Available online at http://scik.org
Adv. Fixed Point Theory, 2023, 13:11
https://doi.org/10.28919/afpt/8084
ISSN: 1927-6303

# A NEW ITERATIVE ALGORITHM FOR GENERALIZED $(\alpha, \beta)$-NONEXPANSIVE MAPPING IN CAT(0) SPACE

SAMIR DASHPUTRE[1,*], RAKESH TIWARI[2], JAYNENDRA SHRIVAS[2]

[1]Department of Mathematics, Shahid Durwasha Nishad Govt. College, Arjunda, Dist.- Balod (C.G.) 491225, India

[2]Department of Mathematics, Govt. V.Y.T. PG Autonomous College, Durg (C.G.) 491001, India



**Abstract.** In this paper, we provide certain fixed point results for a generalized $(\alpha, \beta)$-nonexpansive mapping, as well as a new iterative algorithm for approximating the fixed point of this class of mappings in the setting of CAT(0) spaces. Furthermore, we establish strong and $\Delta$-converges theorem for generalized $(\alpha, \beta)$-nonexpansive mapping in CAT(0) space. Finally, we present a numerical example to illustrate our main result and then display the efficiency of the proposed algorithm compared to different iterative algorithms in the literature. Our results obtained in this paper improve, extend and unify some related results in the literature.

**Keywords:** CAT(0) space; generalized $(\alpha, \beta)$-nonexpansive mapping; strong and $\Delta$-convergence theorems.

**2020 AMS Subject Classification:** 47H09, 47H10, 54H25.

## 1. Introduction

Let $K$ be a nonempty subset of a metric space $(X, d)$ and $\Omega\colon K \to K$ be a nonlinear mapping. The fixed point set of $\Omega$ is denoted by $F(\Omega)$, that is, $F(\Omega) = \{x \in K\colon x = \Omega x\}$. Remember that a selfmap $\Omega$ on a metric space subset $K$ is called nonexpansive if

$$d(\Omega x, \Omega y) \leq d(x, y) \ \forall x, y \in K. \tag{1.1}$$

*Corresponding author

E-mail address: samir231973@gmail.com

Received July 03, 2023

Nowadays, the study of fixed points for nonexpansive operators is an important and active research field. One of Gohde's [10] earlier results states that nonexpansive operators always admit a fixed point on closed bounded and convex subsets in the framework of uniform convexity of Banach space. Kirk [14, 15] was the first to introduce fixed point theory of nonexpansive operators in the context of nonlinear CAT(0) spaces. Suzuki [21] made a significant breakthrough in 2008 by introducing a weak notion of nonexpansive operators. It is worth noting that a selfmap $\Omega$ of a metric space subset $K$ is said to satisfy Condition (C) (also known as Suzuki map) if for any $x,y \in K$, we have

$$\frac{1}{2}d(x,\Omega x) \leq d(x,y) \implies d(\Omega x,\Omega y) \leq d(x,y). \tag{1.2}$$

*Remark* 1.1. It is clear that every nonexpansive map is Suzuki nonexpansive. However, an example in [21] shows that there exists maps which are Suzuki nonexpansive but not nonexpansive.

In 2011, Aoyama and Kohsaka [3] proposed the class of $\alpha$-nonexpansive maps as follows:

A selfmap $\Omega$ of a metric space subset $K$ is said to satisfy $\alpha$-nonexpansive maps if one can find a real number $\alpha \in [0,1)$ for any $x,y \in K$, we have

$$d(\Omega x,\Omega y)^2 \leq \alpha d(x,\Omega y)^2 + \alpha d(y,\Omega x)^2 + (1-2\alpha)d(x,y)^2. \tag{1.3}$$

In 2017, Pant and Shukla [20] proposed the class of $\alpha$-nonexpansive maps as follows:

A selfmap $\Omega$ of a metric space subset $K$ is said to satisfy generalized $\alpha$-nonexpansive maps if one can find a real number $\alpha \in [0,1)$ for any $x,y \in K$, we have

$$\begin{aligned} \frac{1}{2}d(x,\Omega x) \leq d(x,y) \implies d(\Omega x,\Omega y) &\leq \alpha d(y,\Omega x) + \alpha d(x,\Omega y) \\ &+ (1-2\alpha)d(x,y). \end{aligned} \tag{1.4}$$

*Remark* 1.2. It is clear that every Suzuki nonexpansive map is generalized 0-nonexpansive. However, an example in [20] shows that there exist maps which are generalized $\alpha$-nonexpansive but not Suzuki nonexpansive.

In 2019, Pant and Pandey [19] proposed the class of Reich–Suzuki type nonexpansive maps as follows:

A selfmap $\Omega$ of a metric space subset $K$ is said to satisfy $\beta$-Reich-Suzuki type nonexpansive maps if one can find a real number $\beta \in [0,1)$ for any $x,y \in K$, we have

$$\frac{1}{2}d(x,\Omega x) \leq d(x,y) \implies d(\Omega x,\Omega y) \leq \beta d(x,\Omega x) + \beta d(y,\Omega y) + (1-2\beta)d(x,y). \tag{1.5}$$

*Remark* 1.3. It is clear that every Suzuki nonexpansive map is 0-Reich–Suzuki type nonexpansive. However, an example in [19] shows that there exists maps which are $\beta$-Reich–Suzuki type nonexpansive but not Suzuki nonexpansive.

