Available online at http://scik.org
Adv. Fixed Point Theory, 2023, 13:9
https://doi.org/10.28919/afpt/8080
ISSN: 1927-6303

SOME APPROXIMATE FIXED POINT RESULTS FOR VARIOUS CONTRACTION TYPE MAPPINGS

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Abstract: In this text, we investigate approximate fixed point results for various contraction mappings in a metric space. This manuscript's intention is to demonstrate ε-fixed point results on metric spaces (not necessarily complete) by using contraction mappings such as B-contraction, convex contraction, and so on. The findings are extensions of several others, including the Kannan-type mapping, the Chatterjea-type mapping, and the S. A. M. Mohsenalhosseini-type mapping, etc. A few examples are included to illustrate the results. Finally, we discuss some applications of approximate fixed point results in the field of applied mathematics rigorously.

Keywords: fixed point; approximate fixed point; B-contraction; Bianchini contraction; convex contraction; diameter approximate fixed point.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Researchers from various fields have contributed to the development of science and technology by using fixed point theory. It is common knowledge that large scale problems involving fixed point theory can be solved quickly. As a result, many researchers have focused on creating fixed point theory approaches in recent years and have presented numerous effective techniques

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Received July 01, 2023

for locating fixed points in complex issues. It also allows for nonlinear analysis. Furthermore, fixed point theory is used to address a variety of problems in applied mathematics. These are now crucial in many mathematics related fields and their applications, notably in economics, astronomy, dynamical systems, decision theory, and parameter estimation. In the early 1900s, the mathematician Brouwer [6], called the father of fixed point theory, proved the fixed point theorem for continuous mapping on finite dimensional spaces. In 1922, Banach [1] proved the famous Banach contraction principle. Then, using contraction mappings on metric spaces, numerous experts expanded the Banach principle and provided numerous theorems. A fixed point theorem for operators that are not required to be continuous was established by Kannan (see [17], [18]). Chatterjea [7] has researched a similar kind of contraction condition and proposed his fixed point theorem. Zamfirescu [30] combined the above two operators and found a new contraction operator for proving the fixed point theorem. Ćirić [8] invented generalised contractions and found a fixed point theorem by using them. Following that, researchers Hardy and Rogers [12] proved a new fixed point theorem using a new contraction operator. In [26], Reich introduced his contraction operator and proposed fixed point findings. Similarly, Bianchini [5] proved fixed point theorem by using another contraction mapping. The unique fixed point theorem for weakly B-contraction mapping was proved by Marudai et al. in [21].

Let us consider a selfmap $M : T \to T$. A fixed point is a point (say, t_0) which is equal to Mt_0. That is, $d(Mt_0,t_0) = 0$. Assume that a mapping has a fixed point, t_0. In which case the point (t_0,t_0) is located on its diagram. The conditions for fixed points existence are very strict. As a result, there is no assurance that fixed points will always exist. In the absence of exact fixed points, approximate fixed points may be used because the fixed point methods have overly strict limitations. This is the main reason to find an approximate fixed point (ε-fixed point). One can see, the point Mt_0 is ”very near” to the point t_0. An approximate fixed point is a point that is nearly located at its respective fixed point. Here, the distance is less than ε, i.e., $d(Mt_0,t_0) < \varepsilon$. Initially, In 2003, Tijs at el. [29] proved the existence of approximate fixed points turns out to be still guaranteed under various weakened versions of the well-known fixed point theorems of Brouwer, Kakutani, and Banach (refer, Theorems 2.1 and 2.2). Moreover, he proved approximate fixed point results for contraction maps and nonexpansive maps in Theorems 3.1

and 4.1, respectively. After that, the author Berinde [2] proved approximate fixed point results (Qualitative theorems) by using various operators (Kannan, Chatterjea, Zamfirescu, and weak contractions) in metric spaces (not necessarily complete). Further, he found the diameter of the approximate fixed points (Quantitative theorems) by using two main lemmas (see also [3], [4]). Dey and Saha [11] extended these results, and they found the diameter of the approximate fixed point for the Reich operator tends to zero when ε approaches zero. In the same manner, S. A. M. Mohsenialhosseini (see, [23], [24], and [25]) derived some new approximate fixed point results for cyclical contraction mappings. Also, he extended these results to a family of contraction mappings and found a common fixed point for the Mohseni-Saheli contraction mapping.

The first scholars to investigate a generalisation of the Banach fixed point theorem while simultaneously using a contraction condition of the rational type were Dass and Gupta [9]. Jaggi [16], used a contraction condition of the rational type to prove a fixed point theorem in complete metric spaces. Later, Harjani et al. [13] extended Jaggi's findings to partially ordered metric spaces. Rational contraction conditions have been heavily employed in both the fixed point and common fixed point locations. Also, the authours Tijani et al. [28] proved approximate fixed point results using rational operators: Let (T,d) be a metric space and $M : T \to T$ be a selfmap. Then there exists $l, l_1, l_2 \in (0,1)$ with $l_1 + l_2 < 1$ such that:

$$d(Mt,Mr) \leq \frac{l[d(t,Mt)d(t,Mr) + d(r,Mr)d(r,Mt)]}{d(t,Mr) + d(r,Mt)}, \text{for all } t,r \in T. \tag{1.1}$$

$$d(Mt,Mr) \leq \frac{l_1 d(r,Mr)[1 + d(t,Mt)]}{1 + d(t,r)} + l_2 d(t,r), \text{for all } t,r \in T. \tag{1.2}$$

$$d(Mt,Mr) \leq \frac{l_1 d(r,Mr)d(t,Mt)}{d(t,r)} + l_2 d(t,r), \text{ for all } t,r \in T. \tag{1.3}$$

Moreover, the author Istratescu (refer [14], [15], and [22]) proved fixed point theorems by using various convex contraction mappings. The scope of this paper is to establish approximate fixed point results in a metric space (not necessarily complete) by using contraction mappings such as B-contraction [21], convex contraction [14], etc. Furthermore, many articles provided (see [10], [19], and [27]) some definitions, which helped more to find approximate fixed point results with examples. These findings are extensions of numerous results, including Kannan mapping, Chatterjea mapping and S. A. M. Mohsenialhosseini mapping, and so on. The conclusions are expansions of a few popular fixed point theorems from previous literature.

