Available online at http://scik.org
Adv. Fixed Point Theory, 2023, 13:7
https://doi.org/10.28919/afpt/8060
ISSN: 1927-6303

SOME APPLICATIONS VIA $(\alpha,\varphi)-K$-TYPE CONTRACTION FIXED POINT THEOREMS IN PARTIAL b-METRIC SPACES

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Abstract: In the context of partial b-metric space, we demonstrate several common fixed point solutions for $(\alpha,\varphi)-K$-type contractive mappings in this study. We also look at a few integral equations applications. In order to support our conclusion, we also provided an example.

Keywords: partial b-metric space; weakly compatible mapping; rational contraction; tripled fixed points.
2020 AMS Subject Classification: 54H25, 47H10, 54E50.

1. INTRODUCTION

Due to its numerous applications in homotopy theory, integral, integro-differential, impulsive differential equations, approximation theory, and has been researched in multiple metric spaces, fixed point theory is one of the most fruitful roles in nonlinear analysis.

Czerwik [1], [2] expanded results pertaining to the b-metric spaces in 1993. The ideas of partial metric space, in which any points self-distance might not be zero, was first suggested by

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Received June 15, 2023

Matthews [3] in 1994. By inventing partial b-metric spaces in 2013, Shukla [5] expanded both the ideas of b-metric and partial metric spaces.

A few tripled fixed point findings for contractive type mappings with mixed monotone qualities in partially ordered metric spaces were established by Berinde and Borcut [6]. Borcut et al. [7] have created the concept of a tripled coincidence point for a pair of nonlinear contractive mappings. Aydi et al. [8] have studied the common tripled fixed point theorem for ω-compatible mappings in abstract metric spaces. Recently, Samet et al. [9] demonstrated fixed point theorems for such sorts of mappings in the complete metric spaces and developed the notion of α-contractive and α-admissible mappings. As may be seen in ([10]- [20]), numerous researchers have established tripleed fixed point results for various spaces.

A common tripled fixed point theorem for two mappings meeting $(\alpha,\varphi)-K$-type contractive requirements in partial b-metric space is demonstrated in this paper. We also look at several integral equations applications, and examples are also given.

First we recall some basic definitions and results.

2. PRELIMINARIES

Definition 2.1:([5]) Let $\Im$ be a nonempty set and $\kappa \geq 1$ be a given real number. A function $P_b : \Im \times \Im \to [0,\infty)$ is called a partial b- metric if for all $\wp_1, \wp_2, \wp_3 \in \Im$, the following conditions are satisfied:

(P_b1) $\wp_1 = \wp_2$ if and only if $P_b(\wp_1,\wp_1) = P_b(\wp_1,\wp_2) = P_b(\wp_2,\wp_2)$;

(P_b2) $P_b(\wp_1,\wp_1) \leq P_b(\wp_1,\wp_2)$;

(P_b3) $P_b(\wp_1,\wp_2) = P_b(\wp_2,\wp_1)$;

(P_b4) $P_b(\wp_1,\wp_2) \leq \kappa(P_b(\wp_1,\wp_3) + P_b(\wp_3,\wp_2) - P_b(\wp_3,\wp_3))$.

The pair $(\Im, P_b)$ is called a partial b-metric space.

Remark 2.2: Since a partial metric space is a special case of a partial b-metric space $(\Im, P_b)$ when $\kappa = 1$, the class of partial b-metric space $(\Im, P_b)$ is actually bigger than the class of partial metric space. Additionally, because a b-metric space is a particular instance of a partial b-metric space $(\Im, P_b)$ when the self-distance $P(\wp_1,\wp_1) = 0$, the class of partial b-metric space $(\Im, P_b)$ is effectively bigger than the class of b-metric space. The examples below demonstrate that a partial b-metric on $\Im$ need not be a partial metric, nor a b-metric on $\Im$ should be either [4], [5].

Example 2.3:([5]) Let $\Im=[0,1)$. Create a function P_b with the following formula $P_b(\partial_1;\partial_2)=[\max\{\partial_1,\partial_2\}]^2+|\partial_1-\partial_2|^2$, for all $\partial_1,\partial_2\in\Im$. When $\kappa=2>1$, the pair $(\Im,P_b)$ is referred to as a partial b-metric space. P_b, however, is neither a partial metric on $\Im$ nor a b-metric.

Definition 2.4:([4]) Every partial b-metric P_b defines a b-metric d_{P_b}, where $d_{P_b}(\partial_1,\partial_2)=2P_b(\partial_1,\partial_2)-P_b(\partial_1,\partial_1)-P_b(\partial_2,\partial_2)$, for all $\partial_1,\partial_2\in\Im$

Definition 2.5:([4]) A sequence $\{\wp_p\}$ in a partial b-metric space $(\Im,P_b)$ is said to be:

(i) P_b -convergent to a point $\wp\in\Im$ if $\lim_{p\to\infty}P_b(\wp,\wp_p)=P_b(\wp,\wp)$

(ii) a P_b -Cauchy sequence if $\lim_{p,q\to\infty}P_b(\wp_p,\wp_q)$ exists and is finite;

(iii) A partial b-metric space $(\Im,P_b)$ is said to be P_b -complete if every P_b-Cauchy sequence $\{\wp_p\}$ in $\Im$ is P_b converges to a point $\wp\in\Im$ such that

$$\lim_{p,q\to\infty}P_b(\wp_p,\wp_q)=\lim_{p\to\infty}P_b(\wp,\wp_p)=P_b(\wp,\wp).$$

Lemma 2.6:([4]) If and only if a sequence $\{\wp_n\}$ is a b-Cauchy sequence in the b-metric space $(\Im,d_{P_b})$, then it is a P_b-Cauchy sequence in a partial b-metric space $(\Im,P_b)$.

Lemma 2.7:([4]) If and only if the b-metric space $(\Im,d_{P_b})$ is b-complete, a partial b-metric space $(\Im,P_b)$ qualifies as P_b-complete. Additionally,

$\lim_{n,m\to\infty}d_{P_b}(\chi_n,\chi_m)=0\iff\lim_{m\to\infty}P_b(\chi_m,\chi)=\lim_{n\to\infty}P_b(\chi_n,\chi)=P_b(\chi,\chi).$

Definition 2.8:([6]) Let $\Im$ be a nonempty set. An element $(\partial_1,\partial_2,\partial_3)\in\Im$ is called a tripled fixed point of a given mapping $\mathscr{H}:\Im^3\to\Im$ if $\mathscr{H}(\partial_1,\partial_2,\partial_3)=\partial_1$, $\mathscr{H}(\partial_2,\partial_3,\partial_1)=\partial_2$ and $\mathscr{H}(\partial_3,\partial_1,\partial_2)=\partial_3$

Definition 2.9:([7]) Let $\mathscr{H}:\Im^3\to\Im$ and $\mathscr{V}:\Im\to\Im$ be two mappings. An element $(\partial_1,\partial_2,\partial_3)$ is said to be a tripled coincident point of $\mathscr{H}$ and $\mathscr{V}$ if

$$\mathscr{H}(\partial_1,\partial_2,\partial_3)=\mathscr{V}\partial_1,\ \mathscr{H}(\partial_2,\partial_3,\partial_1)=\mathscr{V}\partial_2,\ \mathscr{H}(\partial_3,\partial_1,\partial_2)=\mathscr{V}\partial_3.$$

Definition 2.10:([7]) Let $\mathscr{H}:\Im^3\to\Im$ and $\mathscr{V}:\Im\to\Im$ be two mappings. An element $(\partial_1,\partial_2,\partial_3)$ is said to be a tripled common point of $\mathscr{H}$ and $\mathscr{V}$ if

$\mathscr{H}(\partial_1,\partial_2,\partial_3)=\mathscr{V}\partial_1=\partial_1$, $\mathscr{H}(\partial_2,\partial_3,\partial_1)=\mathscr{V}\partial_2=\partial_2$ and $\mathscr{H}(\partial_3,\partial_1,\partial_2)=\mathscr{V}\partial_3=\partial_3$.

Definition 2.11:([8]) Let $(\Im, P_b)$ be a partial b- metric space. A pair $(\mathscr{H}, \mathscr{V})$ is called weakly compatible if $\mathscr{V}(\mathscr{H}(\partial_1, \partial_2, \partial_3)) = \mathscr{H}(\mathscr{V}\partial_1, \mathscr{V}\partial_2, \mathscr{V}\partial_3)$ whenever for all $\partial_1, \partial_2, \partial_3 \in \Im$ such that $\mathscr{H}(\partial_1, \partial_2, \partial_3) = \mathscr{V}\partial_1$, $\mathscr{H}(\partial_2, \partial_3, \partial_1) = \mathscr{V}\partial_2$ and $\mathscr{H}(\partial_3, \partial_1, \partial_2) = \mathscr{V}\partial_3$.

Definition 2.12:([9]) Let $\mathscr{T} : \Im^3 \to \Im$ and $\alpha : \Im^3 \to R^+$. We say that $\mathscr{T}$ is an α-admissible if for all $\ell_1, \ell_2, \ell_3 \in \Im$, we have

$\alpha(\ell_1, \ell_2, \ell_3) \geq 1$ implies $\alpha(\mathscr{T}(\ell_1, \ell_2, \ell_3), \mathscr{T}(\ell_2, \ell_3, \ell_1), \mathscr{T}(\ell_3, \ell_1, \ell_2)) \geq 1$

Definition 2.13:([9]) Let $\mathscr{T} : \Im^3 \to \Im$, $\int : \Im \to \Im$ and $\alpha : \Im^3 \to R^+$ be a mappings. We say that $\mathscr{T}$ and $\int$ are an α-admissible if $\forall\ \ell_1, \ell_2, \ell_3 \in \Im$, we have

$\alpha(\int\ell_1, \int\ell_2, \int\ell_3) \geq 1$ implies $\alpha(\mathscr{T}(\ell_1, \ell_2, \ell_3), \mathscr{T}(\ell_2, \ell_3, \ell_1), \mathscr{T}(\ell_3, \ell_1, \ell_2)) \geq 1$.

Let Δ be a family of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(a) φ is nondecreasing;

(b) $\varphi(s) < s$ for $s \in [0, \infty)$

Now we prove our main result.