**Definition 1.4.** A selfmap $\Omega$ of a CAT(0) space subset $K$ is said to be generalized $(\alpha,\beta)$-nonexpansive, if there exists real number $\alpha,\beta \in \mathbb{R}^+$ satisfying $\alpha+\beta<1$ such that, for all $x,y \in K$

$$\frac{1}{2}d(x,\Omega x) \leq d(x,y) \implies d(\Omega x,\Omega y) \leq \alpha d(x,\Omega y) + \alpha d(y,\Omega x) + \beta d(x,\Omega x) + \beta d(y,\Omega y) + (1-2\alpha-2\beta)d(x,y). \tag{1.6}$$

The following proposition gives many examples of generalized $(\alpha,\beta)$-nonexpansive mpas.

*Remark* 1.5. Let $\Omega$ be a selfmap on a subset $K$ of a CAT(0) space $X$. Then, the following hold:

(1) If $\Omega$ is Suzuki nonexpansive, then $\Omega$ is generalized $(0,0)$-nonexpansive.
(2) If $\Omega$ is generalized $\alpha$-nonexpansive, then $\Omega$ is generalized $(\alpha,0)$- nonexpansive.
(3) If $\Omega$ is $\beta$-Reich–Suzuki type nonexpansive, then $\Omega$ is generalized $(0,\beta)$-nonexpansive.

## 2. PRELIMINARIES

Let $(X,d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a mapping $K$ from a closed interval $[0,r] \subset R$ to X such that

$$c(0)=x,\ c(r)=y,\ d(c(t),c(s))=|t-s|$$

for all $s,t \in [0,r]$. In particular, $K$ is an isometry and $d(x,y)=r$. The image of $K$ is call a geodesic segment (or metric segment) joining $x$ and $y$. When it is unique, this geodesic is denoted by $[x,y]$. We denote the point $w \in [x,y]$ such that $d(x,w)=\alpha d(x,y)$ by $w=(1-\alpha)x \oplus \alpha y$, where $\alpha \in [0,1]$.

The space $(X,d)$ is called a geodesic space if any two points of $X$ are joined by a geodesic and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x,y \in X$. A subset $D \subseteq X$ is said to be convex if D includes geodesic segment joining every two points of itself. A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic metric space $(X,d)$ consist of three points (the vertices of $\Delta$) and a geodesic segment between each pair of vertices (the edges of $\Delta$). A comparison triangle for geodesic triangle (or $\Delta(x_1,x_2,x_3)$) in $(X,d)$ is a triangle $\bar{\Delta}(x_1,x_2,x_3) = \Delta(\bar{x_1},\bar{x_2},\bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{x_i},\bar{x_j}) = d(x_i,x_j)$$

for $i,j \in \{1,2,3\}$. A geodesic metric space is said to be a CAT(0) space if all geodesic triangle of appropriate size satisfy the following CAT(0) comparison axiom:

Let $\Delta$ be a geodesic triangle in $C$ and let $\bar{\Delta} \subset \mathbb{R}^2$ be comparison triangle for $\Delta$. Then $\Delta$ is said to satisfy the CAT(0) inequality if for all $x,y \in \Delta$ and all comparison points $\bar{x},\bar{y} \in \bar{\Delta}$,

$$d(x,y) \leq d_{\mathbb{R}^2}(\bar{x},\bar{y}).$$

If $x,y_1,y_2$ are points of a CAT(0) space and $y_0$ is the midpoint of the segment $[y_1,y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the CAT(0) inequality impels

$$d^2(x,\frac{y_1 \oplus y_2}{2}) \leq \frac{1}{2}d^2(x,y_1) + \frac{1}{2}d^2(x,y_2) - \frac{1}{4}d^2(y_1,y_2).$$

this inequality is the (CN) inequality of Bruhat and Tits [7]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

It is well known that all complete, simply combined Riemannian manifold having non-positive section curvature is a CAT(0) space. For other examples, Euclidean buildings [6], Pre-Hilbert spaces, $\mathbb{R}$-trees [5], the complex Hilbert ball with a hyperbolic metric ([9]) is a CAT(0) space. Further, complete CAT(0) spaces are called Hadamard spaces.

Now, we give some elementary properties about CAT(0) spaces as follows:

**Lemma 2.1.** [8] *Let $X$ be a CAT(0) space, $x,y,z \in X$ and $t \in [0,1]$. Then*

$$d(tx \oplus (1-t)y,z) \leq td(x,z) + (1-t)d(y,z).$$

*Let $\{x_n\}$ be a bounded sequence in $X$, complete CAT(0) spaces. For $x \in X$ set:*

$$r(x,\{x_n\}) = \lim_{n\to\infty} sup\ d(x,x_n).$$

*The asymptotic radius $r(\{x_n\})$ is given by*

$$r(\{x_n\}) = inf\{r(x,x_n) \colon x \in K\},$$

*and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as:*

$$A(\{x_n\}) = \{x \in K \colon r(x,x_n) = r(\{x_n\})\}.$$

*Remark* 2.2. The cardinality of the set $A(\{x_n\})$ in any CAT(0) space is always equal to one, (see e.g., [8]).

The ([8], Proposition 2.1) tells us that in the setting of CAT(0) spaces, for every bounded sequence, namely, $\{x_n\} \subset K$, the set $A(\{x_n\})$ is essentially the subset of $K$ provided that $K$ is convex and bounded. It is well-known that $\{x_n\}$ has a subsequence which $\Delta$-converges to some point provided that the sequence is bounded.