This manuscripts remaining portions are displayed as follows. In Section 2, we recall the notations, basic notions, and essential definitions needed throughout the paper. In Section 3, we prove the main concept related to approximate fixed point results using various contraction, rational contraction, and convex contraction mappings. In Section 4, we go one step further and find the applications of approximate fixed point results in a wide range of applied mathematical topics. Finally, In Section 5, we reach a conclusion.

2. Preliminaries

In this section, some notations, basic notions, essential definitions and lemmas from earlier works are recalled. These are then employed throughout the remainder of the main results of the manuscript.

Definition 2.1. [2] *Let (T,d) be a metric space and $M : T \to T$, $\varepsilon > 0$. Then $t \in T$ is said to be an ε-fixed point (approximate fixed point) of M if*

$$d(t,Mt) < \varepsilon.$$

Remark 2.2. *Let $F_\varepsilon(M) = \{t \in T : d(t,Mt) < \varepsilon\}$ denotes the set of all ε-fixed point of M for a given $\varepsilon > 0$.*

Definition 2.3. [2] *Let us consider the map $M : T \to T$. Then M has an approximate fixed point property (a.f.p.p) if for every $\varepsilon > 0$,*

$$F_\varepsilon(M) \neq \emptyset.$$

Lemma 2.4. [2] *Let (T,d) be a metric space, $M : T \to T$ such that M is asymptotic regular, i.e., $d(M^n(t),M^{n+1}(t)) \longrightarrow 0$ as $n \longrightarrow +\infty$, for all $t \in T$. Then, $F_\varepsilon(M) \neq 0$, for every $\varepsilon > 0$.*

Lemma 2.5. [2] *Let K be a closed subset of a metric space (T,d) and $M : K \to T$ be a compact map. Then M has a fixed point if and only if it has an approximate fixed point property.*

Remark 2.6. [2] *In the following, by $D(K)$ for a set $K \neq \emptyset$ we will understand the diameter of the set K, i.e.,*

$$D(K) = sup\{d(t,r) : t,r \in K\}.$$

Definition 2.7. [2] *Let (T,d) be a metric space, $M : T \to T$ a operator and $\varepsilon > 0$. We define the diameter of the set $F_\varepsilon(M)$, i.e.,*

$$D(F_\varepsilon(M)) = sup\{d(t,r) : t,r \in F_\varepsilon(M)\}.$$

Lemma 2.8. [2] *Let (T,d) be a metric space, $M : T \to T$ an operator and $\varepsilon > 0$. We assume that:*

(i) $F_\varepsilon(M) \neq \emptyset$; and

(ii) for all $\theta > 0$, there exists $\phi(\theta) > 0$ such that

$d(t,r) - d(Mt,Mr) \leq \theta$ implies $d(t,r) \leq \phi(\theta)$, for all $t,r \in F_\varepsilon(M)$.

Then:

$$D(F_\varepsilon(M)) \leq \phi(2\varepsilon).$$

Definition 2.9. [21] *Let (T,d) be a metric space. A selfmap $M : T \to T$ is said to be a B-contraction if there exists $l_1,l_2,l_3 \in [0,1)$ with $2l_1 + l_2 + 2l_3 < 1$ such that*

$$d(Mt,Mr) \leq l_1[d(t,Mt) + d(r,Mr)] + l_2 d(t,r) + l_3[d(t,Mr) + d(r,Mt)], \text{ for all } t,r \in T.$$

Definition 2.10. [5] *Let (T,d) be a metric space. A selfmap $M : T \to T$ is said to be a Bianchini contraction if there exists $l \in (0,1)$ such that*

$$d(Mt,Mr) \leq lB(t,r),$$

where $B(t,r) = \max\{d(t,Mt), d(r,Mr)\}$, for all $t,r \in T$.

Definition 2.11. [14] *A continuous mapping $M : T \to T$ is said to be a convex contraction of order 2 if there exists constants $l_0,l_1 \in [0,1)$ such that the following conditions hold:*

(i) $l_0 + l_1 < 1$; and

(ii) $d(M^2t, M^2r) < l_0 d(t,r) + l_1 d(Mt,Mr)$, for all $t,r \in T$.

Definition 2.12. [14] *Let $M : T \to T$ be a continuous map. Then M is said to be n-convex contraction if there exists $l_0, l_1, ..., l_{n-1} \in (0,1)$ such that the following conditions hold:*

(i) $l_0 + l_1 + ... + l_{n-1} < 1$; and

(ii) $d(M^n t, M^n r) \leq l_0 d(t,r) + l_1 d(Mt, Mr) + ... + l_{n-1} d(M^{n-1}t, M^{n-1}r)$, for all $t, r \in T$.

3. MAIN RESULTS

In this section, we prove some approximate fixed point theorems for various contraction mappings on metric spaces, including the B-contraction, the Bianchini contraction, and the convex contraction mappings and their related consequences.

