim_{\xi'\to+0} \sup_{s\in(\xi,\xi+\xi')} \varphi(s) < \xi,$$

a contradiction unless $\xi = 0$. Thus, (v_n) is a Cauchy sequence. Now the E is complete there exists $z \in E$ such that $v_n \to z$ as $n \to \infty$. If g is lower semi-continuous at the point z then we have

$$\rho(z, F(z)) = g(z) \leq \liminf_{n\to\infty} g(v_n) = \liminf_{n\to\infty} \rho(v_n, F(v_n)) = 0.$$

The compactness of $F(z)$ implies $z \in F(z)$. □

The following example illustrates the validity of our Theorem 3.12.

Example 3.13. Let $E = [0,1]$ and $\rho(x,y) = |x-y|$ for $x, y \in E$ be a usual metric on E. Define the $F : E \to C(E)$ by

$$F(x) = \begin{cases} \left[0, \dfrac{1}{n+2}\right], & \text{if } x = \dfrac{1}{n},\ n \in \mathbb{N}, \\ \left\{\dfrac{x}{2}\right\}, & \text{otherwise.} \end{cases}$$

Then F is asymptotic regular on E and the mapping

$$g(x) = \begin{cases} \dfrac{2}{n(n+2)}, & \text{if } x = \dfrac{1}{n},\ n \in \mathbb{N}, \\ \left\{\dfrac{x}{2}\right\}, & \text{otherwise,} \end{cases}$$

is lower semi-continuous at $x = 0$. If $x,\ y \neq \frac{1}{n}$, then $H(F(x), F(y)) = \frac{|x-y|}{2} < |x-y|$. Let $x = \frac{1}{n}$ and $y \neq \frac{1}{n}$ then $H(F(x), F(y)) = H\left(\left[0, \frac{1}{(n+2)}\right], \left\{\frac{y}{2}\right\}\right) \leq \frac{y}{2} < \mu\rho(y, F(y)) = \mu\left|y - \frac{y}{2}\right|$ for $\mu \geq 2$. Similarly, when $x = \frac{1}{n}$ and $y = \frac{1}{m}$, we find that $H(F(x), F(y)) = \left|\frac{1}{n+2} - \frac{1}{m+2}\right| < \rho(x,y) = \left|\frac{1}{n} - \frac{1}{m}\right|$. Finally, for $x, y = \frac{1}{n}$, we have $\frac{1}{2}\rho(x, F(x)) > \rho(x,y) = 0$ implies (3.2) is obviously true. Hence all the assumptions of Theorem 3.12 are verified and F has a fixed point at $x = 0$.

If we take $F = f$, a single valued self mapping on E in Theorem 3.12 then we get a generalized version of [24, Theorem 5].

Corollary 3.14. *Let (E,ρ) be a complete metric space and $f: E \to E$ is an asymptotically regular at some point $v_0 \in E$ such that for all $x, y \in E$,*

$$\frac{1}{2}\rho(x, f(x)) \le \rho(x,y) \implies \rho(f(x), f(y)) \le \varphi(L(x,y)),$$

where $L(x,y)$ is defined in Theorem 1.2. Then the sequence of iterations $(v_n) \in E$ starting from v_0, converges to a point $z \in E$. If $g(x) = \rho(x, f(x))$ for all $x \in E$ is lower semi-continuous at the point z then z is a fixed point of f.

The following result is an extension of Theorem 2.2 in setting of multivalued mappings.

Theorem 3.15. *Let (E,ρ) be a metric space and let $F: E \to CB(E)$ be a multivalued asymptotically regular mapping at some point $v_0 \in E$ such that*

$$\frac{1}{2}\rho(x, F(x)) \le \rho(x,y) \implies \delta(F(x), F(y)) \le \varphi(D(x,y)) \text{ for } x \ne y \text{ with } x, y \in \overline{O(v_0, F)}, \tag{3.5}$$

where $D(x,y)$ is defined in Theorem 3.12. If E is f-orbitally complete then any sequence $(v_n) \subseteq O(v_0, F)$ is convergent to a point $z \in \overline{O(v_0, F)}$. If $g(x) = \rho(x, F(x))$ for all $x \in E$, is F-orbitally lower semi-continuous at the point z then z is a fixed point of F.