**Definition 2.3.** [8] A sequence $\{x_n\}$ in CAT(0) space is said to be $\Delta$-converges to $x \in K$ if $x$ is the unique asymptotic center for every subsequence $\{a_n\}$ of $\{x_n\}$. In this case we write $\Delta - lim_n x_n = x$ and read as $x$ is the $\Delta - limit$ of $\{x_n\}$.

Notice that a bounded sequence $\{x_n\}$ in a CAT(0) space is known as regular if and only if for every subsequence, namely, $\{a_n\}$ of $\{x_n\}$ one has $r(\{x_n\}) = r\{a_n\}$. It is wellknown that, in the setting of CAT(0) spaces each regular sequence $\Delta$-converges and consequently each bounded sequence has a $\Delta$-convergent subsequence.

**Definition 2.4.** [8] Let $\Omega$ be a selfmap on a subset $K$ of a given CAT(0) space and $f$ be a selfmap of $[0,\infty)$. We say that $\Omega$ has condition $(I)$ if the following holds:

(1) $f(g) = 0$ if and only if $g = 0$.
(2) $f(g) > 0$ for every $g > 0$.
(3) $d(x,\Omega x) \geq f(d(x, f(\Omega)))$.

We now present some propositions and Lemmas, which characterize the generalized $(\alpha,\beta)$-nonexpansive mapping.

**Proposition 2.5.** [8] Suppose $K$ is a nonempty subset of a given CAT(0) space. If $\Omega\colon K \to K$ has generalized $(\alpha,\beta)$-nonexpansive mapping. Then for every fixed point p of $\Omega$, one has

$$d(p,\Omega x) \le d(p,x), \tag{2.1}$$

for each $x \in K$.

**Lemma 2.6.** [16] *Suppose $K$ is nonempty closed convex subset of a given CAT(0) space. If $\Omega\colon K \to K$ has generalized $(\alpha,\beta)$-nonexpansive mapping and the sequence $\{x_n\} \subseteq K$ satisfy $\lim_{n\to\infty} d(\Omega x_n, x_n) = 0$ and $\Delta - \lim_n x_n = p$, then $p = \Omega p$.*

**Lemma 2.7.** [16] *Let $K$ be a nonempty subset of a given CAT(0) space. If $\Omega\colon K \to K$ has the generalized $(\alpha,\beta)$-nonexpansive mapping. Then the set $F(\Omega)$ always closed.*

**Lemma 2.8.** [23] *Let $\Omega$ be a selfmap on a subset $K$ of a CAT(0) space $X$. If $\Omega$ is generalized $(\alpha,\beta)$-nonexpansive, then for each $x, y \in K$ :*

*(1)* $d(\Omega x, \Omega^2 x) \le d(x, \Omega x)$.

*(2) Either* $\frac{1}{2} d(x,\Omega x) \le d(x,y)$ *or* $\frac{1}{2} d(\Omega x, \Omega^2 x) \le d(\Omega x, y)$.

*(3) Either*

$d(\Omega x,\Omega y) \le \alpha d(x,\Omega y) + \alpha d(y,\Omega x) + \beta d(x,\Omega x) + \beta d(y,\Omega y) + (1-2\alpha-2\beta) d(x,y)$ *or* $d(\Omega^2 x,\Omega y) \le \alpha d(\Omega x,\Omega y) + \alpha d(y,\Omega^2 x) + \beta d(\Omega x,\Omega^2 x) + \beta d(y,\Omega y) + (1-2\alpha-2\beta) d(\Omega x, y)$.

**Lemma 2.9.** [23] *Let $\Omega$ be a selfmap on a subset $K$ of a CAT(0) space $X$. If $\Omega$ is generalized $(\alpha,\beta)$-nonexpansive, then for each $x, y \in K$, we have*

$$d(x_n,\Omega p) \le \Big(\frac{3+\alpha+\beta}{1-\alpha-\beta}\Big) d(x_n,\Omega p) + d(x_n, p).$$

**Lemma 2.10.** [16] *Let $X$ be a CAT(0) space and $\{t_n\}$ be any real sequence such that $0 < a \le a_n \le b < 1$ for $n \ge 1$. Let $\{y_n\}$ and $\{z_n\}$ be any two sequences of $X$ such that $\lim_{n\to\infty} sup\, d(y_n, x) \le q$, $\lim_{n\to\infty} sup\, d(z_n, x) \le q$ and $\lim_{n\to\infty} d(a_n y_n \oplus (1-a_n) z_n, x) = p$ hold for some $q \ge 0$. Then $\lim_{n\to\infty} d(y_n, z_n) = 0$.*

## 3. NEW ITERATION PROCESS AND ITS CONVERGENCE ANALYSIS

Over the last few years many iterative processes have been obtained in different domains to approximate fixed points of various classes of mappings. Mann iteration [17], Ishikawa iteration [13], Halpern iteration [11], Thakur et al. [22] and Ullah et al. [23] are the few basic iteration processes.