Theorem 3.1. *Let (T,d) be a metric space and $M : T \to T$ be a contraction mapping. Then M has an approximate fixed point (ε-fixed point).*

Proof. Fix $t_0 \in T$ and a sequence $\{t_n\}$ is defined by $t_{n+1} = Mt_n,$ for all $n \geq 0$. Which implies that $\{t_n\}$ is a Cauchy sequence. That is, for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $p, q \geq k_0$ implies $d(t_p, t_q) < \varepsilon$. In particular, if $n \geq k_0, d(t_n, t_{n+1}) < \varepsilon$. That is, $d(t_n, Mt_n) < \varepsilon$. Therefore, $t_n \in F_\varepsilon(M) \neq \emptyset,$ for all $\varepsilon > 0$. Hence, M has an approximate fixed point (ε-fixed point). □

Example 3.2. *Let $T = (0,1)$ and a selfmap $M : T \to T$ is defined by $Mt = \frac{t}{2}$. Then, M is a contraction. Since $(0,1)$ is not complete and hence M does not have a fixed point. But $\{t_n\}$ is defined by $t_{n+1} = Mt_n$ with $t_0 = \frac{1}{2}$. Note that $t_{n+1} = \frac{1}{2^{n+2}}$. Then, $F_\varepsilon(M) \neq \emptyset$, for all $\varepsilon > 0$. Hence, M has an ε-fixed point.*

Theorem 3.3. *If $M, S : T \to T$ such that M is contraction and S is nonexpansive then $(M \circ S)$ has an approximate fixed point (ε-fixed point).*

Proof. Given M is contraction, there exist $l \in [0,1)$ such that $d(Mt, Mr) \leq ld(t,r)$. Consider,

$$\begin{aligned} d(M \circ S(t), M \circ S(r)) &= d(M(St), M(Sr)) \\ &\leq ld(St, Sr) \\ &\leq ld(t,r) \end{aligned}$$

That is, $(M \circ S)$ is a contraction. Then, by Theorem 3.1, $(M \circ S)$ has an approximate fixed point (ε-fixed point). □

Theorem 3.4. *Let a mapping $M : T \to T$ be a B-contraction operator on a metric space (T, d). Then M has an ε-fixed point and*

$$D(F_\varepsilon(M)) \leq \frac{2\varepsilon(l_1 + l_3 + 1)}{1 - l_2 - 2l_3}, \text{ for all } \varepsilon > 0.$$

Proof. Given that M is a B-contraction operator. Let $\varepsilon > 0$ and $t \in T$. Define a sequence $\{t_n\}$ such that $t_{n+1} = Mt_n,$ for all $n \geq 0$. Consider,

$$\begin{aligned}
d(M^n t, M^{n+1} t) &= d(M(M^{n-1} t), M(M^n t)) \\
&\leq l_1[d(M^{n-1} t, M^n t) + d(M^n t, M^{n+1} t)] + l_2 d(M^n t, M^{n-1} t) \\
&\quad + l_3[d(M^{n-1} t, M^{n+1} t) + d(M^n t, M^n t)] \\
&= l_1[d(M^{n-1} t, M^n t) + d(M^n t, M^{n+1} t)] + l_2 d(M^{n-1} t, M^n t) \\
&\quad + l_3[d(M^{n-1} t, M^n t) + d(M^n t, M^{n+1} t)] \\
&= \left(\frac{l_1 + l_2 + l_3}{1 - l_1 - l_3}\right) d(M^{n-1} t, M^n t) \\
&= \lambda d(M^{n-1} t, M^n t), \text{ where } \lambda = \frac{l_1 + l_2 + l_3}{1 - l_1 - l_3} \\
&\leq \lambda^2 d(M^{n-2} t, M^{n-1} t) \\
&\dots \\
&\leq \lambda^n d(t, Mt)
\end{aligned}$$

Since $d(M^n t, M^{n+1} t) \longrightarrow 0$ as $n \longrightarrow +\infty$, for every $t \in T$. That is, $\{t_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_\varepsilon(M) \neq \emptyset$, for all $\varepsilon > 0$. Therefore, M has an ε-fixed point. Clearly condition (i) of Lemma 2.8 is proved. Now only to prove, the condition (ii) of Lemma 2.8. For that, take $\theta > 0$ and $t, r \in F_\varepsilon(M)$. Assume also that $d(t, r) - d(Mt, Mr) \leq \theta$. Show that $\phi(\theta) > 0$ exists. Consider,

$$d(t, r) \leq d(Mt, Mr) + \theta$$

Now, it follows

$$(1-l_2-2l_3)d(t,r) \leq 2l_1\varepsilon + 2l_3\varepsilon + \theta$$

That is,

$$d(t,r) \leq \frac{2l_1\varepsilon + 2l_3\varepsilon + \theta}{1-l_2-2l_3} = \gamma$$

So, for all $\theta > 0$, there exists $\phi(\theta) = \gamma > 0$, such that

$$d(t,r) - d(Mt,Mr) \leq \theta \text{ implies } d(t,r) \leq \phi(\theta)$$

By Lemma 2.8, $D(F_\varepsilon(M)) \leq \phi(2\varepsilon)$, for all $\varepsilon > 0$. Hence,

$$D(F_\varepsilon(M)) \leq \frac{2\varepsilon(l_1+l_3+1)}{1-l_2-2l_3}, \text{ for all } \varepsilon > 0.$$

□

Theorem 3.5. *Let $M : T \to T$ be a Bianchini contraction on a metric space (T,d). Then M has an ε-fixed point and $D(F_\varepsilon(M)) \leq \varepsilon(l+2)$, for all $\varepsilon > 0$.*

Proof. Given M is Bianchini contraction. Let $\varepsilon > 0$ and $t_0 \in M$. Define a sequence $\{t_n\}$ such that $t_{n+1} = Mt_n$, for all $n \geq 0$.

Case 1. *If $B(t,r) = d(t,Mt)$. Then, Definition 2.10 becomes:*

$$\begin{aligned} d(Mt,Mr) &\leq ld(t,Mt) \\ \textit{Substitute } r = Mt,\ d(Mt,M^2t) &\leq ld(t,Mt) \\ \textit{Again substitute } t = Mt,\ d(M^2t,M^3t) &\leq ld(Mt,M^2t) \\ &= l^2 d(t,Mt) \\ &\vdots \\ d(M^nt,M^{n+1}t) &\leq l^n d(t,Mt) \end{aligned}$$

Case 2. *If $B(t,r) = d(r,Mr)$. Then, Definition 2.10 becomes:*

$$\begin{aligned} d(Mt,Mr) &\leq ld(r,Mr) \\ \textit{Substitute } r = Mt,\ d(Mt,M^2t) &\leq ld(Mt,M^2t) \end{aligned}$$