Proof. Let (v_n) be a sequence in $O(v_0, F)$. Since F is asymptotically regular at v_0 implies

$$\lim_{n\to\infty} \rho(v_n, v_{n+1}) = 0 = \lim_{n\to\infty} \rho(v_n, F(v_n)).$$

We claim that (v_n) is a Cauchy sequence. If not then by Lemma 2.1, there exist $\xi > 0$ and two sequences $m(k),\ n(k)$ with $k \le m(k) < n(k)$ such that

$$\rho(v_{m(k)}, v_{n(k)}) \ge \xi \tag{3.6}$$

and

$$\rho(v_{n(k)}, v_{n(k)+1}) < \xi \quad \text{for all } n(k) \ge k \in \mathbb{N}. \tag{3.7}$$

Furthermore, by (3.6) and (3.7), we have

$$\frac{1}{2}\rho(v_{n(k)}, F(v)_{n(k)}) \le \rho(v_{m(k)}, v_{n(k)}).$$

Then, by triangle inequality and contraction (3.5), we have

$$\begin{aligned}\rho(v_{m(k)},v_{n(k)}) &\leq \rho(v_{m(k)},v_{m(k)+1})+\rho(v_{m(k)+1},v_{n(k)+1})+\rho(v_{n(k)},v_{n(k)+1})\\ &\leq \rho(v_{m(k)},v_{m(k)+1})+\delta(F(v)_{m(k)},F(v)_{n(k)})+\rho(v_{n(k)},v_{n(k)+1})\\ &\leq \rho(v_{m(k)},v_{m(k)+1})+\varphi(D(v_{m(k)},v_{n(k)}))+\rho(v_{n(k)},v_{n(k)+1}).\end{aligned}$$

Letting $k\to+\infty$ and from Lemma 2.1, we get

$$\xi \leq \lim_{k\to+\infty}\varphi(D(v_{m(k)},v_{n(k)})).$$

Also, we note that $\lim\limits_{k\to+\infty} D(v_{m(k)},v_{n(k)})=\xi$ and by $\limsup\limits_{s\to t^+}\varphi(s)<t$ for all $t>0$, we get

$$\xi \leq \lim_{k\to+\infty}\varphi(D(v_{m(k)},v_{n(k)}) \leq \lim_{\xi'\to+0}\ \sup_{s\in(\xi,\xi+\xi')}\varphi(s)<\xi,$$

a contradiction. Thus (v_n) is a Cauchy sequence in $O(v_0,F)\subseteq E$ and by f-orbitally completeness, it converges to a point $z\in E$. Since z is the limit point of a sequence $(v_n)\in O(v_0,F)$ therefore $z\in\overline{O(v_0,F)}$. If $g(v)=\rho(v,F(v))$ is F-orbitally lower semi continuous at the point z then

$$\rho(z,F(z))=g(z)\leq \liminf_{n\to\infty} g(v_n)=\liminf_{n\to\infty}\rho(v_n,F(v)_n)=0.$$

The closedness of $F(z)$ implies $z\in F(z)$ and z is a fixed point of F. $\square$

Example 3.16. Let $E=\left\{\frac{1}{3^n}:n\in\mathbb{N}\right\}\cup\{0,1\}\cup\left\{2-\frac{1}{2^n}:n\in\mathbb{N}\right\}$ and ρ be a usual metric on E. Define $F:E\to E$ by

$$F(x)=\begin{cases}\left\{0,\ \frac{1}{3^{n+1}}\right\}, & \text{when } x=\frac{1}{3^n},\ n\in\mathbb{N},\\ \{x\}, & \text{when } x=\{0,\ 1\},\\ \left\{0,\ 2-\frac{1}{2^{n+1}}\right\}, & \text{when } x=2-\frac{1}{2^n}.\end{cases}$$

Then

$$\rho(x,F(x))=\begin{cases}\frac{2}{3^{n+1}}, & \text{when } x=\frac{1}{3^n},\ n\in\mathbb{N},\\ 0, & \text{when } x=\{0,1\},\\ \frac{1}{2^{n+1}}, & \text{if } x=2-\frac{1}{2^n},\end{cases}$$

is continuous mapping on E. One can easily verify that F is asymptotically regular at $v_0 = \dfrac{1}{3}$ and $\overline{O\left(\dfrac{1}{3}, F\right)} = \left\{\dfrac{1}{3^n} : n \in \mathbb{N}\right\} \cup \{0\}$.
If $x = \dfrac{1}{3^n}$ and $y = 0$ or $x = \dfrac{1}{3^n}$ and $y = \dfrac{1}{3^m}$ for $m > n \in \mathbb{N}$ then

$$\delta(F(x), F(y)) = \frac{1}{3^{n+1}} \leq \frac{1}{2}\rho(x, F(x)).$$

Taking $\varphi(t) = \dfrac{t}{2}$, we get $\delta(F(x), F(y)) \leq \varphi(\rho(x, f(x)))$. Hence F satisfies all the assumptions of Theorem 3.15 and F has a fixed point in E. Here F has two fixed points at $x = 0$ and 1 in E.