Mann [17] described one of the earlier iteration processes as follows:

$$\begin{aligned} &x_1 \in K \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Omega x_n,\ n \geq 1. \end{aligned} \tag{3.1}$$

The Mann iteration can be seen as a subset of the Ishikawa iteration process, which was described by Ishikawa in [13] as follows:

$$\begin{aligned} &x_1 \in K \\ &y_n = (1-\beta_n)x_n + \beta_n \Omega x_n \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Omega y_n,\ n \geq 1. \end{aligned} \tag{3.2}$$

Agarwal et al. [2] is the slightly modification of the Ishikawa iteration and was defined as follows:

$$\begin{aligned} &x_1 \in K \\ &y_n = (1-\beta_n)x_n + \beta_n \Omega x_n \\ &x_{n+1} = (1-\alpha_n)\Omega x_n + \alpha_n \Omega y_n,\ n \geq 1. \end{aligned} \tag{3.3}$$

We can infer from [2] that the Agarwal iterative process is superior to the earlier processes, namely the Picard, Mann and Ishikawa iterative processes, by a significant margin.

In 2016, Thakur et al. [22] proposed the iterative process listed below:

$$\begin{aligned} &x_1 \in K \\ &z_n = (1-\beta_n)x_n + \beta_n \Omega x_n \\ &y_n = \Omega((1-\alpha_n)x_n + \alpha_n z_n) \\ &x_{n+1} = \Omega y_n,\ n \geq 1. \end{aligned} \tag{3.4}$$

Thakur et al. [22] demonstrated that the sequence $\{x_n\}$ defined by the iterative process (3.4) converges (in certain circumstances) to a fixed point of a given Suzuki map. Furthermore, they built a new example of Suzuki mappings $\Omega$ and demonstrated that the iterative process (3.4) converges to a fixed point faster than earlier iterative processes proposed by Picard, Mann [17], Ishikawa [13], Noor [18], S [2] and Abbas [1].

In 2020, Ullah et al. [23] introduced a new iterative process, which they call it "$K$" iteration process, as follows:

$$
\begin{aligned}
&x_1 \in K \\
&z_n = (1-\beta_n)x_n + \beta_n \Omega x_n \\
&y_n = \Omega((1-\alpha_n)\Omega x_n + \alpha_n \Omega z_n) \\
&x_{n+1} = \Omega y_n, \ n \geq 1.
\end{aligned} \tag{3.5}
$$

**Question:** Is it possible to develop an iteration process whose rate of convergence is even faster than the iteration processes defined above?

To answer this, we introduce the new iteration process as follows:

Let $K$ be a nonempty, closed and convex subset of a complete CAT(0) space $X$ and $\Omega\colon K \to K$ be a mapping. Let $x_1 \in K$ be arbitrary and the sequence $\{x_n\}$ generated iteratively by

$$
\begin{aligned}
&x_1 \in K \\
&z_n = \Omega((1-\alpha_n)x_n \oplus \alpha_n \Omega x_n) \\
&y_n = \Omega((1-\beta_n)z_n \oplus \beta_n \Omega z_n) \\
&x_{n+1} = \Omega((1-\gamma_n)y_n \oplus \gamma_n \Omega y_n), n \geq 1
\end{aligned} \tag{3.6}
$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1). Motivated by what has been said above, in this research work, we establish convergence result of a new iteration process for generalized $(\alpha, \beta)$-nonexpansive mapping in the setting of CAT(0) space. A numerical example is provided to demonstrate the fastness of the new iteration process. Our results are improved and generalized form of the earlier results.

## 4. Main Results

This section establishes some significant strong and $\Delta$-convergence results for operators with generalized $(\alpha,\beta)$-nonexpansive mapping. Our results will generalize the results of Ullah et al. [23].

**Theorem 4.1.** *Let $\Omega\colon K \to K$ satisfies the generalized $(\alpha,\beta)$-nonexpansive mapping defined on a nonempty closed convex subset $K$ of a complete CAT(0) space $X$ such that $F(\Omega) \neq \emptyset$. If $\{x_n\}$ is a sequence generated by (3.6) then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(\Omega)$.*

**Proof:** Let $p \in F(\Omega)$. By Proposition 2.5, we have

$$
\begin{aligned}
d(z_n, p) &= d(\Omega((1-\alpha_n)x_n \oplus \alpha_n\Omega x_n), p) \\
&\leq ((1-\alpha_n))d(x_n, p) + \alpha_n d(\Omega x_n, p) \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\
&\leq d(x_n, p).
\end{aligned} \tag{4.1}
$$

Using Proposition 2.5 and (4.1), we get

$$
\begin{aligned}
d(y_n, p) &= d(\Omega((1-\beta_n)z_n \oplus \beta_n\Omega z_n), p) \\
&\leq (1-\beta_n)d(z_n, p) + \beta_n d(\Omega z_n, p) \\
&\leq (1-\beta_n)d(z_n, p) + \beta_n d(z_n, p) \\
&\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\
&\leq d(x_n, p).
\end{aligned} \tag{4.2}
$$

Using Proposition 2.5, (4.1) and (4.2), we get

$$
\begin{aligned}
d(x_{n+1}, p) &= d(\Omega((1-\gamma_n)y_n \oplus \gamma_n\Omega y_n), p) \\
&\leq (1-\gamma_n)d(y_n, p) + \gamma_n d(\Omega y_n, p) \\
&\leq (1-\gamma_n)d(y_n, p) + \gamma_n d(y_n, p) \\
&\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p) \\
&\leq d(x_n, p).
\end{aligned} \tag{4.3}
$$

Thus, $\{d(x_n,p)\}$ is a non-increasing sequence of reals which is bounded below by zero and hence convergent. Therefore, $\lim_{n\to\infty} d(x_n,p)$ exists $\forall\ p \in F(\Omega)$. □

**Theorem 4.2.** *Let $\Omega\colon K \to K$ satisfies the generalized $(\alpha,\beta)$-nonexpansive mapping defined on a nonempty closed convex subset $K$ of a complete CAT(0) space $X$ and $\{x_n\}$ is generated by the algorithm (3.6), then $F(\Omega) \neq \emptyset$ If and only if $\{x_n\}$ is bounded and* $\lim_{n\to\infty} d(\Omega x_n, x_n) = 0$.