3. Main Results

Definition 3.1: Let $(\Im, P_b)$ be a partial b-metric spaces with the coefficient $\kappa \geq 1$ and $\alpha : \Im^3 \to R^+$, two mappings $\mathscr{T} : \Im^3 \to \Im$, $\int : \Im \to \Im$ are called $(\alpha, \varphi) - K$-contraction if it satisfies for all $\ell_1, \ell_2, \ell_3, \wp_1, \wp_2, \wp_3 \in \Im$;

$$(1)\quad \alpha(\int\ell_1, \int\ell_2, \int\ell_3) P_b(\mathscr{T}(\ell_1, \ell_2, \ell_3), \mathscr{T}(\wp_1, \wp_2, \wp_3)) \leq \varphi(\lambda K(\ell_1, \ell_2, \ell_3, \wp_1, \wp_2, \wp_3))$$

where,$\varphi \in \Delta$, $\lambda \in [0, \frac{1}{2\kappa^2})$ and

$$K(\ell_1, \ell_2, \ell_3, \wp_1, \wp_2, \wp_3) = \max \left\{ \begin{array}{c} P_b(\int\ell_1, \int\wp_1), P_b(\int\ell_2, \int\wp_2), P_b(\int\ell_3, \int\wp_3), \\ P_b(\int\ell_1, \mathscr{T}(\ell_1, \ell_2, \ell_3)), P_b(\int\ell_2, \mathscr{T}(\ell_2, \ell_3, \ell_1)), \\ P_b(\int\ell_3, \mathscr{T}(\ell_3, \ell_1, \ell_2)), P_b(\int\wp_1, \mathscr{T}(\wp_1, \wp_2, \wp_3)), \\ P_b(\int\wp_2, \mathscr{T}(\wp_2, \wp_3, \wp_1)), P_b(\int\wp_3, \mathscr{T}(\wp_3, \wp_1, \wp_2)), \\ \frac{P_b(\int\ell_1, \mathscr{T}(\ell_1, \ell_2, \ell_3)) P_b(\int\wp_1, \mathscr{T}(\wp_1, \wp_2, \wp_3))}{2\kappa^2[1 + P_b(\int\ell_1, \int\wp_1)]}, \\ \frac{P_b(\int\ell_2, \mathscr{T}(\ell_2, \ell_3, \ell_1)) P_b(\int\wp_2, \mathscr{T}(\wp_2, \wp_3, \wp_1))}{2\kappa^2[1 + P_b(\int\ell_2, \int\wp_2)]}, \\ \frac{P_b(\int\ell_3, \mathscr{T}(\ell_3, \ell_1, \ell_2)) P_b(\int\wp_3, \mathscr{T}(\wp_3, \wp_1, \wp_2))}{2\kappa^2[1 + P_b(\int\ell_3, \int\wp_3)]}, \end{array} \right\}.$$

Theorem 3.2: Assume that $(\Im,P_b)$ be a partial b- metric space with the coefficient $\kappa\geq 1$ and $\mathscr{T}:\Im^3\to\Im$ and $\int:\Im\to\Im$ be two mappings satisfying $(\alpha,\varphi)-K$-contraction and assume that

(3.2.1) $\mathscr{T}(\Im^3)\subseteq\int(\Im)$ and $\int(\Im)$ is a complete subspace of $\Im$

(3.2.2) $\mathscr{T}$ and $\int$ are α- admissible mappings,

(3.2.3) $\exists\,\ell_0,\wp_0,\aleph_0\in\Im\ni\alpha(\mathscr{T}(\ell_0,\wp_0,\aleph_0),\mathscr{T}(\wp_0,\aleph_0,\ell_0),\mathscr{T}(\int\aleph_0,\int\ell_0,\int\wp_0)\geq 1,$

(3.2.4) $(\mathscr{T},\int)$ is a weakly compatible pair.

Then $\mathscr{T}$ and $\int$ have a unique common tripled fixed point in $\Im$.

Proof Let $\ell_0,\wp_0,\aleph_0$ be arbitrary points in $\Im$. From (3.2.1), there exist sequences $\{\ell_n\}$, $\{\wp_n\}$,$\{\aleph_n\}$, $\{u_n\}$, $\{v_n\}$and $\{w_n\}$ in $\Im$ such that, for all $n\geq 0$

$$\mathscr{T}(\ell_n,\wp_n,\aleph_n)=\int\ell_{n+1}=u_n\quad \mathscr{T}(\wp_n,\aleph_n,\ell_n)=\int\wp_{n+1}=v_n\quad \mathscr{T}(\aleph_n,\ell_n,\wp_n)=\int\aleph_{n+1}=w_n.$$

For simplification we denote $\Omega_n=\max\left\{\begin{array}{c}P_b(u_n,u_{n+1}),P_b(v_n,v_{n+1}),\\ P_b(w_n,w_{n+1})\end{array}\right\}$.

Case(i): If for some n_0, we have

$$u_{n_0}=u_{n_0+1}=\mathscr{T}(\ell_{n_0},\wp_{n_0},\aleph_{n_0})=\int\ell_{n_0+1}\quad v_{n_0}=v_{n_0+1}=\mathscr{T}(\wp_{n_0},\aleph_{n_0},\ell_{n_0})=\int\wp_{n_0+1}$$
$$w_{n_0}=w_{n_0+1}=\mathscr{T}(\aleph_{n_0},\ell_{n_0},\wp_{n_0})=\int\aleph_{n_0+1},$$

then $(u_{n_0},v_{n_0},w_{n_0})$ is a common tripled fixed point of $\mathscr{T}$ and $\int$.

Case (ii): Suppose that $u_n\neq u_{n+1}$, $v_n\neq v_{n+1}$ and $w_n\neq w_{n+1}$ for all $n\geq 0$. Since $\mathscr{T}$ and $\int$ are α-admissible, we have

$$\alpha(\int\ell_0,\int\wp_0,\int\aleph_1)=\alpha(\int\ell_0\int\wp_0,\mathscr{T}(\aleph_0,\ell_0,\wp_0))\geq 1$$
$$\Rightarrow\alpha(\mathscr{T}(\ell_0,\wp_0,\aleph_0)\mathscr{T}(\wp_0,\aleph_0,\ell_0),\mathscr{T}(\aleph_1,\ell_1,\wp_1))=\alpha(\int\ell_1,\int\wp_1,\int\aleph_2)\geq 1$$

Recursively, we find that $\alpha(\int\ell_n,\int\wp_n,\int\aleph_{n+1})\geq 1$, for all $n=0,1,...$

From (1), (3.2.2) and (3.2.3), we have that

$$\begin{aligned}P_b(u_n,u_{n+1}) &= P_b(\mathscr{T}(\ell_n,\wp_n,\aleph_n),\mathscr{T}(\ell_{n+1},\wp_{n+1},\aleph_{n+1}))\\ &\leq \alpha(\int\ell_n,\int\wp_n,\int\aleph_n)P_b(\mathscr{T}(\ell_n,\wp_n,\aleph_n),\mathscr{T}(\ell_{n+1},\wp_{n+1},\aleph_{n+1}))\\ &\leq \varphi(\lambda K(\ell_n,\wp_n,\aleph_n,\ell_{n+1},\wp_{n+1},\aleph_{n+1})) \qquad (2)\end{aligned}$$

where,

$$
\begin{aligned}
&K(\ell_n,\wp_n,\aleph_n,\ell_{n+1},\wp_{n+1},\aleph_{n+1})\\
&= \max\left\{\begin{array}{c}
P_b(\int\ell_n,\int\ell_{n+1}),P_b(\int\wp_n,\int\wp_{n+1}),P_b(\int\aleph_n,\int\aleph_{n+1}),\\
P_b(\int\ell_n,\mathscr{T}(\ell_n,\wp_n,\aleph_n)),P_b(\int\wp_n,\mathscr{T}(\wp_n,\aleph_n,\ell_n)),P_b(\int\aleph_n,\mathscr{T}(\aleph_n,\ell_n,\wp_n)),\\
P_b(\int\ell_{n+1},\mathscr{T}(\ell_{n+1},\wp_{n+1},\aleph_{n+1})),P_b(\int\wp_{n+1},\mathscr{T}(\wp_{n+1},\aleph_{n+1},\ell_{n+1})),\\
P_b(\int\aleph_{n+1},\mathscr{T}(\aleph_{n+1},\ell_{n+1},\wp_{n+1})),\\
\frac{P_b(\int\ell_n,\mathscr{T}(\ell_n,\wp_n,\aleph_n))P_b(\int\ell_{n+1},\mathscr{T}(\ell_{n+1},\wp_{n+1},\aleph_{n+1}))}{2\kappa^2[1+P_b(\int\ell_n,\int\ell_{n+1})]},\\
\frac{P_b(\int\wp_n,\mathscr{T}(\wp_n,\aleph_n,\ell_n))P_b(\int\wp_{n+1},\mathscr{T}(\wp_{n+1},\aleph_{n+1},\ell_{n+1}))}{2\kappa^2[1+P_b(\int\wp_n,\int\wp_{n+1})]},\\
\frac{P_b(\int\aleph_n,\mathscr{T}(\aleph_n,\ell_n,\wp_n))P_b(\int\aleph_{n+1},\mathscr{T}(\aleph_{n+1},\ell_{n+1},\wp_{n+1}))}{2\kappa^2[1+P_b(\int\aleph_n,\int\aleph_{n+1})]},
\end{array}\right\}\\
&= \max\left\{\begin{array}{c}
P_b(u_{n-1},u_n),P_b(v_{n-1},v_n),P_b(w_{n-1},w_n),\\
P_b(u_{n-1},u_n),P_b(v_{n-1},v_n),P_b(w_{n-1},w_n),\\
P_b(u_n,u_{n+1}),P_b(v_n,v_{n+1}),P_b(w_n,w_{n+1})\\
\frac{P_b(u_{n-1},u_n)P_b(u_n,u_{n+1})}{2\kappa^2[1+P_b(u_{n-1},u_n)]},\frac{P_b(v_{n-1},v_n)P_b(v_n,v_{n+1})}{2\kappa^2[1+P_b(v_{n-1},v_n)]},\\
\frac{P_b(w_{n-1},w_n)P_b(w_n,w_{n+1})}{2\kappa^2[1+P_b(w_{n-1},w_n)]},
\end{array}\right\}\\
\leq\ &\max\left\{\begin{array}{c}
P_b(u_{n-1},u_n),P_b(v_{n-1},v_n),P_b(w_{n-1},w_n),\\
P_b(u_n,u_{n+1}),P_b(v_n,v_{n+1}),P_b(w_n,w_{n+1})
\end{array}\right\}
\end{aligned}
$$