Now, for $x = 1$ and $y = 0$, we have $\dfrac{1}{2}\rho(x, F(x)) < \rho(x, y)$, $H(F(x), F(y)) = 1 = \rho(x, y)$ and $\rho(x, F(x)) = 0 = \rho(y, F(y))$ which implies F does not satisfy contraction conditions used in [14], [19] and [31].

If we take $F = f$, a single valued self mapping on E in Theorem 3.15 then we get a generalized version of Theorem 1.4, Theorem 1.5 and [23, Theorem 2.2].

Corollary 3.17. *Let (E, ρ) be a metric space and $f : E \to E$ be a asymptotically regular at some point $v_0 \in E$ such that*

$$\frac{1}{2}\rho(x, f(x)) \leq \rho(x, y) \implies \rho(f(x), f(y)) \leq \varphi(L(x, y)) \text{ for } x \neq y \text{ with } x, y \in \overline{O(v_0, f)},$$

where $L(x, y)$ is defined in Theorem 1.2. If E is f-orbitally complete then the sequence of iterations $(f^n(v_0))$ is convergent to a point $z \in E$. If $g(x) = \rho(x, f(x))$ for all $x \in E$ is f-orbitally lower semi-continuous at the point z then z is a fixed point of f.

Example 3.18. Let $E = \{5, 6, 7, 12\}$ and ρ be a usual metric on E. Define $f : E \to E$ by

$$f(5) = 5,\ f(6) = 12,\ f(7) = 5,\ f(12) = 7.$$

One can easily verify that at $x = 8$, f is asymptotically regular and $O(6, f) = E$. Also, for all $x,\ y \in E$

$$\rho(f(x), f(y)) \leq \left(\frac{7}{13}\right) L(x, y).$$

If we take $\varphi(t) = \dfrac{7}{13}t$ for $t > 0$ then we get $\rho(f(x), f(y)) \leq \varphi\big(L(x, y)\big)$. Thus all the assumptions of Corollary 3.17 are fulfilled and f has a fixed point at $z = 5$. However, if we take $x = 5$

and $y=6$ then $\rho(f(x),f(y))>m(x,y)$. Thus the contraction conditions used in Theorem 1.4, Theorem 1.5 and [23, Theorem 2.2] are not satisfied.

4. Application to Volterra-Type Integral Inclusion

In this section, we study the existence result for integral inclusion of Volterra-type. Consider $E=\mathscr{C}([a,b],\mathbb{R})$ the space of all continuous real valued functions. Then (E,ρ) is a complete metric space endowed with

$$\rho(\upsilon,\omega)=\sup_{t\in[a,b]}|\upsilon(t)-\omega(t)|.$$

Consider the integral inclusion

$$\upsilon(t)\in f(t)+\int_a^t G(t,s,\upsilon(s))ds,\ t\in[a,b], \tag{4.1}$$

where $G:[a,b]\times[a,b]\times\mathbb{R}\to C(\mathbb{R})$ and $C(\mathbb{R})$ is the class of non-empty closed and bounded subset of $\mathbb{R}$. Let $G_\upsilon(t,s)=G(t,s,\upsilon(s)),\ (t,s)\in[a,b]\times[a,b],\ \upsilon\in\mathbb{R}$ is a lower semi continuous and for $f\in E$, we define $F:E\to C(E)$ by

$$F(\upsilon(t))=\left\{\upsilon(t)\in E\ :\ \upsilon(t)\in f(t)+\int_a^t G(t,s,\upsilon(s))ds,\ t\in[a,b]\right\}\ \text{for all}\ \ \upsilon\in E. \tag{4.2}$$

A selection for F is a continuous mapping $f:E\to E$ such that $f(\upsilon)\in F(\upsilon)$ (see [18]) and Michael's selection theorem [18] implies there exists a continuous operator $k_\upsilon:[a,b]\times[a,b]\to\mathbb{R}$ such that $k_\upsilon(t,s)\in G(t,s,\upsilon(s))$ for $t,\ s\in[a,b]$ and $\upsilon\in E$. Hence $f(t)+\int_a^t k_\upsilon(t,s)ds\in F(\upsilon(t))$ and $F(\upsilon(t))\neq\emptyset$. Also, it is easy to see, $F(\upsilon(t))$ is a compact. Define

$$d(\upsilon(t),F(\upsilon(t)))=\inf\left\{\sup_{t\ \in[a,b]}|\upsilon(t)-\omega(t)|\ :\ \omega(t)\in F(\upsilon(t))\right\}.$$

Then, the mapping $g:E\to\mathbb{R}$, where $g(\upsilon)=d(\upsilon,F(\upsilon))$ is lower semi-continuous on E.