**Proof:** Suppose that $F(\Omega) \neq \emptyset$ and $p \in F(\Omega)$.

Then by theorem 4.1, it follows that $\lim_{n\to\infty} d(x_n,p)$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} d(x_n,p) = c. \tag{4.4}$$

By the proof of theorem 4.1, we have

$$\lim_{n\to\infty} sup\ d(z_n,p) \leq \lim_{n\to\infty} sup\ d(y_n,p) \leq \lim_{n\to\infty} sup\ d(x_n,p) = c. \tag{4.5}$$

By using Lemma 2.5, we have

$$\lim_{n\to\infty} sup\ d(\Omega x_n,p) \leq \lim_{n\to\infty} sup\ d(x_n,p) = c. \tag{4.6}$$

Again by the proof of theorem 4.1, we have $d(y_n,p) \leq d(x_n,p)$

Therefore,

$$\begin{aligned} d(x_{n+1},p) &= d(\Omega((1-\gamma_n)y_n \oplus \gamma_n \Omega y_n),p) \\ &\leq (1-\gamma_n)d(y_n,p) + \gamma_n d(\Omega y_n,p) \\ &\leq (1-\gamma_n)d(x_n,p) + \gamma_n d(y_n,p). \end{aligned}$$

It follows that

$$\begin{aligned} d(x_{n+1},p) - d(x_n,p) &\leq \frac{d(x_{n+1},p) - d(x_n,p)}{\gamma_n} \\ &\leq d(y_n,p) - d(x_n,p) \\ &\leq (1-\beta_n)d(z_n,p) + \beta_n d(z_n,p) - d(x_n,p) \\ &\leq d(z_n,p) - d(x_n,p). \end{aligned}$$

So, we can get $d(x_{n+1},p) \leq d(z_n,p)$ and form (4.4), we have

$$c \leq \lim_{n\to\infty} inf\ d(z_n,p). \tag{4.7}$$

Hence, from (4.6) and (4.7), we obtain

$$c = \lim_{n\to\infty} d(z_n, p). \tag{4.8}$$

Therefore, from (4.8), we have

$$\begin{aligned} c = \lim_{n\to\infty} d(z_n, p) &= \lim_{n\to\infty} d(\Omega((1-\alpha_n)x_n \oplus \alpha_n \Omega x_n), p) \\ &\leq \lim_{n\to\infty} (1-\alpha_n) d(x_n, p) + \alpha_n d(\Omega x_n, p) \\ &\leq \lim_{n\to\infty} (1-\alpha_n) d(x_n, p) + \lim_{n\to\infty} \alpha_n d(\Omega x_n, p) \\ &\leq c. \end{aligned} \tag{4.9}$$

Hence,

$$\lim_{n\to\infty} (1-\alpha_n) d(x_n, p) + \alpha_n d(\Omega x_n, p) = c. \tag{4.10}$$

Now, from (4.5),(4.7),(4.10) and Lemma 2.10, we conclude that,

$$\lim_{n\to\infty} d(\Omega x_n, x_n) = 0.$$

Conversely, let $p \in A(\{x_n\})$. By Lemma 2.9, we have

$$d(x_n, \Omega p) \leq (\frac{3+\alpha+\beta}{1-\alpha-\beta}) d(x_n, \Omega p) + d(x_n, p). \tag{4.11}$$

This implies that

$$\begin{aligned} r(x_n, \Omega p) &= \lim_{n\to\infty} sup\ d(x_n, \Omega p) \\ &\leq (\frac{3+\alpha+\beta}{1-\alpha-\beta}) \lim_{n\to\infty} sup\ d(x_n, p) \\ &\leq \lim_{n\to\infty} sup\ d(x_n, p) = r(x_n, p). \end{aligned} \tag{4.12}$$

So $\Omega p \in A\{x_n\}$. By the uniqueness of asymptotic centers, one can conclude that $\Omega p = p$. This completes the proof. □

**Theorem 4.3.** *Let $\Omega\colon K \to K$ satisfies the generalized $(\alpha,\beta)$-nonexpansive mapping defined on a nonempty closed convex subset $K$ of a complete CAT(0) space $X$ such that $F(\Omega) \neq \emptyset$. If $\{x_n\}$ is a sequence generated by (3.6). Then $\{x_n\}$ $\Delta$-converges to a fixed point of $\Omega$.*