Which is impossible because $l \in (0,1)$. Therefore, **Case 2** does not exists. Now by **Case 1**, $d(M^n t, M^{n+1} t) \longrightarrow 0$ as $n \longrightarrow +\infty$, for all $t, r \in T$. Thus, $\{t_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_\varepsilon(M) \neq \emptyset$, for all $\varepsilon > 0$. That is, M has an ε-fixed point. Here, as in the previous Theorem 3.4, we have

$$\begin{aligned} d(t,r) &\leq d(Mt,Mr) + \theta \\ &\leq lB(t,r) + \theta \\ &= ld(t,Mt) + \theta \\ &= l\varepsilon + \theta \end{aligned}$$

So, for every $\theta > 0$, there exists $\phi(\theta) = l\varepsilon + \theta > 0$ such that

$$d(t,r) - d(Mt,Mr) \leq \theta \text{ implies } d(t,r) \leq \phi(\theta).$$

By Lemma 2.8, $\delta(F_\varepsilon(M)) \leq \phi(2\varepsilon)$, for all $\varepsilon > 0$. Hence,

$$D(F_\varepsilon(M)) \leq \varepsilon(l+2), \text{ for all } \varepsilon > 0.$$

□

Corollary 3.6. *Let (T,d) be a metric space and $M : T \to T$. Then there exist $l \in (0,1)$ such that $d(Mt,Mr) \leq ld(r,Mr)$, for all $t, r \in T$. Then M has an ε-fixed point and $D(F_\varepsilon(M)) \leq \varepsilon(l+2)$, for all $\varepsilon > 0$.*

Proof. Substituting $B(t,r) = d(r,Mr)$ in Theorem 3.5 completes this corollary. □

Theorem 3.7. *Let (T,d) be a metric space and $M : T \to T$. Then there exists $l \in [0,1)$ such that $d(t,Mt) + d(r,Mr) \neq 0$ and*

$$d(Mt,Mr) \leq \frac{l[d(t,Mt)d(t,Mr) + d(r,Mr)d(r,Mt)]}{d(t,Mr) + d(r,Mt)}, \text{ for all } t, r \in T.$$

Prove that M has an ε-fixed point and

$$D(F_\varepsilon(M)) < \frac{\varepsilon^2(l^2 + 6l + 9) + \varepsilon(l+1)}{2}, \text{ for all } \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$, $t_0 \in M$ and a sequence $\{t_n\}$ is defined by $t_{n+1} = Mt_n,$ for all $n \geq 0$.

Consider,

$$\begin{aligned}
d(M^n t, M^{n+1} t) &= d(M(M^{n-1} t), M(M^n t)) \\
&\leq l \left[\frac{d(M^{n-1} t, M^n t) d(M^{n-1} t, M^{n+1} t) + d(M^n t, M^{n+1} t) d(M^n t, M^n t)}{d(M^{n-1} t, M^{n+1} t) + d(M^n t, M^n t)} \right] \\
&\leq l d(M^{n-1} t, M^n t) \\
&\leq l^2 d(M^{n-2} t, M^{n-1} t) \\
&\cdots \\
&\leq l^n d(t, Mt)
\end{aligned}$$

Since $d(M^n t, M^{n+1} t) \longrightarrow 0$ as $n \longrightarrow +\infty$, for all $t, r \in T$. Which implies $\{t_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_\varepsilon(M) \neq \emptyset$, for all $\varepsilon > 0$. That is, M has an ε-fixed point. Here, using the same procedure as in Theorem 3.4, we get

$$\begin{aligned}
d(t,r) &\leq l \left[\frac{d(t,Mt) d(t,Mr) + d(r,Mr) d(r,Mt)}{d(t,Mr) + d(r,Mt)} \right] + 2\varepsilon \\
&= l \left[\frac{d(t,Mt)[d(t,r) + d(r,Mr)] + d(r,Mr)[d(t,r) + d(t,Mt)]}{d(t,r) + d(r,Mr) + d(t,r) + d(t,Mt)} \right] + 2\varepsilon \\
&= l \left[\frac{\varepsilon[d(t,r) + \varepsilon] + \varepsilon[d(t,r) + \varepsilon]}{2d(t,r) + 2\varepsilon} \right] + 2\varepsilon \\
&= \frac{2l\varepsilon d(t,r) + 2l\varepsilon^2 + 4\varepsilon d(t,r) + 4\varepsilon^2}{2d(t,r) + 2\varepsilon}
\end{aligned}$$

On simplyfying, we get

$$2[d(t,r)]^2 - 2\varepsilon(1+l) d(t,r) \leq 2\varepsilon^2 (l+2)$$

$$2[d(t,r)]^2 \leq 2\varepsilon(1+l) d(t,r) + 2\varepsilon^2 (l+2)$$

Which implies $a = 2, b = -2\varepsilon(1+l)$ and $c = -2\varepsilon^2(l+2)$. Therefore,

$$\begin{aligned}
d(t,r) &\leq \frac{2\varepsilon(1+l) \pm \sqrt{4\varepsilon^2(1+l)^2 + 16\varepsilon^2(l+2)}}{4} \\
&= \frac{2\varepsilon(1+l) + \sqrt{4\varepsilon^2(1+2l+l^2) + 16\varepsilon^2 l + 32\varepsilon^2}}{4} \\
&= \frac{2\varepsilon(1+l) + \sqrt{4\varepsilon^2 + 8\varepsilon^2 l + 4l^2\varepsilon^2 + 16\varepsilon^2 l + 32\varepsilon^2}}{4}
\end{aligned}$$

$$= \frac{2\varepsilon(1+l)+\sqrt{36\varepsilon^2+24\varepsilon^2 l+4l^2\varepsilon^2}}{4}$$
$$= \frac{\varepsilon(1+l)+\sqrt{9\varepsilon^2+6\varepsilon^2 l+l^2\varepsilon^2}}{2}$$
$$< \frac{\varepsilon+\varepsilon l+9\varepsilon^2+6\varepsilon^2 l+l^2\varepsilon^2}{2}$$

