Available online at http://scik.org
Adv. Fixed Point Theory, 2023, 13:12
https://doi.org/10.28919/afpt/8075
ISSN: 1927-6303

APPROXIMATION OF FIXED POINT VIA NEW ITERATIVE PROCESS FOR GENERALIZED NONEXPANSIVE MAPPINGS IN BANACH SPACE

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Abstract. In this paper, we proved strong and week convergence theorem for our proposed iterative process for class of generalized nonexpansive mappings in uniformly convex Banach space. Finally, we present a numerical example to illustrate that our iterative process is faster than the well known iteration process appeared in the literature, the results obtained in this paper improve, extend the results of [6], [9] and many more in this direction.

Keywords: Generalized (α,β)-nonexpansive type 1 mapping, generalized (α,β)-nonexpansive type 2 mapping, fixed point, strong and week convergence theorems.

2020 AMS Subject Classification: 47H09, 47H10.

1. Introduction

The concept of fixed points theory and its application has proven to be a vital tool in the study of nonlinear functional analysis and it is a very useful tool in establishing the existence and uniqueness theorems for nonlinear ordinary, partial and random differential and integral equations in different abstract spaces.

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Received June 30, 2023

Throughout in this paper, we assume that X is Banach space and C be a nonempty subset of X. Let $F(T) = \{Tx = x : x \in C\}$ is denote the set of fixed point of T. Let $T : C \to C$ be mapping, then T is said to be

(i) *contraction*, if $\|Tx - Ty\| \leq k\|x - y\|$, for all $x, y \in C$ and $k \in [0, 1)$, where k is called contraction constant.

(ii) *nonexpansive*, if $k = 1$ that is $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

(iii) *quasi-nonexpansive* if $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$ and $p \in F(T)$.

In 1920 S. Banach [1] proved most important result in complete metric space, it state that *if X is a complete metric space and $T : X \to X$ is contraction mapping, then T has a unique fixed point.* Banach fixed point theorem is not only proved an existence and uniqueness of fixed point but also we will see in the proof of theorem , it provides us with a constructive procedure for getting better and better approximations of the fixed point. This procedure is called Picard iteration.

In 1965, Browder [8], Gohde [10] and Kirk [11] independently prove that *every nonexpansive mapping of a closed convex and bounded subset of uniformly convex Banach space has a fixed point* . Further several other researcher have examine an amount of generalization of nonexpansive in the few decades. In this context, we have define the following generalization of nonexapansive mappings by various researchers as follows:

Definition 1.1. *Let C be a nonempty subset of Banach Space X and $T : C \to C$ be mapping then T is said to be*

(i) mean nonexpansive [2]*, if there exists $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$ such that*

$$\|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|,$$

for all $x, y \in C$,

(ii) satisfy condition (C)(Suzuki type) [3]*, if*

$$\frac{1}{2}\|Tx - x\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$,

(iii) satisfy condition (C_λ) if

$$\lambda\|Tx - x\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$,

(iv) *generalized mean nonexapnsive mapping , if there exists* $\alpha, \beta, \lambda \in [0,1)$, *with* $\alpha + \beta < 1$ *such that for all* $x, y \in C$

$$\lambda \|Tx - x\| \leq \|x - y\| \Longrightarrow \|Tx - Ty\| \leq \alpha \|x - y\| + \beta \|x - Ty\|,$$

(v) α*-nonexpansive mapping* [4] *if there exists* $\alpha < 1$ *such that for all* $x, y \in C$

$$\|Tx - Ty\|^2 \leq \alpha \|Tx - y\|^2 + \alpha \|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2,$$

(vi) *generalized* (α, β)*-nonexpansive type-1, if there exist* $\alpha, \beta, \lambda \in [0,1)$ *with* $\alpha \leq \beta$ *and* $\alpha + \beta < 1$ *such that for all* $x, y \in C$, $\lambda \|Tx - Ty\| \leq \|x - y\|$

$$\|Tx - Ty\| \leq \alpha \|y - Tx\| + \beta \|x - Ty\| + (1 - (\alpha + \beta))\|x - y\|.$$

Remark 1.1. (i) It is worth mentioning that nonexpansive mappings are continuous on their domains but mean nonexpansive, generalized mean nonexpansive, mappings satisfying condition (C), condition (C_λ), need not be continuous. Due to this fact, these mappings are more fascinating and applicable compare to nonexpansive mappings.

(ii) If $\alpha = \beta = 0$ and $\lambda = \frac{1}{2}$, then the generalized (α, β)-nonexpansive type 1 mapping satisfying the condition (C).

(iii) If $\alpha = \beta = 0$ and $\lambda \in [0,1)$, then the generalized (α, β)-nonexpansive type 1 mapping satisfying condition (C_λ).