**Proof:** By Theorem 4.2, the sequence $\{x_n\}$ is bounded. Hence one can take $A(\{x_n\}) = \{c\}$ for some $c \in X$. We are going to prove $A(\{x_n\}) = \{c\}$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Suppose $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $A(\{x_n\}) = \{c\}$. Since $\{x_{n_k}\}$ is bounded, one can find a subsequence $\{x_{n_j}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_j}\}$ $\Delta$-converges to $p$ for some $p \in E$. By Theorem 4.2 and Lemma 2.10, one has $p \in F(\Omega)$ and hence $\lim_{n\to\infty} sup\ d(x_n, p)$ exists. If $p \neq x$, then the singletoness of the cardinality of the asymptotic centers allows us the following

$$\begin{aligned} \lim_{n\to\infty} sup\ d(x_n, p) &= \lim_{j\to\infty} sup\ d(x_{n_j}, p) < \lim_{j\to\infty} sup\ d(x_{n_j}, x) \\ &\leq \lim_{k\to\infty} sup\ d(x_{n_k}, x) < \lim_{k\to\infty} sup\ d(x_{n_k}, p) \\ &= \lim_{n\to\infty} sup\ d(x_n, p), \end{aligned} \tag{4.13}$$

which is contradiction. Therefore, $x = p \in F(\Omega)$. Suppose that $x \neq c$. Then

$$\begin{aligned} \lim_{n\to\infty} sup\ d(x_n, x) &= \lim_{k\to\infty} sup\ d(x_{n_k}, x) \leq \lim_{k\to\infty} sup\ d(x_{n_k}, c) \\ &\leq \lim_{n\to\infty} sup\ d(x_m, c) < \lim_{n\to\infty} sup\ d(x_m, x) \\ &= \lim_{n\to\infty} sup\ d(x_n, x). \end{aligned} \tag{4.14}$$

Thus $\{x_n\}$ $\Delta$-converges to an element $c \in F(\Omega)$. □

**Theorem 4.4.** *Let $\Omega\colon K \to K$ satisfies the generalized $(\alpha, \beta)$-nonexpansive mapping defined on a nonempty closed convex subset $K$ of a complete CAT(0) space $X$ such that $F(\Omega) \neq \emptyset$. If $\{x_n\}$ is a sequence defined by (3.6), then $\{x_n\}$ strongly converges to a fixed point of $\Omega$ if and only if* $\lim_{n\to\infty} d(x_n, F(\Omega)) = 0$.

**Proof:** If the sequence $\{x_n\}$ converges to a point $p \in F(\Omega)$, then

$$\lim_{n\to\infty} inf d(x_n, p) = 0,$$

so

$$\lim_{n\to\infty} d(x_n, F(\Omega)) = 0.$$

For converse part, assume that $\lim_{n\to\infty} inf d(x_n, F(\Omega)) = 0$. From Theorem 4.1, we have

$$d(x_{n+1}, p) \leq d(x_n, p)\ for\ any\ p \in F(\Omega),$$

so we have,

$$d(x_{n+1},F(\Omega)) \le d(x_n,F(\Omega)). \tag{4.15}$$

Thus, $d(x_n,F(\Omega))$ forms a decreasing sequence which is bounded below by zero as well, thus $\lim_{n\to\infty} d(x_n,F(\Omega))$ exists. Since, $\lim_{n\to\infty} inf d(x_n,F(\Omega)) = 0$ so $\lim_{n\to\infty} d(x_n,F(\Omega)) = 0$.

Now, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{x_j\}$ in $F(\Omega)$ such that $d(x_{n_j},x_j) \le \frac{1}{2^j}$ for all $j \in \mathbb{N}$. From the proof of Theorem 4.1, we have

$$\begin{aligned} d(x_{n_{j+1}},x_j) &\le d(x_{n_j},x_j) \\ &\le \frac{1}{2^j}. \end{aligned}$$

Using triangle inequality, we get

$$\begin{aligned} d(x_{n_{j+1}},x_j) &\le d(x_{j+1},x_{n_{j+1}}) + d(x_{n_{j+1}},x_j) \\ &\le \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &\le \frac{1}{2^{j-1}} \\ &\to 0 \text{ } as \text{ } j \to \infty. \end{aligned}$$

So, $\{x_j\}$ is a cauchy sequence in $F(\Omega)$. From Lemma 2.7 $F(\Omega)$ is closed, so $\{x_j\}$ converges to some $x \in F(\Omega)$.

Again, owing to triangle inequality, we have

$$d(x_{n_j},x) \le d(x_{n_j},x_j) + d(x_j,x).$$

Letting $j \to \infty$, we have $\{x_{n_j}\}$ converges strongly to $x \in F(\Omega)$.

Since $\lim_{n\to\infty} inf d(x_n,x)$ exists by Theorem 4.1, therefore $\{x_n\}$ converges to $x \in F(\Omega)$. □

Eventually, we discuss the strong convergence for our scheme (3.6) by using the condition$(I)$ given by definition 2.4.

**Theorem 4.5.** *Let $\Omega\colon K \to K$ satisfies the generalized $(\alpha,\beta)$-nonexpansive mapping defined on a nonempty closed convex subset $K$ of a complete CAT(0) space $X$ such that $F(\Omega) \ne \emptyset$. If $\{x_n\}$ is a sequence defined by (3.6) and $\Omega$ satisfies the Condition (I), then $\{x_n\}$ converges strongly to a fixed point of $\Omega$.*

**Proof:** From (4.15), $\lim_{n\to\infty} d(x_n, F(\Omega))$ exists.

Also, by theorem 4.2 we have $\lim_{n\to\infty} d(x_n, \Omega x_n) = 0$.

It follows from the Condition (I) that

$$\lim_{n\to\infty} f(d(x_n, F(\Omega))) \leq \lim_{n\to\infty} d(x_n, \Omega x_n)$$
$$= 0.$$

So $\lim_{n\to\infty} f(d(x_n, F(\Omega))) = 0$. Since $f$ is a non decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, therefore $\lim_{n\to\infty} d(x_n, \Omega x_n) = 0$.