Hence,

$$D(F_\varepsilon(M)) < \frac{\varepsilon^2(l^2+6l+9)+\varepsilon(l+1)}{2}, \text{ for all } \varepsilon > 0.$$

□

Theorem 3.8. *Let (M,d) be a metric space. Suppose a selfmap $M : T \to T$ is a n-convex contraction. Prove that for every $\varepsilon > 0$, $F_\varepsilon(M) \neq \emptyset$.*

Proof. Let $t_0 \in T$ and define $t_{n+1} = Mt_n,$ for all $n \in \mathbb{N}$. Consider,

$$l = max\{d(t_0,t_1), d(t_1,t_2), ..., d(t_{n-1},t_n)\}.$$

Now,

$$\begin{aligned}
d(t_n,t_{n+1}) &= d(M^n t_0, M^n t_1)\\
&\le l_0 d(t_0,t_1)+l_1 d(t_1,t_2)+...+l_{n-1}d(t_{n-1},t_n)\\
&\le l(l_0+l_1+l_2+...+l_{n-1})\\
d(t_{n+1},t_{n+2}) &= d(M^n t_1, M^n t_2)\\
&\le l_0 d(t_1,t_2)+l_1 d(t_2,t_3)+...+l_{n-2}d(t_{n-1},t_n)+l_{n-1}d(t_n,t_{n+1})\\
&\le l_0 l+...+l_{n-2}l+l_{n-1}l(l_0+l_1+...+l_{n-1})l\\
&\le (l_0+l_1+...+l_{n-1})l
\end{aligned}$$

Similarly,

$$d(t_{n+2},t_{n+3}) \le (l_0+l_1+...+l_{n-1})l$$
$$\cdots$$
$$d(t_{2n-1},t_{2n}) \le (l_0+l_1+...+l_{n-1})l$$

$$\begin{aligned} d(t_{2n},t_{2n+1}) &\leq l_0 d(t_n,t_{n+1}) + l_1 d(t_{n+1},t_{n+2}) + \ldots + l_{n-1} d(t_{2n-1},t_{2n}) \\ &\leq l_0 l(l_0+l_1+..+l_{n-1}) + \ldots + l_{n-1} l(l_0+l_1+..+l_{n-1}) \\ &= l(l_0+l_1+\ldots+l_{n-1})^2 \end{aligned}$$

Again,

$$d(t_{3n},t_{3n+1}) \leq l(l_0+l_1+\ldots+l_{n-1})^3$$

In general,

$$d(t_{n^2},t_{n^2+1}) \leq l(l_0+l_1+\ldots+l_{n-1})^n$$

$$\sum d(t_{n^2},t_{n^2+1}) \leq l \sum (l_0+l_1+\ldots+l_{n-1})^n < +\infty$$

That is, $d(t_{n^2},t_{n^2+1}) \longrightarrow 0$ as $n \longrightarrow +\infty$. Therefore, $t_{n^2} \in F_\varepsilon(M)$, for all $\varepsilon > 0$ provides that $F_\varepsilon(M) \neq \emptyset$, for all $\varepsilon > 0$. Hence, M has an approximate fixed point (ε-fixed point). □

Corollary 3.9. *Let (M,d) be a metric space. Suppose a selfmap $M : T \to T$ is a 2-convex contraction. Prove that for every $\varepsilon > 0$, $F_\varepsilon(M) \neq \emptyset$.*

Proof. Substituting $n = 2$ in Theorem 3.8 completes this corollary. □

Theorem 3.10. *Let D be a nonempty closed bounded convex subset of a Banach space M. Let $T : D \to D$ be a nonexpansive map. Then, $F_\varepsilon(T) \neq \emptyset$.*

Proof. Let $p_0 \in D$ and define $T_n : D \to D$ by $T_n(p) = (1-\alpha_n)p_0 + \alpha_n T(p_n)$, where $\{\alpha_n\} \subseteq [0,1]$ such that $\alpha_n \longrightarrow 1$ as $n \longrightarrow +\infty$. Then,

$$\begin{aligned} \|T_n(p) - T_n(q)\| &= \alpha_n \|T(p) - T(q)\| \\ &\leq \alpha_n \|p - q\| \end{aligned}$$

i.e., T_n is a contraction. Since D is complete, there exists $\{p_n\}$ in D such that $T_n(p_n) = p_n$.

Now,

$$p_n = (1-\alpha_n)p_0 + \alpha_n T(p_n)$$

$$p_n - Tp_n = (1-\alpha_n)p_0 + \alpha_n Tp_n - Tp_n$$

$$
\begin{aligned}
&= (1-\alpha_n)p_0 - Tp_n(1-\alpha_n) \\
\|p_n - Tp_n\| &= (1-\alpha_n)\|p_0 - Tp_n\| \\
&\leq \|1-\alpha_n\|\|p_0 - Tp_n\|
\end{aligned}
$$

Since D is bounded, there exist $M > 0$ such that for all $p \in D, \|p\| \leq M$.

$$
\begin{aligned}
\|p_0 - Tp_n\| &\leq \|p_0\| + \|Tp_n\| \\
&\leq 2M
\end{aligned}
$$

Let $\varepsilon > 0$ be given then $t_n \longrightarrow 1$ as $n \longrightarrow +\infty$. Therefore, given $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that

$$
\begin{aligned}
1 - \alpha_n &< \frac{\varepsilon}{2M} \\
\|p_n - Tp_n\| &< \varepsilon \\
p_n &\in F_\varepsilon(T)
\end{aligned}
$$

Therefore, $F_\varepsilon(T) \neq \emptyset$. Hence, T has an approximate fixed point (ε-fixed point). □

Remark 3.11. *We have proved many approximate fixed point results by using various operators on metric spaces (not necessarily complete). In the following table, one can see the diameters of various contraction operators and the diameters of a few rational contraction operators.*