From (2), we have

$$
P_b(u_n,u_{n+1}) \leq \varphi\left(\lambda\max\left\{\begin{array}{c}
P_b(u_{n-1},u_n),P_b(v_{n-1},v_n),P_b(w_{n-1},w_n),\\
P_b(u_n,u_{n+1}),P_b(v_n,v_{n+1}),P_b(w_n,w_{n+1})
\end{array}\right\}\right) \tag{3}
$$

Similarly, we can prove that

$$
P_b(v_n,v_{n+1}) \leq \varphi\left(\lambda\max\left\{\begin{array}{c}
P_b(u_{n-1},u_n),P_b(v_{n-1},v_n),P_b(w_{n-1},w_n),\\
P_b(u_n,u_{n+1}),P_b(v_n,v_{n+1}),P_b(w_n,w_{n+1})
\end{array}\right\}\right) \tag{4}
$$

and

$$
P_b(w_n,w_{n+1}) \leq \varphi\left(\lambda\max\left\{\begin{array}{c}
P_b(u_{n-1},u_n),P_b(v_{n-1},v_n),P_b(w_{n-1},w_n),\\
P_b(u_n,u_{n+1}),P_b(v_n,v_{n+1}),P_b(w_n,w_{n+1})
\end{array}\right\}\right) \tag{5}
$$

Combining (3), (4) and (5), we get

$$\max\left\{\begin{array}{c} P_b(u_n,u_{n+1}), \\ P_b(v_n,v_{n+1}), \\ P_b(w_n,w_{n+1}) \end{array}\right\} \leq \varphi\left(\lambda \max\left\{\begin{array}{c} P_b(u_{n-1},u_n),P_b(v_{n-1},v_n),P_b(w_{n-1},w_n), \\ P_b(u_n,u_{n+1}),P_b(v_n,v_{n+1}),P_b(w_n,w_{n+1}) \end{array}\right\}\right)$$

If $P_b(u_{n-1},u_n) < P_b(u_n,u_{n+1})$, , $P_b(v_{n-1},v_n) < P_b(v_n,v_{n+1})$ and

$P_b(w_{n-1},w_n) < P_b(w_n,w_{n+1})$, then we have

$$\begin{aligned} \max\left\{\begin{array}{c} P_b(u_n,u_{n+1}), \\ P_b(v_n,v_{n+1}), \\ P_b(w_n,w_{n+1}) \end{array}\right\} &\leq \varphi\left(\lambda \max\left\{ P_b(u_n,u_{n+1}),P_b(v_n,v_{n+1}),P_b(w_n,w_{n+1}) \right\}\right) \\ &< \lambda \max\left\{ P_b(u_n,u_{n+1}),P_b(v_n,v_{n+1}),P_b(w_n,w_{n+1}) \right\} \end{aligned}$$

a contradiction. Accordingly, we conclude that

$$\begin{aligned} \max\left\{\begin{array}{c} P_b(u_n,u_{n+1}), \\ P_b(v_n,v_{n+1}), \\ P_b(w_n,w_{n+1}) \end{array}\right\} &\leq \varphi\left(\lambda \max\left\{ P_b(u_{n-1},u_n),P_b(v_{n-1},v_n),P_b(w_{n-1},w_n) \right\}\right) \\ &< \lambda \max\left\{ P_b(u_{n-1},u_n),P_b(v_{n-1},v_n),P_b(w_{n-1},w_n) \right\} \forall \quad n \geq 1. \end{aligned}$$

Thus

$$\Omega_n < \lambda \Omega_{n-1} \forall n \geq 1. \tag{6}$$

Therefore, $\{\Omega_n\}$ is a decreasing sequence and converges to $\delta \geq 0$. Suppose $\delta > 0$ and letting $n \to \infty$ in (6), we have that $\delta \leq \lambda\delta < \delta$, is a contradiction . Hence $\delta = 0$. Thus $\lim_{n\to\infty} \Omega_n = 0$. It follows that

$$\lim_{n\to\infty} P_b(u_n,u_{n+1}) = \lim_{n\to\infty} P_b(v_n,v_{n+1}) = \lim_{n\to\infty} P_b(w_n,w_{n+1}) = 0. \tag{7}$$

From (7) and (P_b2), we have that

$$\lim_{n\to\infty} P_b(u_n,u_n) = \lim_{n\to\infty} P_b(v_n,v_n) = \lim_{n\to\infty} P_b(w_n,w_n) = 0. \tag{8}$$

From definition of d_{P_b}, (7) and (8), we have that

$$\lim_{n\to\infty} d_b(u_n,u_{n+1}) = \lim_{n\to\infty} d_b(v_n,v_{n+1}) = \lim_{n\to\infty} d_b(w_n,w_{n+1}) = 0. \tag{9}$$

We now demonstrate that the sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ in partial b-metric space $(\Im, P_b)$ are Cauchy sequences. If sufficient to demonstrate that the sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ in b-metric space are Cauchy sequences $(\Im, d_{P_b})$. Assume, however, that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are not Cauchy sequences. This results in n,m for $d_{P_b}(u_n,u_m) \nrightarrow 0$, $d_{P_b}(v_n,v_m) \nrightarrow 0$ and $d_{P_b}(w_n,w_m) \nrightarrow 0$ as $n,m \to \infty$.

Consequently, $\max\{d_{P_b}(u_n,u_m), d_{P_b}(v_n,v_m), d_{P_b}(w_n,w_m)\} \nrightarrow 0$ as $n,m \to \infty$.

Then there exist an $\varepsilon > 0$ and monotonically increases sequences of natural numbers $\{m_k\}$, $\{n_k\}$ such that $n_k > m_k > k$.

$$\max\{d_{P_b}(u_{n_k},u_{m_k}), d_{P_b}(v_{n_k},v_{m_k}), d_{P_b}(w_{n_k},w_{m_k})\} \geq \varepsilon \tag{10}$$

and

$$\max\left\{d_{P_b}(u_{n_k-1},u_{m_k}), d_{P_b}(v_{n_k-1},v_{m_k}), d_{P_b}(w_{n_k-1},w_{m_k})\right\} < \varepsilon. \tag{11}$$

From (10) and (11), we have that

$$\begin{aligned}
\varepsilon &\leq \max\{d_{P_b}(u_{n_k},u_{m_k}), d_{P_b}(v_{n_k},v_{m_k}), d_{P_b}(w_{n_k},w_{m_k})\} \\
&\leq \kappa \max\left\{d_{P_b}(u_{m_k},u_{n_k-1}), d_{P_b}(v_{m_k},v_{n_k-1}), d_{P_b}(w_{m_k},w_{n_k-1})\right\} \\
&\qquad + \kappa \max\left\{d_{P_b}(u_{n_k-1},u_{n_k}), d_{P_b}(v_{n_k-1},v_{n_k}), d_{P_b}(w_{n_k-1},w_{n_k})\right\} \\
&< \kappa\varepsilon + \kappa \max\left\{d_{P_b}(u_{n_k-1},u_{n_k}), d_{P_b}(v_{n_k-1},v_{n_k}), d_{P_b}(w_{n_k-1},w_{n_k})\right\}.
\end{aligned}$$

Using $k \to \infty$ as the upper limit and (9), we may deduce that

$$\varepsilon \leq \lim_{k\to\infty} \sup \max\{d_{P_b}(u_{n_k},u_{m_k}), d_{P_b}(v_{n_k},v_{m_k}), d_{P_b}(w_{n_k},w_{m_k})\} \leq \kappa\varepsilon. \tag{12}$$

Also

$$\begin{aligned}
\varepsilon &\leq \max\{d_{P_b}(u_{n_k},u_{m_k}), d_{P_b}(v_{n_k},v_{m_k}), d_{P_b}(w_{n_k},w_{m_k})\} \\
&\leq \kappa \max\left\{d_{P_b}(u_{m_k},u_{n_k+1}), d_{P_b}(v_{m_k},v_{n_k+1}), d_{P_b}(w_{m_k},w_{n_k+1})\right\} \\
&\qquad + \kappa \max\left\{d_{P_b}(u_{n_k+1},u_{n_k}), d_{P_b}(v_{n_k+1},v_{n_k}), d_{P_b}(w_{n_k+1},w_{n_k})\right\}.
\end{aligned}$$

Using $k \to \infty$ as the upper limit and (9), we may deduce that

$$\frac{\varepsilon}{\kappa} \leq \lim_{k\to\infty} \sup \max\left\{d_{P_b}(u_{m_k},u_{n_k+1}), d_{P_b}(v_{m_k},v_{n_k+1}), d_{P_b}(w_{m_k},w_{n_k+1})\right\}. \tag{13}$$

On other hand

$$\max\left\{d_{P_b}(u_{m_k},u_{n_k+1}),d_{P_b}(v_{m_k},v_{n_k+1})\right\} \leq \kappa\max\left\{d_{P_b}(u_{m_k},u_{n_k}),d_{P_b}(v_{m_k+1},v_{n_k})\right\} +\kappa\max\left\{d_{P_b}(u_{n_k},u_{n_k+1}),d_{P_b}(v_{n_k},v_{n_k+1})\right\}.$$