From the above definition, we have the following facts (See proposition 3.4 [6])

(i) Every nonexpansive mapping is a generalized (α, β)-nonexpansive type 1 mapping.

(ii) Every mean nonexpansive mapping is a generalized (α, β)-nonexpansive type 1 mapping.

(iii) All mappings satisfying condition (C) is an (α, β)-nonexpansive type 1 mapping.

(iv) All mappings satisfying condition (C_λ) is an (α, β)-nonexpansive type 1 mapping.

Akutsah et. al. [6] proved that the converse of above statements are not always true (See example 3.5). In the same paper, authors [6] have proved the following fact about the generalized (α, β)-nonexpansive mapping:

Let C be nonempty subset of Banach Space X and $T : C \to C$ be generalized (α,β)-nonexpansive of Type 1, with $F(T) \neq \phi$, then

(i) T is a quasi-nonexpansive (See proposition 3.6 [6]).

(ii) $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex (See proposition 3.7 [6]).

Beside that Akutsah et. al. established corollaries [3.8], [3.9], [3.10], [3.11], [3.12] for non-expansive, mean nonexpansive, condition C, condition C_λ, generalised mean nonexpansive respectively.

It is well known fact that Picard iteration procedure fails to approximate the fixed point for nonexpansive mapping, for example $T : [0,1] \to [0,1]$ defined by $Tx = 1-x$ for all $x \in [0,1]$ is nonexpansive mapping with $F(T) = \frac{1}{2}$, but for initial guesses $x_0 \neq \frac{1}{2}$, Picard iteration fails to converge to fixed point of T.

2. Main Results

Proposition 2.1. *Let C be a Nonempty subset of Banach Space X and $T : C \to C$ be generalized (α,β)-nonexpansive type-1 mapping with $F(T) \neq \phi$. Then T is quasi-nonexapnsive mapping.*

Lemma 2.2. *Let X be uniformly convex Banach Space and $0 \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$ and $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n)\| = c$ holds for some $r \geq 0$. Then $\|x_n - y_n\| = 0$.*

Theorem 2.3. *Let C be nonempty closed subset of Banach Space X with Opial propert and $T : C \to C$ be a generalized (α,β)-nonexpansive type 1 mapping with $\lambda = \frac{\gamma}{2}$, $\gamma \in [0,1)$. If $\{x_n\}$ converges weakly to x and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tx = x$, that is $I - T$ is demiclosed at zero, where I is the identity mapping.*

In this Section, we prove strong and week convergence theorem for generalized (α,β)-nonexpansive mapping for an iteration defined by (2.1) in the setting of uniformly convex Banach space.

Let C be a nonempty subset of Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive mapping, for each $x_0 \in C$, we define a sequence $\{x_n\}$ in iterative manner as below:

$$\begin{aligned} x_{n+1} &= Ty_n \\ y_n &= T((1-\beta_n)z_n + \beta_n T z_n) \\ z_n &= T((1-\alpha_n)x_n + \alpha_n T x_n), \end{aligned} \tag{2.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

Theorem 2.4. *Let C be a Nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ be a mapping which satisfies generalized (α, β)-nonexpansive type -1 mapping with $F(T) \neq \phi$. If $\{x_n\}$ be a sequence defined by (2.1), then*

(a) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in F(T)$.
(b) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. Let $p \in F(T)$, then by Proposition 2.1, T be a quasi-nonexpansive mapping, using (2.1), we get

$$\begin{aligned} \|z_n - p\| &= \|T((1-\alpha_n)x_n + \alpha_n T x_n) - p\| \\ &\leq \|(1-\alpha_n)x_n + \alpha_n T x_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|T x_n - p\| \\ &\leq \|x_n - p\|, \end{aligned} \tag{2.2}$$

again using (2.1), (2.2) and Proposition 2.1, we get

$$\begin{aligned} \|y_n - p\| &= \|T((1-\beta_n)z_n + \beta_n T z_n) - p\| \\ &\leq \|(1-\beta_n)z_n + \beta_n T z_n - p\| \\ &\leq (1-\beta_n)\|z_n - p\| + \beta_n \|T z_n - p\| \\ &\leq \|z_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{2.3}$$

Again using (2.1), (2.3) and Proposition 2.1, we get

$$\begin{aligned} \|x_{n+1}-p\| &= \|Ty_n-p\| \\ &\leq \|y_n-p\| \\ &\leq \|x_n-p\|. \end{aligned} \tag{2.4}$$

This shows that $\{\|x_n-p\|\}$ is bounded and non-increasing for all $p \in F(T)$. Thus $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-p\|$ exists, hence it complete the proof of part (a).