Now, under the above assumptions, we have the following theorem.

Theorem 4.1. *Let F be an asymptotic regular on E such that the following condition holds:*

$$\frac{1}{2}d(\upsilon(s),F(\upsilon(s)))\leq d(\upsilon(s),\omega(s))$$

implies

$$H\left(G(t,s,\upsilon(s)),G(t,s,\omega(s))\right) \le \varphi(D\big(\upsilon(s),\omega(s)\big)) \tag{4.3}$$

for all $s,t \in [a,b]$, $\upsilon,\omega \in E$, *where* $D(\upsilon(s),\omega(s)) = d(\upsilon(s),\omega(s)) + \mu\{d(\upsilon(s),F(\upsilon(s))) + d(\omega(s),F(\omega(s)))\}$ *and* $\mu \ge 0$. *Then, the integral inclusion* (4.1) *has a solution.*

Proof. Since the F is asymptotic regular and g is lower semi-continuous mapping on E. Now, we will prove that F satisfies condition (3.2) of Theorem 3.12. Let $\upsilon \in E$ and $\upsilon(t) \in F(\upsilon)$. Then by Michael's selection theorem, we have $k_\upsilon(t,s) \in G_\upsilon(t,s)$ for $t,s \in [a,b]$ such that

$$\upsilon(t) = f(t) + \int_a^t k_\upsilon(t,s)ds$$

and for any $\omega \in F(\upsilon)$,

$$\frac{1}{2}d(\upsilon(s),F(\upsilon(s))) \le d(\upsilon(s),\omega(s)).$$

Then condition (4.3) implies there exists $r(t,s) \in G_\omega(t,s)$ such that

$$|k_\upsilon(t,s) - r(t,s)| \le \varphi\left(d(\upsilon(s),\omega(s)) + \mu\{d(\upsilon(s),F(\upsilon(s))) + d(\omega(s),F(\omega(s)))\}\right),$$

for all $t,s \in [a,b]$.

Let us consider a multivalued operator S defined by

$$S(t,s) = G_\omega(t,s) \cap \left\{ \omega \in \mathbb{R} : |k_\upsilon(t,s) - \omega| \le \varphi \left(\begin{array}{l} d(\upsilon(s),\omega(s)) + \\ \mu\left\{d(\upsilon(s),F\upsilon(s)) + d(\omega(s),F(\omega(s)))\right\} \end{array} \right) \right\}$$

for all $t,s \in [a,b]$. Since S is a lower semi continuous, it follows that there exists a continuous mapping $k_\omega(t,s) \in S(t,s)$ for all $t,s \in [a,b]$ such that

$$\omega(t) = f(t) + \int_a^t k_\omega(t,s)ds \ \in F(\omega(s)) \ \text{ for } \ t \in [a,b]$$

and for any $t \in [a,b]$, we get

$$\begin{aligned} d(\upsilon(t),\omega(t)) &= \sup_{t\in[a,b]} |\int_a^t k_\upsilon(t,s)ds - \int_a^t k_\omega(t,s)ds| \le \int_a^t \sup_{t\in[a,b]} |k_\upsilon(t,s) - k_\omega(t,s)|ds \\ &\le \varphi(d\big(\upsilon(s),\omega(s)) + \mu\{d(\upsilon(s),F(\upsilon(s))) + d(\omega(s),F(\omega(s)))\}). \end{aligned}$$

Finally, in view of Lemma 3.1 and interchanging the role of υ and ω, we reach to

$$H(F(\upsilon),F(\omega)) \leq \varphi(d(\upsilon(s),\omega(s))+\mu\{d(\upsilon(s),F(\upsilon(s)))+d(\omega(s),F(\omega(s)))\}).$$

Thus, all the assumptions of Theorem 3.12 are verified and hence the inclusion problem (4.1) has a solution in E. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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