Using $k\to\infty$ as the upper limit and (9), we may deduce that

$$\limsup_{k\to\infty}\max\left\{d_{P_b}(u_{m_k},u_{n_k+1}),d_{P_b}(v_{m_k},v_{n_k+1})\right\}\leq\varepsilon\kappa^2 \tag{14}$$

also, from (5), we have that

$$\begin{aligned}
\varepsilon &\leq \max\{d_{P_b}(u_{n_k},u_{m_k}),d_{P_b}(v_{n_k},v_{m_k}),d_{P_b}(w_{n_k},w_{m_k})\}\\
&\leq \kappa\max\left\{d_{P_b}(u_{m_k},u_{m_k+1}),d_{P_b}(v_{m_k},v_{m_k+1}),d_{P_b}(w_{m_k},w_{m_k+1})\right\}\\
&\quad +\kappa\max\left\{d_{P_b}(u_{m_k+1},u_{n_k}),d_{P_b}(v_{m_k+1},v_{n_k}),d_{P_b}(w_{m_k+1},w_{n_k})\right\}\\
&\leq \left\{\begin{array}{c}
\kappa\max\left\{d_{P_b}(u_{m_k},u_{m_k+1}),d_{P_b}(v_{m_k},v_{m_k+1}),d_{P_b}(w_{m_k},w_{m_k+1})\right\}\\
+\kappa^2\max\left\{d_{P_b}(u_{m_k+1},u_{n_k+2}),d_{P_b}(v_{m_k+1},v_{n_k+2}),d_{P_b}(w_{m_k+1},w_{n_k+2})\right\}\\
+\kappa^2\max\left\{d_{P_b}(u_{n_k+2},u_{n_k}),d_{P_b}(v_{n_k+2},v_{n_k}),d_{P_b}(w_{n_k+2},w_{n_k})\right\}
\end{array}\right\}\\
&\leq \left\{\begin{array}{c}
\kappa\max\left\{d_{P_b}(u_{m_k},u_{m_k+1}),d_{P_b}(v_{m_k},v_{m_k+1}),d_{P_b}(w_{m_k},w_{m_k+1})\right\}\\
+\kappa^2\max\left\{d_{P_b}(u_{m_k+1},u_{n_k+2}),d_{P_b}(v_{m_k+1},v_{n_k+2}),d_{P_b}(w_{m_k+1},w_{n_k+2})\right\}\\
+\kappa^3\max\left\{d_{P_b}(u_{n_k+2},u_{n_k+1}),d_{P_b}(v_{n_k+2},v_{n_k+1}),d_{P_b}(w_{n_k+2},w_{n_k+1})\right\}\\
+\kappa^3\max\left\{d_{P_b}(u_{n_k+1},u_{n_k}),d_{P_b}(v_{n_k+1},v_{n_k}),d_{P_b}(w_{n_k+1},w_{n_k})\right\}
\end{array}\right\}.
\end{aligned}$$

Using $k\to\infty$ as the upper limit and (9), we may deduce that

$$\frac{\varepsilon}{\kappa^2}\leq\limsup_{k\to\infty}\max\left\{d_{P_b}(u_{m_k+1},u_{n_k+2}),d_{P_b}(v_{m_k+1},v_{n_k+2}),d_{P_b}(w_{m_k+1},w_{n_k+2})\right\}.$$

On other hand

$$\max\left\{\begin{array}{c} d_{P_b}(u_{m_k+1},u_{n_k+2}),\\ d_{P_b}(v_{m_k+1},v_{n_k+2}),\\ d_{P_b}(w_{m_k+1},w_{n_k+2})\end{array}\right\} \leq \kappa\max\left\{\begin{array}{c} d_{P_b}(u_{m_k+1},u_{m_k}),d_{P_b}(v_{m_k+1},v_{m_k}),\\ d_{P_b}(w_{m_k+1},w_{m_k})\end{array}\right\} +\kappa\max\left\{\begin{array}{c} d_{P_b}(u_{m_k},u_{n_k+2}),d_{P_b}(v_{m_k},v_{n_k+2}),\\ d_{P_b}(w_{m_k},w_{n_k+2})\end{array}\right\}$$

$$\leq \left\{ \begin{array}{c} \kappa \max \left\{ \begin{array}{c} d_{P_b}(u_{m_k+1}, u_{m_k}), d_{P_b}(v_{m_k+1}, v_{m_k}), \\ d_{P_b}(w_{m_k+1}, w_{m_k}) \end{array} \right\} \\ +\kappa^2 \max \left\{ \begin{array}{c} d_{P_b}(u_{m_k}, u_{n_k}), d_{P_b}(v_{m_k}, v_{n_k}), \\ d_{P_b}(w_{m_k}, w_{n_k}) \end{array} \right\} \\ +\kappa^3 \max \left\{ \begin{array}{c} d_{P_b}(u_{n_k}, u_{n_k+1}), d_{P_b}(v_{n_k}, v_{n_k+1}), \\ d_{P_b}(w_{n_k}, w_{n_k+1}) \end{array} \right\} \\ +\kappa^3 \max \left\{ \begin{array}{c} d_{P_b}(u_{n_k+1}, u_{n_k+2}), d_{P_b}(v_{n_k+1}, v_{n_k+2}), \\ d_{P_b}(w_{n_k+1}, w_{n_k+2}) \end{array} \right\} \end{array} \right\}.$$

Taking upper limit as $k \to \infty$ and from (9), (12) we have that

$$\lim_{k\to\infty} \sup \max \left\{ d_{P_b}(u_{m_k+1}, u_{n_k+2}), d_{P_b}(v_{m_k+1}, v_{n_k+2}) \right\} \leq \varepsilon.\kappa^3. \tag{15}$$

Now

$$\begin{aligned} (16) \quad & P_b\left(u_{m_k+1}, u_{n_k+2}\right) \\ = \; & P_b\left(\mathscr{T}(\ell_{m_k+1}, \wp_{m_k+1}, \aleph_{m_k+1}), \mathscr{T}(\ell_{n_k+2}, \wp_{n_k+2}, \aleph_{n_k+2})\right) \\ \leq \; & \alpha\left(\int \ell_{m_k+1}, \int \wp_{m_k+1}, \int \aleph_{m_k+1}\right) P_b\left(\mathscr{T}(\ell_{m_k+1}, \wp_{m_k+1}, \aleph_{m_k+1}), \mathscr{T}(\ell_{n_k+2}, \wp_{n_k+2}, \aleph_{n_k+2})\right) \\ \leq \; & \varphi\left(\lambda K(\ell_{m_k+1}, \wp_{m_k+1}, \aleph_{m_k+1}, \ell_{n_k+2}, \wp_{n_k+2}, \aleph_{n_k+2})\right) \end{aligned}$$

where,

$$K(\ell_{m_k+1}, \wp_{m_k+1}, \aleph_{m_k+1}, \ell_{n_k+2}, \wp_{n_k+2}, \aleph_{n_k+2})$$

$$= \max \left\{ \begin{array}{c} P_b(\int \ell_{m_k+1}, \int \ell_{n_k+2}), P_b(\int \wp_{m_k+1}, \int \wp_{n_k+2}), P_b(\int \aleph_{m_k+1}, \int \aleph_{n_k+2}), \\ P_b(\int \ell_{m_k+1}, \mathscr{T}(\ell_{m_k+1}, \wp_{m_k+1}, \aleph_{m_k+1})), P_b(\int \wp_{m_k+1}, \mathscr{T}(\wp_{m_k+1}, \aleph_{m_k+1}, \ell_{m_k+1})), \\ P_b(\int \aleph_{m_k+1}, \mathscr{T}(\aleph_{m_k+1}, \ell_{m_k+1}, \wp_{m_k+1})), \\ P_b(\int \ell_{n_k+2}, \mathscr{T}(\ell_{n_k+2}, \wp_{n_k+2}, \aleph_{n_k+2})), P_b(\int \wp_{n_k+2}, \mathscr{T}(\wp_{n_k+2}, \aleph_{n_k+2}, \ell_{n_k+2})), \\ P_b(\int \aleph_{n_k+2}, \mathscr{T}(\aleph_{n_k+2}, \ell_{n_k+2}, \wp_{n_k+2})), \\ \frac{P_b(\int \ell_{m_k+1}, \mathscr{T}(\ell_{m_k+1}, \wp_{m_k+1}, \aleph_{m_k+1})) P_b(\int \ell_{n_k+2}, \mathscr{T}(\ell_{n_k+2}, \wp_{n_k+2}, \aleph_{n_k+2}))}{2\kappa^2\left[1+P_b(\int \ell_{m_k+1}, \int \ell_{n_k+2})\right]}, \\ \frac{P_b(\int \wp_{m_k+1}, \mathscr{T}(\wp_{m_k+1}, \aleph_{m_k+1}, \ell_{m_k+1})) P_b(\int \wp_{n_k+2}, \mathscr{T}(\wp_{n_k+2}, \aleph_{n_k+2}, \ell_{n_k+2}))}{2\kappa^2\left[1+P_b(\int \wp_{m_k+1}, \int \wp_{n_k+2})\right]}, \\ \frac{P_b(\int \aleph_{m_k+1}, \mathscr{T}(\aleph_{m_k+1}, \ell_{m_k+1}, \wp_{m_k+1})) P_b(\int \aleph_{n_k+2}, \mathscr{T}(\aleph_{n_k+2}, \ell_{n_k+2}, \wp_{n_k+2}))}{2\kappa^2\left[1+P_b(\int \aleph_{m_k+1}, \int \aleph_{n_k+2})\right]}, \end{array} \right.$$

$$\leq \max\left\{\begin{array}{c} P_b(u_{m_k},u_{n_k+1}),P_b(v_{m_k},v_{n_k+1}),P_b(w_{m_k},w_{n_k+1}) \\ P_b(u_{m_k},u_{m_k+1}),P_b(v_{m_k},v_{m_k+1}),P_b(w_{m_k},w_{m_k+1}) \\ P_b(u_{n_{k+1}},u_{n_{k+2}}),P_b(v_{n_{k+1}},v_{n_{k+2}}),P_b(w_{n_{k+1}},w_{n_{k+2}}) \\ \frac{P_b\left(u_{m_k},u_{m_k+1}\right).P_b\left(u_{n_k+1},u_{n_k+2}\right)}{2\kappa^2\left[1+P_b(u_{m_k},u_{n_k+1})\right]},\frac{P_b\left(v_{m_k},v_{m_k+1}\right).P_b\left(v_{n_k+1},v_{n_k+2}\right)}{2\kappa^2\left[1+P_b(v_{m_k},v_{n_k+1})\right]} \\ \frac{P_b\left(w_{m_k},w_{m_k+1}\right).P_b\left(w_{n_k+1},w_{n_k+2}\right)}{2\kappa^2\left[1+P_b(w_{m_k},w_{n_k+1})\right]} \end{array}\right\}.$$