Proof of part (b):

From part (a), it clear that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p \in F(T)$, therefore, we suppose that $\lim_{n\to\infty}\|x_n-p\| = r \geq 0$. Now two cases aeries:

Case-I If $r=0$ then $\lim_{n\to\infty}\|x_n-Tx_n\| = 0$ is obvious.Indeed, consequence of triangle inequality and application of Proposition 2.1, we have

$$\lim_{n\to\infty}\|x_n-Tx_n\| \leq \lim_{n\to\infty}\|x_n-p\| + \lim_{n\to\infty}\|Tx_n-p\| \leq 2r = 0$$

Case-II If $r \neq 0$. Now taking $\limsup$ on both side of (2.2), we get

$$\limsup_{n\to\infty}\|z_n-p\| \leq \limsup_{n\to\infty}\|x_n-p\|. \tag{2.5}$$

From (2.1), we have

$$\begin{aligned} \|x_{n+1}-p\| &\leq \|y_n-p\| \\ &\leq \|T((1-\beta_n)z_n+\beta_n Tz_n)-p\| \\ &\leq (1-\beta_n)\|z_n-p\|+\beta_n\|Tz_n-p\| \\ &\leq \|z_n-p\|, \end{aligned} \tag{2.6}$$

taking $\liminf$ on both side of (2.6), we have

$$\liminf_{n\to\infty}\|x_{n+1}-p\| \leq \liminf_{n\to\infty}\|z_n-p\|. \tag{2.7}$$

Using (2.5) and (2.7), we have

$$\lim_{n\to\infty}\|z_n-p\| = r.$$

$$\begin{aligned} r &= \lim_{n\to\infty} \|z_n - p\| \\ &\leq \lim_{n\to\infty} \|T((1-\alpha_n)x_n + Tx_n) - p\| \\ &\leq \lim_{n\to\infty} \|(1-\alpha_n)x_n + Tx_n - p\|. \end{aligned}$$

Using Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

The proof is complete. □

Theorem 2.5. *Let X be a uniformly convex Banach space which satisfies the Opial condition and C be a nonempty closed convex subset of X. Let $T : C \to C$ be a generalized (α,β)-nonexpansive type 1 mapping such that $\lambda = \frac{\gamma}{2} \in [0, \frac{1}{2}]$ with $F(T) \neq \phi$ and $\{x_n\}$ be a sequence defined in (2.1). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. It has been established in Theorem 2.4 that $\lim_{n\to\infty} \|x_n - p\|$ exists and sequence $\{x_n\}$ is bounded. Now X is uniformly convex Banach space, we Have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly in C. Now we established that $\{x_n\}$ has a unique weak sub-sequential limit in $F(T)$. Let x and y be weak limits of a subsequence $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. Then by Theorem 2.4, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed with respect to zero by Theorem 2.3 we have $Tx = x$ with same argument we have $Ty = y$. From Theorem 2.4, we have $\lim_{n\to\infty} \|x_n - p\|$ exists. Now suppose that by the Opial condition

$$\begin{aligned} \lim_{n\to\infty} \|x_n - p\| &= \lim_{k\to\infty} \|x_{n_k} - x\| \\ &< \lim_{k\to\infty} \|x_{n_k} - y\| \\ &= \lim_{n\to\infty} \|x_n - y\| \\ &= \lim_{j\to\infty} \|x_{n_j} - y\| \\ &< \lim_{j\to\infty} \|x_{n_j} - x\| \\ &= \lim_{n\to\infty} \|x_n - x\|. \end{aligned}$$

This is contradiction, so $x = y$. Hence $\{x_n\}$ converges weakly to a fixed point of T. Hence, it complete the proof of the theorem. □

Theorem 2.6. *ł Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a generalized (α,β)-nonexpansive type 1 mapping on C, $\{x_n\}$ be defined by (2.2) and $F(T) \neq \phi$. Then, $\{x_n\}$ converges strongly to a point of F(T) if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x,F(T)) = \inf\{\|x-p\| : p \in F(T)\}$.*