S. No	*Operator(s)*	*Diameter, for all* $\varepsilon > 0, D(F_\varepsilon(T))$
1	*Contraction*	$\leq \dfrac{2\varepsilon}{1-l}$
2	*Kannan*	$\leq 2\varepsilon(l+1)$
3	*Chatterjea*	$\leq \dfrac{2\varepsilon(l+1)}{1-2l}$
4	*B-contraction*	$\leq \dfrac{2\varepsilon(l_1+l_3+1)}{1-l_2-2l_3}$
5	*Bianchini*	$\leq \varepsilon(l+2)$
6	*Hardy-Rogers*	$\leq \dfrac{\varepsilon(l_2+l_3+l_4+l_5+2)}{1-l_1-l_4-l_5}$
7	*Ćirić contraction*	$\leq \dfrac{\varepsilon(l_2+l_3+2l_4+2)}{1-l_1-2l_4}$
8	*Ćirić-Reich-Rus*	$\leq \dfrac{2\varepsilon(l_2+1)}{1-l_1}$
9	*Reich contraction*	$\leq \dfrac{\varepsilon(l_2+l_3+2)}{1-l_1}$
10	*Zamfirescu*	$\leq \dfrac{(1+\delta)2\varepsilon}{1-\delta}$
11	*Weak contraction*	$\leq \dfrac{\varepsilon(2+W)}{1-l-W}$
12	*Mohseni-saheli*	$\leq \dfrac{2\varepsilon(1+l)}{1-2l}$
13	*Mohseni-semi*	$\leq \dfrac{\varepsilon(l+2)}{1-l}$
14	*Contraction (1.1)*	$< \dfrac{\varepsilon^2(l^2+6l+9)+\varepsilon(l+1)}{2}$
15	*Contraction (1.2)*	$< \dfrac{6\varepsilon+4\varepsilon^2(1+l_1-l_1l_2)+4\varepsilon(l_1-l_2-l_1l_2)}{2(1-l_2)}$
16	*Contraction (1.3)*	$< \varepsilon\left(\dfrac{2}{1-l_2}+l_1\right)$

4. Applications

Approximate fixed point theory covers a wide range of applications in applied mathematics, particularly differential geometry, numerical analysis, and so on. By reading [20] and its references therein, one can find a variety of applications involving approximate fixed point results in the field of applied mathematics. The two examples below demonstrate how to apply approximate fixed point findings in differential equations.

Example 4.1. *Consider* $u^{(4)}t = u(t) - 8e^t(1+t)$ *with the boundary conditions* $u(0) = u(1) = 0$ *and* $u''(0) = 0; u''(1) = -4e$*. Here the exact solution is* $u_E t = t(1-t)e^t$*. Define* $\theta : C^{(3)}[0,1] \longrightarrow C^{(3)}[0,1]$ *by:*

$$\theta(u) = u + \int_0^1 G(t,s)[u^{(4)}(s) - u(s) + 8e^s(1+s)]ds \tag{4.1}$$

Here, u is a solution of $u^{(4)}(t) = 0$*, and* $u(t) = \frac{2e}{3}(t - t^3)$*. So,*

$$\begin{aligned}\theta(u) &= \frac{2e(t-t^3)}{3} + \int_0^1 G(t,s)u^4(s)ds - \int_0^1 G(t,s)[u(s) - 8e^s(1+s)]ds \\ &= \frac{2e(t-t^3)}{3} - \int_0^1 G(t,s)[u(s) - 8e^s(1+s)]ds\end{aligned}$$

Consider,

$$\begin{aligned}|\theta(u) - \theta(v)| &= \left|\int_0^1 G(t,s)[u(s) - 8e^s(1+s)]ds - \int_0^1 G(t,s)[v(s) - 8e^s(1+s)]ds\right| \\ &= \left|\int_0^1 G(t,s)[u(s) - v(s)]ds\right|\end{aligned}$$

$$|\theta(u) - \theta(v)| \leq \left[\int_0^1 |G(t,s)|^2 ds\right]^{\frac{1}{2}} \left[\int_0^1 |u(s) - v(s)|^2 ds\right]^{\frac{1}{2}} \tag{4.2}$$

where,

$$G(t,s) = \frac{1}{6}\begin{cases} s(t-1)(s^2 - 2t + t^2) & 0 \leq s \leq t \\ t(s-1)(t^2 - 2s + s^2) & t \leq s \leq 1 \end{cases}$$

Now,

$$\int_0^1 |G(t,s)|^2 ds \leq \frac{1}{36}\left(\int_0^t s^2(t-1)^2(s^2-2t+t^2)^2 ds + \int_t^1 t^2(s-1)^2(t^2-2s+s^2)^2 ds\right)$$
$$= \frac{1}{36}\left[\frac{8t^2}{105} - \frac{4t^4}{15} + \frac{8t^6}{15} - \frac{16t^7}{35} + \frac{4t^8}{35}\right]$$

Therefore,

$$\int_0^1 |G(t,s)|^2 ds \leq \frac{1}{36}\left(\frac{17}{2240}\right)$$

Then (4.2) *becomes,*

$$|\theta(u)-\theta(v)| \leq \frac{1}{48}\sqrt{\frac{17}{35}}\left[\int_0^1 |u(s)-v(s)|^2 ds\right]^{\frac{1}{2}}$$
$$\leq \left[\int_0^1 |u(s)-v(s)|^2 ds\right]^{\frac{1}{2}}$$
$$= \|u-v\|$$

Hence, θ is a contraction operator. Then, by Theorem 3.1, θ has an ε-fixed point.