Therefore, from (17), we have

$$\begin{aligned} & P_b\left(u_{m_k+1},u_{n_k+2}\right) \\ \leq\ & \varphi\left(\lambda\max\left\{\begin{array}{c} P_b(u_{m_k},u_{n_k+1}),P_b(v_{m_k},v_{n_k+1}),P_b(w_{m_k},w_{n_k+1}) \\ P_b(u_{m_k},u_{m_k+1}),P_b(v_{m_k},v_{m_k+1}),P_b(w_{m_k},w_{m_k+1}) \\ P_b(u_{n_{k+1}},u_{n_{k+2}}),P_b(v_{n_{k+1}},v_{n_{k+2}}),P_b(w_{n_{k+1}},w_{n_{k+2}}) \\ \frac{P_b\left(u_{m_k},u_{m_k+1}\right).P_b\left(u_{n_k+1},u_{n_k+2}\right)}{2\kappa^2\left[1+P_b(u_{m_k},u_{n_k+1})\right]},\frac{P_b\left(v_{m_k},v_{m_k+1}\right).P_b\left(v_{n_k+1},v_{n_k+2}\right)}{2\kappa^2\left[1+P_b(v_{m_k},v_{n_k+1})\right]} \\ \frac{P_b\left(w_{m_k},w_{m_k+1}\right).P_b\left(w_{n_k+1},w_{n_k+2}\right)}{2\kappa^2\left[1+P_b(w_{m_k},w_{n_k+1})\right]} \end{array}\right\}\right) \\ <\ & \lambda\max\left\{\begin{array}{c} P_b(u_{m_k},u_{n_k+1}),P_b(v_{m_k},v_{n_k+1}),P_b(w_{m_k},w_{n_k+1}) \\ P_b(u_{m_k},u_{m_k+1}),P_b(v_{m_k},v_{m_k+1}),P_b(w_{m_k},w_{m_k+1}) \\ P_b(u_{n_{k+1}},u_{n_{k+2}}),P_b(v_{n_{k+1}},v_{n_{k+2}}),P_b(w_{n_{k+1}},w_{n_{k+2}}) \\ \frac{P_b\left(u_{m_k},u_{m_k+1}\right).P_b\left(u_{n_k+1},u_{n_k+2}\right)}{2\kappa^2\left[1+P_b(u_{m_k},u_{n_k+1})\right]},\frac{P_b\left(v_{m_k},v_{m_k+1}\right).P_b\left(v_{n_k+1},v_{n_k+2}\right)}{2\kappa^2\left[1+P_b(v_{m_k},v_{n_k+1})\right]} \\ \frac{P_b\left(w_{m_k},w_{m_k+1}\right).P_b\left(w_{n_k+1},w_{n_k+2}\right)}{2\kappa^2\left[1+P_b(w_{m_k},w_{n_k+1})\right]} \end{array}\right\}. \end{aligned}$$

Thus

$$\max\left\{\begin{array}{c} P_b\left(u_{m_k+1},u_{n_k+2}\right), \\ P_b\left(v_{m_k+1},v_{n_k+2}\right) \\ P_b\left(w_{m_k+1},w_{n_k+2}\right) \end{array}\right\} < \lambda\max\left\{\begin{array}{c} P_b(u_{m_k},u_{n_k+1}),P_b(v_{m_k},v_{n_k+1}),P_b(w_{m_k},w_{n_k+1}) \\ P_b(u_{m_k},u_{m_k+1}),P_b(v_{m_k},v_{m_k+1}),P_b(w_{m_k},w_{m_k+1}) \\ P_b(u_{n_{k+1}},u_{n_{k+2}}),P_b(v_{n_{k+1}},v_{n_{k+2}}),P_b(w_{n_{k+1}},w_{n_{k+2}}) \\ \frac{P_b\left(u_{m_k},u_{m_k+1}\right).P_b\left(u_{n_k+1},u_{n_k+2}\right)}{2\kappa^2\left[1+P_b(u_{m_k},u_{n_k+1})\right]}, \\ \frac{P_b\left(v_{m_k},v_{m_k+1}\right).P_b\left(v_{n_k+1},v_{n_k+2}\right)}{2\kappa^2\left[1+P_b(v_{m_k},v_{n_k+1})\right]} \\ \frac{P_b\left(w_{m_k},w_{m_k+1}\right).P_b\left(w_{n_k+1},w_{n_k+2}\right)}{2\kappa^2\left[1+P_b(w_{m_k},w_{n_k+1})\right]} \end{array}\right\}.$$

Taking upper limit as $k \to \infty$ and from (7), (14) and (15) we have that

$$\varepsilon\kappa^3 \leq \lambda\varepsilon\kappa^2$$

Sub Case (i): If $\kappa = 1$

$$\varepsilon \leq \lambda\varepsilon < \varepsilon, \text{ is contradiction.}$$

Sub Case (ii): If $\kappa > 1$

$$\varepsilon\kappa^3 \leq \lambda\varepsilon\kappa^2.$$

It follows that $\kappa \leq \lambda < 1$, is contradiction. Hence, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are Cauchy sequences in $(\Im, d_b)$. Suppose $\int(\Im)$ is complete subspace of $\Im$. Then $\{\int\ell_{n+1}\}$, $\{\int\wp_{n+1}\}$ and $\{\int\aleph_{n+1}\}$ are convergent to $\backepsilon$, $\mho$, $\eth$ in $(\int(\Im), d_{P_b})$, thus $d_{P_b}(\int\ell_{n+1}, \backepsilon) = 0, d_{P_b}(\int\wp_{n+1}, \mho) = 0$, $d_{P_b}(\int\aleph_{n+1}, \eth) = 0$ for some $\backepsilon = \int\ell$, $\mho = \int\wp$ and $\eth = \int\aleph$. We have that

$$P_b(\backepsilon, \backepsilon) = \lim_{n,m\to\infty} P_b(\int\ell_n, \int\ell_m) = \lim_{n\to\infty} P_b(\int\ell_n, \backepsilon) = \lim_{n\to\infty} P_b(\int\ell_{n+1}, \backepsilon) = 0. \tag{17}$$

and

$$P_b(\mho, \mho) = \lim_{n,m\to\infty} P_b(\int\wp_n, \int\wp_m) = \lim_{n\to\infty} P_b(\int\wp_n, \mho) = \lim_{n\to\infty} P_b(\int\wp_{n+1}, \mho) = 0. \tag{18}$$

also

$$P_b(\eth, \eth) = \lim_{n,m\to\infty} P_b(\int\aleph_n, \int\aleph_m) = \lim_{n\to\infty} P_b(\int\aleph_n, \eth) = \lim_{n\to\infty} P_b(\int\aleph_{n+1}, \eth) = 0. \tag{19}$$

Now, assume that $(\mathscr{T}, \int)$ is α-admissible mappings. Therefore, there is a sub sequence $\{u_{n_k}\}$, $\{v_{n_k}\}$ and $\{w_{n_k}\}$ of $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ respectively such that $\alpha\left(\ell_{n_k}, \wp_{n_k}, \aleph_{n_k+1}\right) \geq 1$ for all $k \in N$ and $\alpha(\int\ell, \int\wp, \int\aleph) \geq 1$. Now we claim that $\mathscr{T}(\ell, \wp, \aleph) = \backepsilon$, $\mathscr{T}(\wp, \aleph, \ell) = \mho$ and $\mathscr{T}(\aleph, \ell, \wp) = \eth$.

From (1), we have

$$P_b\left(\mathscr{T}(\ell, \wp, \aleph), \mathscr{T}(\ell_n, \wp_n, \aleph_n)\right) \leq \alpha\left(\int\ell, \int\wp, \int\aleph\right) P_b\left(\mathscr{T}(\ell, \wp, \aleph), \mathscr{T}(\ell_n, \wp_n, \aleph_n)\right)$$

$$\leq \varphi\left(\lambda K(\ell, \wp, \aleph, \ell_n, \wp_n, \aleph_n)\right)$$

$$< \lambda \max \left\{ \begin{array}{c} P_b(\int\ell,\int\ell_n), P_b(\int\wp,\int\wp_n), P_b(\int\aleph,\int\aleph_n), \\ P_b(\int\ell,\mathscr{T}(\ell,\wp,\aleph)), P_b(\int\wp,\mathscr{T}(\wp,\aleph,\ell)), P_b(\int\aleph,\mathscr{T}(\aleph,\ell,\wp)), \\ P_b(\int\ell_n,\mathscr{T}(\ell_n,\wp_n,\aleph_n)), P_b(\int\wp_n,\mathscr{T}(\wp_n,\aleph_n,\ell_n)), P_b(\int\aleph_n,\mathscr{T}(\aleph_n,\ell_n,\wp_n)), \\ \frac{P_b(\int\ell,\mathscr{T}(\ell,\wp,\aleph))P_b(\int\ell_n,\mathscr{T}(\ell_n,\wp_n,\aleph_n))}{2\kappa^2[1+P_b(\int\ell,\int\ell_n)]}, \\ \frac{P_b(\int\wp,\mathscr{T}(\wp,\aleph,\ell))P_b(\int\wp_n,\mathscr{T}(\wp_n,\aleph_n,\ell_n))}{2\kappa^2[1+P_b(\int\wp,\int\wp_n)]}, \\ \frac{P_b(\int\aleph,\mathscr{T}(\aleph,\ell,\wp))P_b(\int\aleph_n,\mathscr{T}(\aleph_n,\ell_n,\wp_n))}{2\kappa^2[1+P_b(\int\aleph,\int\aleph_n)]}, \end{array} \right\}$$

$$< \lambda \max \left\{ \begin{array}{c} P_b(\backepsilon,\int\ell_n), P_b(\mho,\int\wp_n), P_b(\eth,\int\aleph_n), \\ P_b(\backepsilon,\mathscr{T}(\ell,\wp,\aleph)), P_b(\mho,\mathscr{T}(\wp,\aleph,\ell)), P_b(\eth,\mathscr{T}(\aleph,\ell,\wp)), \\ P_b(\int\ell_n,\mathscr{T}(\ell_n,\wp_n,\aleph_n)), P_b(\int\wp_n,\mathscr{T}(\wp_n,\aleph_n,\ell_n)), P_b(\int\aleph_n,\mathscr{T}(\aleph_n,\ell_n,\wp_n)), \\ \frac{P_b(\backepsilon,\mathscr{T}(\ell,\wp,\aleph))P_b(\int\ell_n,\mathscr{T}(\ell_n,\wp_n,\aleph_n))}{2\kappa^2[1+P_b(\backepsilon,\int\ell_n)]}, \\ \frac{P_b(\mho,\mathscr{T}(\wp,\aleph,\ell))P_b(\int\wp_n,\mathscr{T}(\wp_n,\aleph_n,\ell_n))}{2\kappa^2[1+P_b(\mho,\int\wp_n)]}, \\ \frac{P_b(\eth,\mathscr{T}(\aleph,\ell,\wp))P_b(\int\aleph_n,\mathscr{T}(\aleph_n,\ell_n,\wp_n))}{2\kappa^2[1+P_b(\eth,\int\aleph_n)]}, \end{array} \right\}.$$