Proof. Let $\{x_n\}$ converges to p a fixed point of T. Then $\lim_{n\to\infty} d(x_n, p) = 0$, and since $0 \leq d(x_n, F(T)) \leq d(x_n, p)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Conversaly, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. It follows from Theorem 2.4 that $\lim_{n\to\infty} ||x_n - p||$ exists and that $\lim_{n\to\infty} d(x_n, F(T))$ exists for all $p \in F(T)$. By our hypothesis, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Suppose $\{x_{n_k}\}$ is any arbitrary subsequence of $\{x_n\}$ and $\{r_k\}$ is a sequence in F(T) such that for all $n \in N$,

$$\|x_{n_k} - r_k\| < \frac{1}{2^k},$$

it follows that

$$\|x_{n+1} - r_k\| \leq \|x_n - r_k\| < \frac{1}{2^k},$$

hence

$$\begin{aligned}\|r_{k+1} - r_k\| &\leq \|r_{k+1} - x_{n+1}\| + \|x_{n+1} - r_k\| \\ &< \frac{1}{2^{(k+1)}} + \frac{1}{2^k} = \frac{1}{2^{k-1}}.\end{aligned}$$

Now, we recall the definition of condition I introduced by [12]. □

Definition 2.7. Let C be a subset of a Banach Space X. A mapping $T : C \to C$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ such that $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ and that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T))$ denotes distance from x to $F(T)$.

Theorem 2.8. *Let C be a nonempty closed convex subset of a uniformly convex banch space X. Let T be a generalised (α,β)-nonexpansive type 1 mapping, $\{x_n\}$ be defined by (2.1) and $F(T) \neq \phi$. Let T satisfy condition (I). Then, $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Using 2.4, we obtain that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Using the fact that for all $x \in C$,

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} f(d(x_n, F(T))) \\ &\leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0. \end{aligned}$$

and that

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0.$$

Since, f is nondecreasing with $f(0) = 0$ and $f(t) > 0$ for $t \in (0, \infty)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus using Theorem 2.6, we obtain that $\{x_n\}$ converges strongly to $p \in F(T)$. □

3. Numerical Example

Define a mapping $T : [0,1] \to [0,1]$ as

$$T(x) = \begin{cases} 1-x & \text{if } x \in [0, \frac{1}{8}), \\ \dfrac{x+7}{8} & \text{if } x \in [\frac{1}{8}, 1]. \end{cases} \tag{3.1}$$

In [6] it is proved that T is a generalised (α, β)-nonexpansive mapping since it satisfy condition (C).

In the same manner, we numerically compare our new iteration process defined by 2.1 with generalised M-iteration process.

Case I: Taking, $\alpha_n = \frac{1}{\sqrt{n^3+4}}, \beta_n = \frac{2}{\sqrt{n^3+5}}$ and $x_0 = 0.5$.

Case II: Taking, $\alpha_n = \frac{1}{202}, \beta_n = \frac{1}{300}$ and $x_0 = 0.8$.

SR - Iteration	M - Iteration
0.5	**0.5**
0.9998805917	0.9647409429
0.9999999595	0.997512751
1	0.9998438479
1	0.9999921931
1	0.9999996839
1	0.9999999892
1	0.9999999997
1	1
1	1
1	1

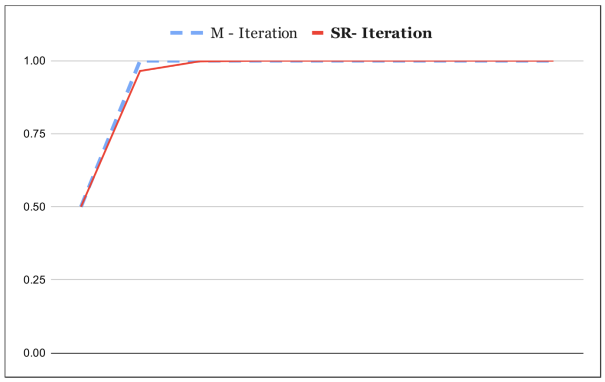

FIGURE 1. Graph.

SR - Iteration	M - Iteration
0.8	**0.8**
0.9996122015	0.9967936472
0.9999992481	0.9999485965
0.9999999985	0.9999991759
1	0.9999999868
1	0.9999999998
1	1
1	1
1	1
1	1
1	1

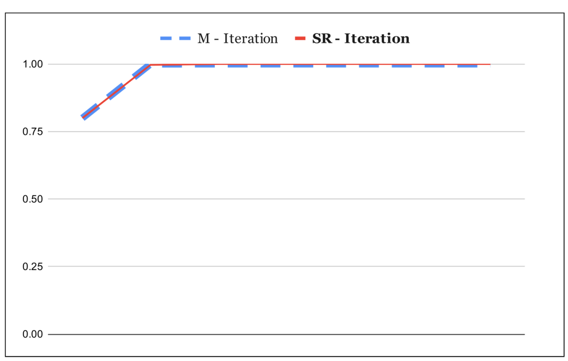

FIGURE 2. Graph.

4. CONCLUSION

We conclude that via numerical example our iterative process is faster than the well known iteration appeared in the literature and our results obtained in this paper improve, extend the results of [6], [9] and many more in this direction.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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