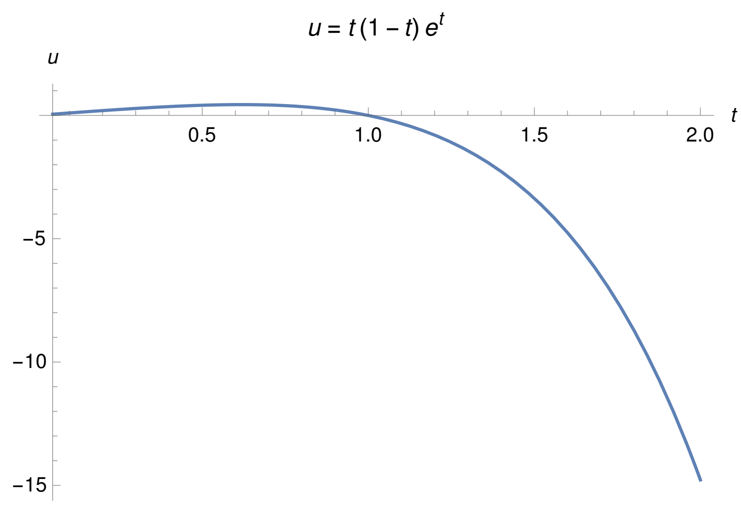

FIGURE 1. solution curve of $u^{(4)}t = u(t) - 8e^t(1+t)$

Example 4.2. *Consider $u^{(4)}(t) = u(t) + 4e^t$ with the boundary condition $u(0) = 1, u'(0) = 2, u(1) = 2e, u'(1) = 3e$. Here, the exact solution is $u_E t = (1+t)e^t$. Define $\lambda : C^{(3)}[0,1] \longrightarrow$*

$C^{(3)}[0,1]$ by:

$$\lambda(u) = u + \int_0^1 G(t,s)[u^4 s - u(s) - 4e^s]ds \tag{4.3}$$

where u is the solution of $u^4(t) = 0$. That is,

$$u(t) = (4-e)t^3 + (3e-7)t^2 + 2t + 1.$$

So, (4.3) *becomes*

$$\begin{aligned}\lambda(u) &= (4-e)t^3 + (3e-7)t^2 + 2t + 1 + \int_0^1 G(t,s)u^4(s)ds \\ &\quad - \int_0^1 G(t,s)[u(s) + 4e^s]ds \\ &= (4-e)t^3 + (3e-7)t^2 + 2t + 1 - \int_0^1 G(t,s)[u(s) + 4e^s]ds\end{aligned}$$

Consider,

$$\begin{aligned}|\lambda(u) - \lambda(v)| &= \left| -\int_0^1 G(t,s)[u(s) + 4e^s]ds + \int_0^1 G(t,s)[v(s) + 4e^s]ds \right| \\ &= \left| \int_0^1 G(t,s)[v(s) - u(s)]ds \right|\end{aligned}$$

$$|\lambda(u) - \lambda(v)| \leq \left(\int_0^1 |G(t,s)|^2 ds \right)^{\frac{1}{2}} \left(\int_0^1 |u(s) - v(s)|^2 ds \right)^{\frac{1}{2}} \tag{4.4}$$

where,

$$G(t,s) = \frac{1}{6} \begin{cases} s^2(t-1)^2(3t-2ts-s) & 0 \leq s \leq t \\ t^2(s-1)^2(3s-2ts-t) & t \leq s \leq 1 \end{cases}$$

Now,

$$\begin{aligned}\int_0^1 |G(t,s)|^2 ds &\leq \frac{1}{36} \left(\int_0^t s^4(t-1)^4(3t-2ts-s)^2 ds + \int_t^1 t^4(s-1)^4(3s-2ts-t)^2 ds \right) \\ &= \frac{1}{36} \left(\frac{3t^4}{35} - \frac{11t^5}{35} + \frac{13t^6}{35} - \frac{2t^7}{35} - \frac{t^8}{5} + \frac{t^9}{7} - \frac{t^{10}}{35} \right) \\ &= \frac{1}{36} \left(\frac{-1}{35} \right) (-1+t)^7 t^4 (3 + 10t + 20t^2) \\ &\leq \frac{1}{36} \left(\frac{13}{35840} \right)\end{aligned}$$

Then (4.4) *yields,*

$$\begin{aligned}
|\lambda(u)-\lambda(v)| &\leq \left(\frac{13}{1290240}\right)^{\frac{1}{2}}\left[\int_0^1 |u(s)-v(s)|^2 ds\right]^{\frac{1}{2}}\\
&= \frac{1}{192}\sqrt{\frac{13}{35}}\left(\int_0^1 |u(s)-v(s)|^2 ds\right)^{\frac{1}{2}}\\
&\leq \left(\int_0^1 |u(s)-v(s)|^2 ds\right)^{\frac{1}{2}}\\
&= \|u-v\|
\end{aligned}$$

Hence, λ is a contraction operator. Then, by Theorem 3.1, λ has an ε-fixed point.

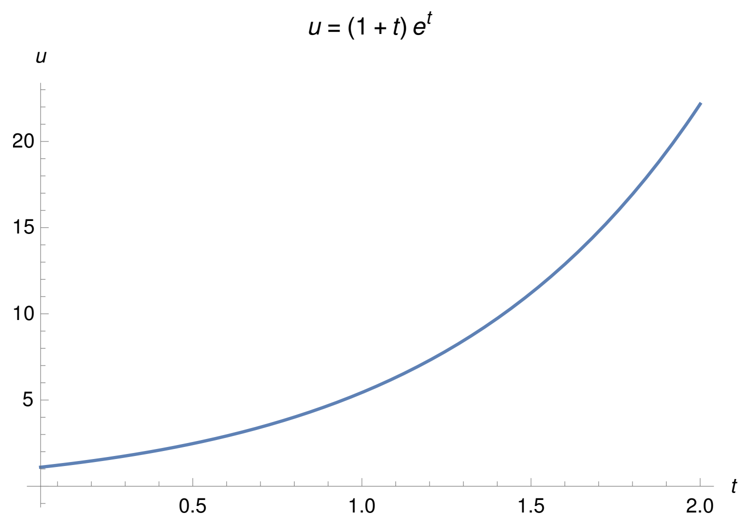

FIGURE 2. solution curve of $u^{(4)}(t) = u(t) + 4e^t$

5. CONCLUSION

This work provides a series of contraction and rational type contraction mappings to demonstrate several approximate fixed point theorems on metric spaces (not necessarily complete). It is essential to note that all of the conclusions made in the current paper generate better constrained approximations of fixed points, mostly in minimising condition $\varepsilon \longrightarrow 0$. In order to confirm the presence of an approximate fixed points, alternative discoveries presented in the later can be demonstrated in a lower environment. Thus, the concept of an approximate fixed points (ε-fixed points) is just as significant as the concept of fixed points.