Letting $n \to \infty$ in the above inequality, then we obtain that

$$P_b(\mathscr{T}(\ell,\wp,\aleph),a) < \lambda \max\left\{ P_b(\backepsilon,\mathscr{T}(\ell,\wp,\aleph)), P_b(\mho,\mathscr{T}(\wp,\aleph,\ell)), P_b(\eth,\mathscr{T}(\aleph,\ell,\wp)) \right\}.$$

Similarly, we can prove

$$P_b(\mathscr{T}(\wp,\aleph,\ell),b) < \lambda \max\left\{ P_b(\backepsilon,\mathscr{T}(\ell,\wp,\aleph)), P_b(\mho,\mathscr{T}(\wp,\aleph,\ell)), P_b(\eth,\mathscr{T}(\aleph,\ell,\wp)) \right\}$$

and

$$P_b(\mathscr{T}(\aleph,\ell,\wp),\eth) < \lambda \max\left\{ P_b(\backepsilon,\mathscr{T}(\ell,\wp,\aleph)), P_b(\mho,\mathscr{T}(\wp,\aleph,\ell)), P_b(\eth,\mathscr{T}(\aleph,\ell,\wp)) \right\}.$$

Therefore,

$$\max\left\{ \begin{array}{c} P_b(\mathscr{T}(\ell,\wp,\aleph),\backepsilon), \\ P_b(\mathscr{T}(\wp,\aleph,\ell),\mho), \\ P_b(\mathscr{T}(\aleph,\ell,\wp),\eth) \end{array} \right\} < \lambda \max\left\{ \begin{array}{c} P_b(\backepsilon,\mathscr{T}(\ell,\wp,\aleph)), P_b(\mho,\mathscr{T}(\wp,\aleph,\ell)), \\ P_b(\eth,\mathscr{T}(\aleph,\ell,\wp)) \end{array} \right\}.$$

It follows that $\mathscr{T}(\ell,\wp,\aleph) = \backepsilon = \int\ell$, $\mathscr{T}(\wp,\aleph,\ell) = \mho = \int\wp$ and $\mathscr{T}(\aleph,\ell,\wp) = \eth = \int\aleph$.

Since $(\mathscr{T},\int)$ is a weakly compatible pair, We have $\mathscr{T}(\backepsilon,\mho,\eth) = \int\backepsilon$, $\mathscr{T}(\mho,\eth,\backepsilon) = \int\mho$ and

$\mathscr{T}(\eth,\backepsilon,\mho)=\int\eth.$

From (1), we have that

$$
\begin{aligned}
&P_b\left(\mathscr{T}(\backepsilon,\mho,\eth),\mathscr{T}(\ell_n,\wp_n,\aleph_n)\right)\le \alpha\left(\int\backepsilon,\int\mho,\int\eth\right)P_b\left(\mathscr{T}(\backepsilon,\mho,\eth),R(\ell_n,\wp_n,\aleph_n)\right)\\
\le\;& \varphi\left(\lambda K(\backepsilon,\mho,\eth,\ell_n,\wp_n,\aleph_n)\right)\\
<\;& \lambda\max\left\{\begin{array}{c}
P_b(\int\backepsilon,\int\ell_n),P_b(\int\mho,\int\wp_n),P_b(\int\eth,\int\aleph_n),\\
P_b(\int\backepsilon,\mathscr{T}(\backepsilon,\mho,\eth)),P_b(\int\mho,\mathscr{T}(\mho,\eth,\backepsilon)),P_b(\int\eth,\mathscr{T}(\eth,\backepsilon,\mho)),\\
P_b(\int\ell_n,\mathscr{T}(\ell_n,\wp_n,\aleph_n)),P_b(\int\wp_n,\mathscr{T}(\wp_n,\aleph_n,\ell_n)),P_b(\int\aleph_n,\mathscr{T}(\aleph_n,\ell_n,\wp_n)),\\
\frac{P_b(\int\backepsilon,\mathscr{T}(\backepsilon,\mho,\eth))P_b(\int\ell_n,\mathscr{T}(\ell_n,\wp_n,\aleph_n))}{2\kappa^2[1+P_b(\int\backepsilon,\int\ell_n)]},\\
\frac{P_b(\int\mho,\mathscr{T}(\mho,\eth,\backepsilon))P_b(\int\wp_n,\mathscr{T}(\wp_n,\aleph_n,\ell_n))}{2\kappa^2[1+P_b(\int\mho,\int\wp_n)]},\\
\frac{P_b(\int\eth,\mathscr{T}(\eth,\backepsilon,\mho))P_b(\int\aleph_n,\mathscr{T}(\aleph_n,\ell_n,\wp_n))}{2\kappa^2[1+P_b(\int\eth,\int\aleph_n)]},
\end{array}\right\}.
\end{aligned}
$$

Letting $n\to\infty$,in the above inequality, we have that

$$
P_b\left(\mathscr{T}(\backepsilon,\mho,\eth),\backepsilon\right)<\lambda\max\left\{\;P_b(\mathscr{T}(\backepsilon,\mho,\eth),\backepsilon),P_b(\mathscr{T}(\mho,\eth,\backepsilon),\mho),P_b(\mathscr{T}(\eth,\backepsilon,\mho),\eth)\;\right\}.
$$

Therefore,

$$
\max\left\{\begin{array}{c}
P_b(\mathscr{T}(\backepsilon,\mho,\eth),\backepsilon),\\
P_b(\mathscr{T}(\mho,\eth,\backepsilon),\mho),\\
P_b(\mathscr{T}(\eth,\backepsilon,\mho),\eth)
\end{array}\right\}<\lambda\max\left\{\begin{array}{c}
P_b(\mathscr{T}(\backepsilon,\mho,\eth),\backepsilon),P_b(\mathscr{T}(\mho,\eth,\backepsilon),\mho),\\
P_b(\mathscr{T}(\eth,\backepsilon,\mho),\eth)
\end{array}\right\}.
$$

It follows that $\mathscr{T}(\backepsilon,\mho,\eth)=\backepsilon=\int\backepsilon$, $\mathscr{T}(\mho,\eth,\backepsilon)=\mho=\int\mho$ and $\mathscr{T}(\eth,\backepsilon,\mho)=\eth=\int\eth$. Therefore $(\backepsilon,\mho,\eth)$ is a common tripled fixed point of $\mathscr{T}$ and $\int$ for uniqueness let us suppose $(\backepsilon^*,\mho^*,\eth^*)$ be another common tripled fixed point of $\mathscr{T}$ and $\int$ such that $\backepsilon\neq\backepsilon^*$, $\mho\neq\mho^*$ and $\eth\neq\eth^*$.

Now from (1), we have that

$$
\begin{aligned}
P_b(\backepsilon,\backepsilon^*) &= P_b\left(\mathscr{T}(\backepsilon,\mho,\eth),\mathscr{T}(\backepsilon^*,\mho^*,\eth^*)\right)\\
&\le \alpha\left(\int\backepsilon,\int\mho,\int\eth\right)P_b\left(\mathscr{T}(\backepsilon,\mho,\eth),\mathscr{T}(\backepsilon^*,\mho^*,\eth^*)\right)\\
&\le \varphi\left(\lambda K(\backepsilon,\mho,\eth,\backepsilon^*,\mho^*,\eth^*)\right)
\end{aligned}
$$

$$
\begin{aligned}
&< \lambda \max \left\{ \begin{array}{c} P_b(\int\backepsilon,\int\backepsilon^*), P_b(\int\mho,\int\mho^*), P_b(\int\eth,\int\eth^*), \\ P_b(\int\backepsilon,\mathscr{T}(\backepsilon,\mho,\eth)), P_b(\int\mho,\mathscr{T}(\mho,\eth,\backepsilon)), P_b(\int\eth,\mathscr{T}(\eth,\backepsilon,\mho)), \\ P_b(\int\backepsilon^*,\mathscr{T}(\backepsilon^*,\mho^*,\eth^*)), P_b(\int\mho^*,\mathscr{T}(\mho^*,\eth^*,\backepsilon^*)), \\ P_b(\int\eth^*,\mathscr{T}(\eth^*,\backepsilon^*,\mho^*)), \\ \frac{P_b(\int\backepsilon,\mathscr{T}(\backepsilon,\mho,\eth))P_b(\int\backepsilon^*,\mathscr{T}(\backepsilon^*,\mho^*,\eth^*))}{2\kappa^2[1+P_b(\int\backepsilon,\int\backepsilon^*)]}, \\ \frac{P_b(\int\mho,\mathscr{T}(\mho,\eth,\backepsilon))P_b(\int\mho^*,\mathscr{T}(\mho^*,\eth^*,\backepsilon^*))}{2\kappa^2[1+P_b(\int\mho,\int\mho^*)]}, \\ \frac{P_b(\int\eth,\mathscr{T}(\eth,\backepsilon,\mho))P_b(\int\eth^*,\mathscr{T}(\eth^*,\backepsilon^*,\mho^*))}{2\kappa^2[1+P_b(\int\eth,\int\eth^*)]}, \end{array} \right\} \\
&< \lambda \max \left\{ P_b(\backepsilon,\backepsilon^*), P_b(\mho,\mho^*), P_b(\eth,\eth^*), \right\}.
\end{aligned}
$$

Therefore,

$$
\max\{P_b(\backepsilon,\backepsilon^*), P_b(\mho,\mho^*), P_b(\eth,\eth^*)\} < \lambda \max\{P_b(\backepsilon,\backepsilon^*), P_b(\mho,\mho^*), P_b(\eth,\eth^*)\}.
$$

It is a contradiction. Hence $(\backepsilon,\mho,\eth)$ is the unique common tripled fixed point of $\mathscr{T}$ and $\int$.