By Theorem 4.4, the sequence $\{x_n\}$ converges strongly to a point of $F(\Omega)$. $\square$

## 5. Numerical Example

The following example shows that there exist maps which are generalized $(\alpha, \beta)$-nonexpansive but neither generalized $\alpha$-nonexpansive nor $\beta$-Reich–Suzuki type.

**Example:** Let $K = \mathbb{R}^+$ which is closed and convex subset of CAT(0) space $X = \mathbb{R}$, endowed with the usual metric. Define a mapping $\Omega \colon \mathbb{R}^+ \to \mathbb{R}^+$.

$$\Omega x = \begin{cases} 0, \ if \ x \in [0.\frac{1}{2}] \\ \frac{x}{2}, \ if \ x \in (\frac{1}{2}, \infty). \end{cases}$$

for all $x \in K$. We shall prove that $\Omega$ is generalized $(\frac{1}{4}, \frac{1}{4})$-nonexpansive.

We shall divide the proof into three cases.

(i) If $0 \leq x, y \leq \frac{1}{2}$, then we have

$$\frac{1}{4}d(x, \Omega y) + \frac{1}{4}d(y, \Omega x) + \frac{1}{4}d(x, \Omega x) + \frac{1}{4}d(y, \Omega y) \geq 0 = d(\Omega x, \Omega y).$$

(ii) If $\frac{1}{2} < x, y < \infty$, then we have

$$\begin{aligned} \frac{1}{4}d(x, \Omega y) + \frac{1}{4}d(y, \Omega x) + \frac{1}{4}d(x, \Omega x) + \frac{1}{4}d(y, \Omega y) &= \frac{1}{4}|x - \frac{y}{2}| + \frac{1}{4}|y - \frac{x}{2}| \\ &+ \frac{1}{4}|x - \frac{x}{2}| + \frac{1}{4}|y - \frac{y}{2}| \\ &\geq \frac{1}{4}|\frac{3x}{2} - \frac{3y}{2}| + \frac{1}{4}|\frac{x}{2} - \frac{y}{2}| \\ &\geq \frac{1}{4}|\frac{4x}{2} - \frac{4y}{2}| \end{aligned}$$

$$= \frac{1}{2}|x - y|$$
$$= d(\Omega x, \Omega y).$$

(iii) If $\frac{1}{2} < x < \infty$ and $0 \leq y \leq \frac{1}{2}$, then we have

$$\begin{aligned}\frac{1}{4}d(x,\Omega y)+\frac{1}{4}d(y,\Omega x)+\frac{1}{4}d(x,\Omega x)+\frac{1}{4}d(y,\Omega y) &= \frac{1}{4}|x|+\frac{1}{4}|y-\frac{x}{2}| \\ &\quad +\frac{1}{4}|x-\frac{x}{2}|+\frac{1}{4}|y| \\ &= \frac{1}{4}|x|+\frac{1}{4}|y-\frac{x}{2}|+\frac{1}{4}|\frac{x}{2}|+\frac{1}{4}|y| \\ &\geq \frac{1}{4}|\frac{4x}{2}| = \frac{1}{2}|x| \\ &= d(\Omega x,\Omega y).\end{aligned}$$

Hence, $\Omega$ is generalized $(\frac{1}{4}, \frac{1}{4})$-nonexpansive. However, for $x = \frac{1}{2}$ and $y = \frac{4}{5}$, we have $\frac{1}{2}d(\Omega x, \Omega y) < d(x, y)$. However,

(i) $d(\Omega x, \Omega y) > d(x, y)$.

(ii) $d(\Omega x, \Omega y) > \frac{1}{4}d(x, \Omega y) + \frac{1}{4}d(y, \Omega x) + (1 - 2(\frac{1}{4}))d(x, y)$.

(iii) $d(\Omega x, \Omega y) > \frac{1}{4}d(x, \Omega x) + \frac{1}{4}d(y, \Omega y) + (1 - 2(\frac{1}{4}))d(x, y)$.

Hence, $\Omega$ is neither generalized $\frac{1}{4}$-nonexpansive nor $\frac{1}{4}$-Reich-Suzuki type. We obtained the influence of initial point for the New iteration algorithm (3.6) by $\alpha_n = 0.90, \beta_n = 0.65, \gamma_n = 0.85$ and $x_1 = 1000$.

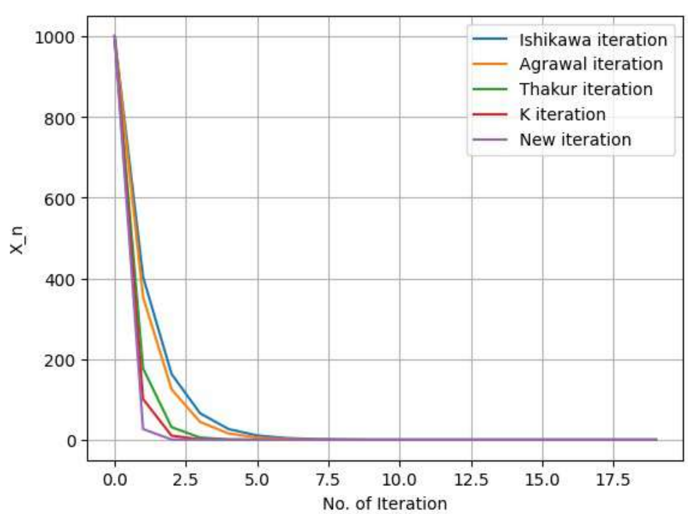


FIGURE 1. Convergence of Ishikawa, Agrawal, Thakur, K and New iterations

TABLE 1. Convergence of our iteration (3.6) for fixed point 0.