ACKNOWLEDGEMENT

The first author wishes to thank Bharathidasan University, India, for its financial support under URF scheme. Also, all the authors thank the anonymous referee(s) of the paper for their valuable recommendations. Once again, we thank the editor for giving us the opportunity to reset the manuscript in a nice way.

AUTHOR CONTRIBUTIONS

All authors contributed equally, read and approved the final manuscript.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] S. Banach, Sur les operations dans les ensembles abstract et leur application aux equation integrals, Fund. Math. 3 (1922), 133-181.

[2] M. Berinde, Approximate fixed point theorems, Stud. Univ. "Babes-Bolyai", Math. 51 (2006), 11-25.

[3] V. Berinde, Iterative approximation of fixed points, Editura Efemeride, Baia Mare, (2002).

[4] V. Berinde, On the approximation of fixed points of weak contractive mappings, Carpathian J. Math. 19 (2003), 7-22. https://www.jstor.org/stable/43996763.

[5] R.M.T. Bianchini, Su un problema di S. Reich riguardante la teoria dei punti fissi, Boll. Un. Mat. Ital. 5 (1972), 103-108. https://cir.nii.ac.jp/crid/1571417124680382336.

[6] L.E. Brouwer, Úber abbildung von mannigfaltigkeiten, Math. Ann. 71 (1911), 97-115.

[7] S.K. Chatterjea, Fixed point theorems, C.R. Acad. Bulg. Sci. 25 (1972), 727-730.

[8] L.B. Ćirić, Generalized contractions and fixed point theorems, Publ. Inst. Math. (Bulgr). 12 (1971), 19-26.

[9] B.K. Dass, S. Gupta, An extension of Banach contraction principle through rational expressions, Indian J. Pure Appl. Math. 6 (1975), 1455–1458.

[10] P. Debnath, N. Konwar, S. Radenovic, eds., Metric fixed point theory: applications in science, engineering and behavioural sciences, Springer Nature Singapore, 2021. https://doi.org/10.1007/978-981-16-4896-0.

[11] D. Dey, M. Saha, Approximate fixed point of Reich operator, Acta Math. Univ. Comenianae, 82 (2013), 119-123.

[12] G.E. Hardy, T.D. Rogers, A generalization of a fixed point theorem of Reich, Can. Math. Bull. 16 (1973), 201-206. https://doi.org/10.4153/cmb-1973-036-0.

[13] J. Harjani, B. Lopez, K. Sadarangani, A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space, Abstr. Appl. Anal. 2010 (2010), 190701. https://doi.org/10.1155/2010/190701.

[14] V.I. Istratescu, Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters. – I, Ann. Mat. Pura Appl. 130 (1982), 89-104. https://doi.org/10.1007/bf01761490.

[15] V.I. Istratescu, Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters, II, Ann. Mat. Pura Appl. 134 (1983), 327-362. https://doi.org/10.1007/bf01773511.

[16] D.S. Jaggi, Some unique fixed point theorems, Indian J. Pure Appl. Math. 8 (1977), 223-230.

[17] R. Kannan, Some results on fixed points, Bull. Cal. Math. Soc. 60 (1968), 71-76.

[18] R. Kannan, Some results on fixed points II, Amer. Math. Mon. 76 (1969), 405-408.

[19] M.S. Khan, A fixed point theorems for metric spaces, Rend. Ist. Mat. Univ. Trieste. 8 (1976), 69-72. http://hdl.handle.net/10077/6497.

[20] S.A. Khuri, I. Louhichi, A novel Ishikawa-Green's fixed point scheme for the solution of BVPs, Appl. Math. Lett. 82 (2018), 50-57. https://doi.org/10.1016/j.aml.2018.02.016.

[21] M. Marudai, V.S. Bright, Unique fixed point theorem weakly B-contractive mappings, Far East J. Math. Sci. 98 (2015), 897-914.

[22] Z.D. Mitrović, H. Aydi, N. Mlaiki, et al. Some new observations and results for convex contractions of Istratescu's type, Symmetry. 11 (2019), 1457. https://doi.org/10.3390/sym11121457.

[23] S.A.M. Mohsenalhosseini, H. Mazaheri, M.A. Dehghan, Approximate best proximity pairs in metric space, Abstr. Appl. Anal. 2011 (2011), 596971. https://doi.org/10.1155/2011/596971.

[24] S.A.M. Mohsenialhosseini, Approximate fixed point theorems of cyclical contraction mapping, Math. Morav. 21 (2017), 125-137. https://doi.org/10.5937/matmor1701125m.

[25] S.A.M. Mohsenalhosseini, M. Saheli, Some of Family of contractive type maps and approximate common fixed point, J. Fixed Point Theory. 2021 (2021), 2.

[26] S. Reich, Some remarks concerning contraction mappings, Can. Math. Bull. 14 (1971), 121-124. https://doi.org/10.4153/cmb-1971-024-9.

[27] H. A. Shahrazad, S. Radenović, N. Shahzad, Fixed point theorems for mappings with convex diminishing diameters on cone metric spaces, Appl. Math. Lett. 24 (2011), 2162-2166. https://doi.org/10.1016/j.aml.2011.06.019.

[28] K. Tijani, S. Olayemi, Approximate fixed point results for rational-type contraction mappings, Int. J. Math. Anal. Optim.: Theory Appl. 7 (2021), 76-86. https://doi.org/10.6084/M9.FIGSHARE.15047445.

[29] S. Tis, A. Torre, R. Branzei, Approximate fixed point theorems, Libertas Math. 23 (2003), 35-39.

[30] T. Zamfirescu, Fix point theorems in metric spaces, Arch. Math. 23 (1972), 292-298. https://doi.org/10.1007/bf01304884.