Example 3.3: Let $\Im=[0,1]$ and $\mathscr{T}:\Im^3\to\Im$ be as $\mathscr{T}(\ell,\wp,\aleph)=\frac{\ell^2+\wp^2+\aleph^2}{16(\ell+\wp+\aleph+1)}$ and $\int:\Im\to\Im$ by $\int(\ell)=\frac{\ell}{4}$, define $\varphi:[0,\infty)\to[0,\infty)$ as $\varphi(t)=\frac{2t}{5}$ and
$\alpha:\Im^3\to R^+$ as $\alpha(\ell,\wp,\aleph)=\begin{cases} 1 & \text{for} \quad (\ell,\wp,\aleph)\in[0,1] \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$,
$P_b:\Im^2\to[0,\infty)$ such that $P_b(\ell;\wp)=[\max\{\ell,\wp\}]^2+|\ell-\wp|^2$, for all $\ell,\wp\in\Im$ is a complete partial b-metric on $\Im$.We show that $\mathscr{T},\int$ is an α-admissible mappings.
Let $\ell,\wp,\aleph\in\Im$, if $\alpha(\int\ell,\int\wp,\int\aleph)\geq 1$ then $\ell,\wp,\aleph\in\Im$. On the other hand, for all $\ell,\wp,\aleph\in[0,1]$ then $\mathscr{T}(\ell,\wp,\aleph)\leq 1$.It follows that $\alpha(\mathscr{T}(\ell,\wp,\aleph),\mathscr{T}(\wp,\aleph,\ell),\mathscr{T}(\aleph,\ell,\wp))\geq 1$.Therefore, the predication holds. In support of the above argument $\alpha(\int 0,\int 0,\int 0)>1$.Now, if $\{\ell_n\}$, $\{\wp_n\}$ and $\{\aleph_n\}$ are a sequence in $\Im$ such that $\alpha(\int\ell_n,\int\wp_n,\int\aleph_n)>1$ and $\ell_n\to\ell,\wp_n\to\wp$, $\aleph_n\to\aleph\in\Im$ for all $n\in N\cup\{0\}$, then $\ell_n,\wp_n,\aleph_n\subseteq[0,1]$ and hence $\ell,\wp,\aleph\in[0,1]$ which implies $\alpha(\int\ell,\int\wp,\int\aleph)\geq 1$. Let $\ell,\wp,\aleph\in[0,1]$. Then

$$
\begin{aligned}
&P_b(\mathscr{T}(\ell,\wp,\aleph),\mathscr{T}(\backepsilon,\mho,\eth)) \\
= \ & \left[\max\left\{\frac{\ell^2+\wp^2+\aleph^2}{16(\ell+\wp+\aleph+1)},\frac{\backepsilon^2+\mho^2+\eth^2}{16(\backepsilon+\mho+\eth+1)}\right\}\right]^2
\end{aligned}
$$

$$
\begin{aligned}
& +\left|\frac{\ell^2+\wp^2+\aleph^2}{16(\ell+\wp+\aleph+1)}-\frac{\backepsilon^2+\mho^2+\eth^2}{16(\backepsilon+\mho+\eth+1)}\right|^2 \\
= & \frac{1}{256}\left\{\begin{array}{c}
\left[\max\left\{\begin{array}{c}\frac{\ell^2}{\ell+\wp+\aleph+1}, \\ \frac{\backepsilon^2}{\backepsilon+\mho+\eth+1}\end{array}\right\}+\max\left\{\begin{array}{c}\frac{\wp^2}{\ell+\wp+\aleph+1}, \\ \frac{\mho^2}{\backepsilon+\mho+\eth+1}\end{array}\right\}+\max\left\{\begin{array}{c}\frac{\aleph^2}{\ell+\wp+\aleph+1}, \\ \frac{\eth^2}{\backepsilon+\mho+\eth+1}\end{array}\right\}\right]^2 \\
+\left|\frac{\ell^2}{(\ell+\wp+\aleph+1)}-\frac{\backepsilon^2}{(\backepsilon+\mho+\eth+1)}\right|^2+\left|\frac{\wp^2}{(\ell+\wp+\aleph+1)}-\frac{\mho^2}{(\backepsilon+\mho+\eth+1)}\right|^2 \\
+\left|\frac{\aleph^2}{(\ell+\wp+\aleph+1)}-\frac{\eth^2}{(\backepsilon+\mho+\eth+1)}\right|^2
\end{array}\right\} \\
\leq & \frac{1}{256}\left\{\begin{array}{c}
\left[\max\left\{\frac{\ell^2}{\ell+1},\frac{\backepsilon^2}{\backepsilon+1}\right\}+\max\left\{\frac{\wp^2}{\wp+1},\frac{\mho^2}{\mho+1}\right\}+\max\left\{\frac{\aleph^2}{\aleph+1},\frac{\eth^2}{\eth+1}\right\}\right]^2 \\
+\left|\frac{\ell^2}{(\ell+1)}-\frac{\backepsilon^2}{(\backepsilon+1)}\right|^2+\left|\frac{\wp^2}{(\wp+1)}-\frac{\mho^2}{(\mho+1)}\right|^2+\left|\frac{\aleph^2}{(\aleph+1)}-\frac{\eth^2}{(\eth+1)}\right|^2
\end{array}\right\} \\
\leq & \frac{1}{256}\left\{\begin{array}{c}
\left[\max\left\{\frac{\ell}{\ell+1},\frac{\backepsilon}{\backepsilon+1}\right\}\right]^2+\left|\frac{\ell}{(\ell+1)}-\frac{\backepsilon}{(\backepsilon+1)}\right|^2 \\
+\left[\max\left\{\frac{\wp}{\wp+1},\frac{\mho}{\mho+1}\right\}\right]^2+\left|\frac{\wp}{(\wp+1)}-\frac{\mho}{(\mho+1)}\right|^2 \\
+\left[\max\left\{\frac{\aleph}{\aleph+1},\frac{\eth}{\eth+1}\right\}\right]^2+\left|\frac{\aleph}{(\aleph+1)}-\frac{\eth}{(\eth+1)}\right|^2
\end{array}\right\} \\
\leq & \frac{1}{16}\left\{\begin{array}{c}
\left[\max\left\{\frac{\ell}{4},\frac{\backepsilon}{4}\right\}\right]^2+\left|\frac{\ell}{4}-\frac{\backepsilon}{4}\right|^2 \\
+\left[\max\left\{\frac{\wp}{4},\frac{\mho}{4}\right\}\right]^2+\left|\frac{\wp}{4}-\frac{\mho}{4}\right|^2 \\
+\left[\max\left\{\frac{\aleph}{4},\frac{\eth}{4}\right\}\right]^2+\left|\frac{\aleph}{4}-\frac{\eth}{4}\right|^2
\end{array}\right\} \\
= & \frac{1}{16}\left[P_b(\int\ell,\int\backepsilon)+P_b(\int\wp,\int\mho)+P_b(\int\aleph,\int\eth)\right] \\
\leq & \frac{2}{5}\left(\frac{1}{4}\max\left\{P_b(\int\ell,\int\backepsilon),P_b(\int\wp,\int\mho),P_b(\int\aleph,\int\eth)\right\}\right) \\
\leq & \varphi\left(\lambda K(\ell,\wp,\aleph,\backepsilon,\mho,\eth)\right)
\end{aligned}
$$

Hence all conditions of Theorem 3.2 are holds and $(0,0,0)$ is the unique common tripled fixed point of $\mathscr{T}$ and $\int$.

3.1. Application to Integral Equations.

As an application of Theorem 3.2, we examine the existence of a singular solution to an initial value problem in this section.

Theorem 3.1.1 Consider the initial value problem

$$\ell'(t)=\Gamma(\delta,\ell(\delta),\ell(\delta),\ell(\delta)),\ \delta\in I=[0,1],\ \ell(0)=\ell_0 \tag{20}$$

where $\Gamma : I\times\left[\frac{\ell_0}{2\kappa^2},\infty\right)^3\to\left[\frac{\ell_0}{2\kappa^2},\infty\right)$ and $\ell_0\in\mathbb{R}$ and

$$\int_0^\delta \Gamma(\tau,\ell(\tau),\wp(\tau),\aleph(\tau))d\tau \le \max\left\{\begin{array}{c}\frac{1}{8}\int_0^\delta \Gamma(\tau,\ell(\tau),\ell(\tau),\ell(\tau))d\tau-\frac{7\ell_0}{16\kappa^2},\\ \frac{1}{8}\int_0^\delta \Gamma(\tau,\wp(\tau),\wp(\tau),\wp(\tau))d\tau-\frac{7\ell_0}{16\kappa^2},\\ \frac{1}{8}\int_0^\delta \Gamma(\tau,\aleph(\tau),\aleph(\tau),\aleph(\tau))d\tau-\frac{7\ell_0}{16\kappa^2},\end{array}\right\}.$$

Define $\varphi:[0,\infty)\to[0,\infty)$ as $\varphi(t)=\frac{t}{2}$. And consider the following conditions:

(a) If there exist a function $\eta:\left[\frac{\ell_0}{2\kappa^2},\infty\right)^3\to R^+$ such that there is an $\ell_1\in C\left(I,\left[\frac{\ell_0}{2\kappa^2},\infty\right)\right)$, for all $\delta\in I$, we have: $\eta\left(\ell_1(\delta),\ell_1(\delta),\int_0^\delta\Gamma(\tau,\ell_1(\tau),\ell_1(\tau),\ell_1(\tau))d\tau\right)\ge 0$,

(b) For all $\delta\in I$ and for all $\ell,\wp,\aleph\in C\left(I,\left[\frac{\ell_0}{2\kappa^2},\infty\right)\right)$,

$$\eta(\ell(\delta),\wp(\delta),\aleph(\delta))\ge 0\Rightarrow \eta(A,B,C)\ge 0$$

where, $A=\frac{\ell_0}{\kappa^2}+\int_0^\delta\Gamma(\tau,\ell(\tau),\ell(\tau),\ell(\tau))d\tau$, $B=\frac{\wp_0}{\kappa^2}+\int_0^\delta\Gamma(\tau,\wp(\tau),\wp(\tau),\wp(\tau))d\tau$ and $C=\frac{\aleph_0}{\kappa^2}+\int_0^\delta\Gamma(\tau,\aleph(\tau),\aleph(\tau),\aleph(\tau))d\tau$;

(c) For any point ℓ of a sequence $\{\ell_n\}$ of points in $C\left(I,\left[\frac{\ell_0}{2\kappa^2},\infty\right)\right)$ with $\eta(\ell_n,\ell_n,\ell_{n+1})\ge 0$, $\lim_{n\to\infty}\inf\eta(\ell_n,\ell_n,\ell)\ge 0$.

Then, equation (20) has a unique solution in $C\left(I,\left[\frac{\ell_0}{2\kappa^2},\infty\right)\right)$.