| No. of iteration | Ishikawa iteration | Agrawal iteration | Thakur iteration | K iteration | New iteration |
|---|---|---|---|---|---|
| 0 | 1000 | 1000 | 1000 | 1000 | 1000 |
| 1 | 403.7500000 | 353.7500000 | 176.8750000 | 100.9375000 | 26.68359375 |
| 2 | 163.0140625 | 125.1390625 | 31.28476563 | 10.18837891 | 0.712014175 |
| 3 | 65.81692773 | 44.26794336 | 5.533492920 | 1.028389496 | 0 |
| 4 | 26.57358457 | 15.65978496 | 0.978736560 | 0 | 0 |
| 5 | 10.72908477 | 5.539648931 | 0 | 0 | 0 |
| 6 | 4.331867976 | 1.959650809 | 0 | 0 | 0 |
| 7 | 1.748991695 | 0.693226474 | 0 | 0 | 0 |
| 8 | 0.706155397 | 0.034661324 | 0 | 0 | 0 |
| 9 | 0.007061842 | 0 | 0 | 0 | 0 |
| 10 | 0.007061554 | 0 | 0 | 0 | 0 |

## 6. CONCLUSION

In this work, we present some fixed point results for a generalized $(\alpha,\beta)$-nonexpansive mappings and also proposed a new iterative algorithm for approximating the fixed point of this class of mappings in the framework of CAT(0) spaces. Our numerical experiment shows that our iterative algorithm is better compare to some existing iterative algorithms in the literature.

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

## REFERENCES

[1] M. Abbas, T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, Mat. Vesnik, 67 (2014), 223–234. http://hdl.handle.net/2263/39674.

[2] R.P. Agarwal, D. O'Regan, D. R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 8 (2007), 61–79.

[3] K. Aoyama, F. Kohsaka, Fixed point theorem for $\alpha$-nonexpansive mappings in Banach spaces, Nonlinear Anal.: Theory Methods Appl. 74 (2011), 4387–4391. https://doi.org/10.1016/j.na.2011.03.057.

[4] M. Basarir and A. Sahin, On the strong and $\Delta$-convergence of S-iteration process for generalized nonexpansive mappings on CAT(0) spaces, Thai J. Math. 12 (2014), 549–559.

[5] M.R. Bridson, A. Haefliger, Metric spaces of non-positive curvature, Springer, Berlin Heidelberg, 2013.

[6] K.S. Brown, Buildings. In: Buildings. Springer, New York, 1989: pp. 76–98. https://doi.org/10.1007/978-1-4612-1019-1_4.

[7] F. Bruhat, J. Tits, Groupes réductifs sur un corps local, Publ. Math., Inst. Hautes Étud. Sci. 44 (1972), 5–251.

[8] S. Dhompongsa, B. Panyanak, On $\Delta$-convergence theorems in CAT(0) spaces, Computers Math. Appl. 56 (2008), 2572–2579. https://doi.org/10.1016/j.camwa.2008.05.036.

[9] K. Goebel, R. Simeon, Uniform convexity, hyperbolic geometry and nonexpansive mappings, M. Dekker, (1984). https://cir.nii.ac.jp/crid/1130000793816060288.

[10] D. Gohde, Zum prinzip der kontraktiven abbildung, Mathematische Nachrichten, 30 (1965), 251–258.

[11] B. Halpern, Fixed points of nonexpanding maps, 73 (1967), 957–961.

[12] A. Amini-Harandi, M. Fakhar, H.R. Hajisharifi, Approximate fixed points of $\alpha$-nonexpansive mappings, J. Math. Anal. Appl. 467 (2018), 1168–1173. https://doi.org/10.1016/j.jmaa.2018.07.058.

[13] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974), 147–150.

[14] W.A. Kirk, Geodesic geometry and fixed point theory. In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), pp. 195–225. Univ. Sevilla Secr. Publ., Seville (2003).

[15] W.A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal.: Theory Methods Appl. 68 (2008), 3689–3696. https://doi.org/10.1016/j.na.2007.04.011.

[16] T. Laokul, B. Panyanak, Approximating fixed points of nonexpansive mappings in CAT(0) spaces, Int. J. Math. Anal. 3 (2009), 1305–1315.

[17] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953), 506–510.

[18] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000), 217–229. https://doi.org/10.1006/jmaa.2000.7042.

[19] R. Pant, R. Pandey, Existence and convergence results for a class of nonexpansive type mappings in hyperbolic spaces, Appl. Gen. Topol. 20 (2019), 281–295.

[20] R. Pant, R. Shukla, Approximating fixed points of generalized $\alpha$-nonexpansive mappings in banach spaces, Numer. Funct. Anal. Optim. 38 (2017), 248–266. https://doi.org/10.1080/01630563.2016.1276075.

[21] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. 340 (2008), 1088–1095. https://doi.org/10.1016/j.jmaa.2007.09.023.

[22] D. Thakur, B.S. Thakur, M. Postolache, Convergence theorems for generalized nonexpansive mappings in uniformly convex Banach spaces, Fixed Point Theory Appl. 2015 (2015), 144. https://doi.org/10.1186/s13663-015-0397-z.

[23] K. Ullah, J. Ahmad, M. de la Sen, On generalized nonexpansive maps in Banach spaces, Computation. 8 (2020), 61. https://doi.org/10.3390/computation8030061.