Proof The integral equation corresponding to initial value problem (20) is

$$\ell(t)=\ell_0+\kappa^2\int_0^\delta\Gamma(\tau,\ell(\tau),\ell(\tau),\ell(\tau))d\tau.$$

Let $\mathfrak{I}=C\left(I,\left[\frac{\ell_0}{2\kappa^2},\infty\right)\right)$ and $P_b(\ell,\wp)=\left[\max\{\ell-\frac{\ell_0}{2\kappa^2},\wp-\frac{\ell_0}{2\kappa^2}\}\right]^2+|\ell-\wp|^2$ for $\ell,\wp\in\mathfrak{I}$. Define $\int:\mathfrak{I}\to\mathfrak{I}$ by $\int(\ell)(\delta)=\frac{\ell(\delta)}{4}$ and $\mathscr{T}:\mathfrak{I}^3\to\mathfrak{I}$ by $\mathscr{T}(\ell,\wp,\aleph)(\delta)=\frac{\ell_0}{\kappa^2}+\int_0^\delta\Gamma(\tau,\ell(\tau),\wp(\tau),\aleph(\tau))d\tau$.

Now

$$
\begin{aligned}
&P_b(\mathscr{T}(\ell,\wp,\aleph)(\delta),\mathscr{T}(\backepsilon,\mho,\eth)(\delta))\\
&=\left[\max\left\{\mathscr{T}(\ell,\wp,\aleph)-\tfrac{\ell_0}{2\kappa^2},\mathscr{T}(\backepsilon,\mho,\eth)-\tfrac{\backepsilon_0}{2\kappa^2}\right\}\right]^2+|\mathscr{T}(\ell,\wp,\aleph)-\mathscr{T}(\backepsilon,\mho,\eth)|^2\\
&=\left[\max\left\{\tfrac{\ell_0}{2\kappa^2}+\int_0^\delta\Gamma(\tau,\ell(\tau),\wp(\tau),\aleph(\tau))d\tau,\tfrac{\backepsilon_0}{2\kappa^2}+\int_0^\delta\Gamma(\tau,\backepsilon(\tau),\mho(\tau),\eth(\tau))d\tau\right\}\right]^2\\
&\quad+|\int_0^\delta\Gamma(\tau,\ell(\tau),\wp(\tau),\aleph(\tau))d\tau-\int_0^\delta\Gamma(\tau,\backepsilon(\tau),\mho(\tau),\eth(\tau))d\tau|^2\\
&\leq\left[\max\left\{\begin{array}{c}\tfrac{\ell_0}{2\kappa^2}+\max\left\{\begin{array}{c}\tfrac{1}{8}\int_0^\delta\Gamma(\tau,\ell(\tau),\ell(\tau),\ell(\tau))d\tau-\tfrac{7\ell_0}{16\kappa^2},\\ \tfrac{1}{8}\int_0^\delta\Gamma(\tau,\wp(\tau),\wp(\tau),\wp(\tau))d\tau-\tfrac{7\ell_0}{16\kappa^2},\\ \tfrac{1}{8}\int_0^\delta\Gamma(\tau,\aleph(\tau),\aleph(\tau),\aleph(\tau))d\tau-\tfrac{7\ell_0}{16\kappa^2},\end{array}\right\},\\ \tfrac{\backepsilon_0}{2\kappa^2}+\max\left\{\begin{array}{c}\tfrac{1}{8}\int_0^\delta\Gamma(\tau,\backepsilon(\tau),\backepsilon(\tau),\backepsilon(\tau))d\tau-\tfrac{7\backepsilon_0}{16\kappa^2},\\ \tfrac{1}{8}\int_0^\delta\Gamma(\tau,\mho(\tau),\mho(\tau),\mho(\tau))d\tau-\tfrac{7\backepsilon_0}{16\kappa^2},\\ \tfrac{1}{8}\int_0^\delta\backepsilon(\tau,\eth(\tau),\eth(\tau),\eth(\tau))d\tau-\tfrac{7\backepsilon_0}{16\kappa^2},\end{array}\right\}\end{array}\right\}\right]^2\\
&+|\max\left\{\begin{array}{c}\tfrac{1}{8}\int_0^\delta\Gamma(\tau,\ell(\tau),\ell(\tau),\ell(\tau))d\tau-\tfrac{7\ell_0}{16\kappa^2}\\ \tfrac{1}{8}\int_0^\delta\Gamma(\tau,\wp(\tau),\wp(\tau),\wp(\tau))d\tau-\tfrac{7\ell_0}{16\kappa^2}\\ \tfrac{1}{8}\int_0^\delta\Gamma(\tau,\aleph(\tau),\aleph(\tau),\aleph(\tau))d\tau-\tfrac{7\ell_0}{16\kappa^2}\end{array}\right\}-\max\left\{\begin{array}{c}\tfrac{1}{8}\int_0^\delta\Gamma(\tau,\backepsilon(\tau),\backepsilon(\tau),\backepsilon(\tau))d\tau-\tfrac{7\backepsilon_0}{16\kappa^2}\\ \tfrac{1}{8}\int_0^\delta\backepsilon(\tau,\mho(\tau),\mho(\tau),\mho(\tau))d\tau-\tfrac{7\backepsilon_0}{16\kappa^2}\\ \tfrac{1}{8}\int_0^\delta\Gamma(\tau,\eth(\tau),\eth(\tau),\eth(\tau))d\tau-\tfrac{7\backepsilon_0}{16\kappa^2}\end{array}\right\}|^2\\
&=\tfrac{1}{4\kappa^4}\left[\max\left\{\begin{array}{c}\max\left\{\tfrac{\ell(\delta)}{4}-\tfrac{\ell_0}{8},\tfrac{\wp(\delta)}{4}-\tfrac{\ell_0}{8},\tfrac{\aleph(\delta)}{4}-\tfrac{\ell_0}{8},\right\}\\ \max\left\{\tfrac{\backepsilon(\delta)}{4}-\tfrac{\backepsilon_0}{8},\tfrac{\mho(\delta)}{4}-\tfrac{\backepsilon_0}{8},\tfrac{\eth(\delta)}{4}-\tfrac{\backepsilon_0}{8}\right\}\end{array}\right\}\right]^2\\
&+\tfrac{1}{4\kappa^4}|\max\left\{\tfrac{\ell(\delta)}{4}-\tfrac{9\ell_0}{8},\tfrac{\wp(\delta)}{4}-\tfrac{9\ell_0}{8},\tfrac{\aleph(\delta)}{4}-\tfrac{9\ell_0}{8}\right\}-\max\left\{\tfrac{\backepsilon(\delta)}{4}-\tfrac{9\backepsilon_0}{8},\tfrac{\mho(\delta)}{4}-\tfrac{9\backepsilon_0}{8},\tfrac{\eth(\delta)}{4}-\tfrac{9\backepsilon_0}{8}\right\}|^2\\
&\leq\tfrac{1}{2}\left(\tfrac{1}{2\kappa^3}\max\left\{\begin{array}{c}\left[\max\left\{\tfrac{\ell(\delta)}{4}-\tfrac{\ell_0}{2\kappa^2},\tfrac{\backepsilon(\delta)}{4}-\tfrac{\backepsilon_0}{2\kappa^2}\right\}\right]^2+|\tfrac{\ell(\delta)}{4}-\tfrac{\backepsilon(\delta)}{4}|^2,\\ \left[\max\left\{\tfrac{\wp(\delta)}{4}-\tfrac{\ell_0}{2\kappa^2},\tfrac{\mho(\delta)}{4}-\tfrac{\backepsilon_0}{2\kappa^2}\right\}\right]^2+|\tfrac{\wp(\delta)}{4}-\tfrac{\mho(\delta)}{4}|^2\\ \left[\max\left\{\tfrac{\aleph(\delta)}{4}-\tfrac{\ell_0}{2\kappa^2},\tfrac{\eth(\delta)}{4}-\tfrac{\backepsilon_0}{2\kappa^2}\right\}\right]^2+|\tfrac{\aleph(\delta)}{4}-\tfrac{\eth(\delta)}{4}|^2\end{array}\right\}\right)\\
&\qquad=\tfrac{1}{2}\left(\tfrac{1}{2\kappa^3}\max\left\{P_b(\int\ell(\delta),\int\backepsilon(\delta)),P_b(\int\wp(\delta),\int\mho(\delta)),P_b(\int\aleph(\delta),\int\eth(\delta))\right\}\right)\\
&\qquad\leq\varphi\left(\lambda K(\ell(\delta),\wp(\delta),\aleph(\delta),\backepsilon(\delta),\mho(\delta),\eth(\delta))\right).
\end{aligned}
$$

Thus

$$P_b(\mathscr{T}(\ell,\wp,\aleph)(\delta),\mathscr{T}(\backepsilon,\mho,\eth)(\delta))\leq\varphi\left(\lambda K(\ell(\delta),\wp(\delta),\aleph(\delta),\backepsilon(\delta),\mho(\delta),\eth(\delta))\right)\forall\ell,\wp,\aleph\in\Im$$

with $\eta(\ell(\delta),\wp(\delta),\aleph(\delta))\geq 0$ for all $\delta\in I$.

Define $\alpha : \Im^3 \to R^+$ as $\alpha(\ell,\wp,\aleph) = \begin{cases} 1 & \text{for} \quad \eta(\ell(\delta),\wp(\delta),\aleph(\delta)) \geq 0, \delta \in I \\ 0 & \text{for} \quad \text{otherwise.} \end{cases}$

Then obviously, $\mathscr{T}$ and $\int$ are α-admissible, for all,$\ell,\wp,\aleph \in \Im$

$$\alpha((\int\ell,\int\wp,\int\aleph)(\delta))P_b(\mathscr{T}(\ell,\wp,\aleph)(\delta),\mathscr{T}(\backepsilon,\mho,\eth)(\delta)) \leq \varphi(\lambda K(\ell(\delta),\wp(\delta),\aleph(\delta),\backepsilon(\delta),\mho(\delta),\eth(\delta)))$$

It follows from Eq. (20), $\mathscr{T}$ and $\int$ have a unique fixed point in $\Im$.

4. Conclusions

This study uses contractive mappings of the $(\alpha,\varphi)-K$-type in the context of partial b-metric space to present some fixed point results and appropriate examples that illustrate the main findings. Additionally, integral equation applications are given.

Funding

No funding.

Author contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgment

The authors are very thanks to the reviewers and editors for valuable comments, remarks and suggestions for improving the content of the paper.

Conflict of Interests

The authors declare that there is no conflict of interests.

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