Available online at http://scik.org
Adv. Fixed Point Theory, 2023, 13:13
https://doi.org/10.28919/afpt/8138
ISSN: 1927-6303

SOME RESULTS ON ZEROS OF THE MONIC POLYNOMIAL OF THE FROBENIUS COMPANION MATRIX

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Abstract. In this work, we intend to present some results related to the zeros of the monic polynomial of the Frobenius companion matrix. These results would certainly contribute to obtaining some new upper bounds for the zeros of such polynomials.

Keywords: Frobenius companion matrix; zeros of polynomials; matrix analysis.

2020 AMS Subject Classification: 65F05, 65F50.

1. Introduction

Today, mathematics is used throughout the world as an essential tool in many fields, including natural science, engineering, medicine, and the social sciences [1, 2, 3, 4]. Discovering the zeros of polynomials is considered a traditional problem that has attracted the care of lot of mathematicians. Such a problem, which is still one of the important topics to both numerical and complex analysts, has several implementations in different fields of mathematics. In the matrix analysis field, the Frobenius companion matrix plays a significant connection between

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Received July 20, 2023

the geometry of polynomials and matrix analysis. This kind of matrix has been employed for determining the zeros' location of polynomials by several matrix schemes [5, 6, 7, 8, 9, 10, 11].

Consider $P(z) = z^n + a_n z^{n-1} + \cdots + a_2 z^2 + a_1$ is a complex monic polynomial with $n \geq 2$ and $a_1 \neq 0$. Assume $z_1, z_2, z_3, \cdots, z_n$ be the zeros of the polynomial p arranged in such a way that $|z_1| \geq |z_2| \geq |z_3| \geq \cdots |z_n|$. The Frobenius companion matrix C_p of p is then defined as:

$$C_p = \begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{1}$$

It is well-known that the characteristic polynomial of C_p is p itself. Thus, the zeros of p are exactly the eigenvalues of C_p [12]. In this work, some results connected with the zeros of the monic polynomial of the Frobenius companion matrix are presented. We think that such results would certainly contribute to obtaining some new upper bounds for the zeros of such polynomials.

2. Main Results

This section presents some results connected with the zeros of the monic polynomial of the Frobenius companion matrix. As a result, some possible cases that could be occurred will be dealt with.

Theorem 1. *Let z_i's be the zeros of p such that $z_i \neq 0$, for $i = 1, 2, \cdots, n$. Let m and r be two integers and $L = m - r$. If $m, n \geq 0$, then:*

$$\sum_{i=1}^{n} |z_i|^L \leq \sum_{i=1}^{n-1} \frac{|z_i|^m}{|z_{i+1}|^r} + \frac{|z_n|^m}{|z_1|^r}, \tag{2}$$

i.e., (1) can be rewritten as:

$$|z_1|^L + |z_2|^L + \cdots + |z_n|^L \leq \frac{|z_1|^m}{|z_2|^r} + \frac{|z_2|^m}{|z_3|^r} + \cdots + \frac{|z_{n-1}|^m}{|z_n|^r} + \frac{|z_n|^m}{|z_1|^r}.$$

Proof. In order to prove this result, we should concern with four cases. These cases are:

Case 1:: If $m=r$, then $L=0$, and so the right-hand side, with the help of using Arithmetic geometric mean inequality, becomes:

$$\left(\frac{|z_1|}{|z_2|}\right)^r+\left(\frac{|z_2|}{|z_3|}\right)^r+\cdots+\left(\frac{|z_{n-1}|}{|z_n|}\right)^r+\left(\frac{|z_n|}{|z_1|}\right)^r \geq n\sqrt[n]{\left(\frac{|z_1|}{|z_2|}\frac{|z_2|}{|z_3|\cdots\frac{|z_{n-1}|}{|z_n|}\frac{|z_n|}{|z_1|}}\right)^n}$$
$$=n=\sum_{i=1}^{n}|z_i|^0=\sum_{i=1}^{n}|z_i|^L,$$

i.e., inequality (1) is therefore hold.

Case 2:: If $m>r>0$, then $L=m-r>0$. Now, by taking the expression $L\frac{|z_1|^m}{|z_2|^r}+r|z_2|^L$, we can have:

$$L\frac{|z_1|^m}{|z_2|^r}+r|z_2|^L=\underbrace{\left(\frac{|z_1|^m}{|z_2|^r}+\cdots++\frac{|z_1|^m}{|z_2|^r}\right)}_{L \text{ times}}+\underbrace{\left(|z_2|^L+|z_2|^L+\cdots+|z_2|^L\right)}_{r \text{ times}}.$$

Using Arithmetic geometric mean inequality implies:

$$L\frac{|z_1|^m}{|z_2|^r}+r|z_2|^L \geq m\sqrt[m]{\frac{|z_1|^{mL}}{|z_2|^{rL}}.|z_2|^{rL}}=m|z_1|^L,$$

i.e.,

$$m|z_1|^L \leq L\frac{|z_1|^m}{|z_2|^r}+r|z_2|^L. \tag{3}$$

In the same way, we can obtain other similar inequalities:

$$m|z_2|^L \leq L\frac{|z_2|^m}{|z_3|^r}+r|z_3|^L, \tag{4}$$

$$m|z_3|^L \leq L\frac{|z_3|^m}{|z_4|^r}+r|z_4|^L, \tag{5}$$

$$\vdots$$

$$m|z_{n-1}|^L \leq L\frac{|z_{n-1}|^m}{|z_n|^r}+r|z_n|^L, \tag{6}$$

$$m|z_n|^L \leq L\frac{|z_n|^m}{|z_1|^r}+r|z_1|^L. \tag{7}$$

Adding (3),(4), (5), (6) and (7) gives:

$$m\left(|z_1|^L+|z_2|^L+\cdots+|z_n|^L\right)$$
$$\leq L\left(\frac{|z_1|^m}{|z_2|^r}+\frac{|z_2|^m}{|z_3|^r}+\cdots+\frac{|z_{n-1}|^m}{|z_n|^r}+\frac{|z_n|^m}{|z_1|^r}\right)+r\left(|z_1|^L+|z_2|^L+\cdots+|z_n|^L\right).$$

Since $L>0$, then inequality (1) is hold as well.

Case 3:: If $0<m<r$, then $L<0$. Now, letting $L^*=-L$ implies $L^*=r-m>0$. So, by applying the same previous procedure on the expression

$$L^*\frac{|z_1|^m}{|z_2|^r}+\frac{m}{|z_1|^{L^*}},$$

we can get inequality (1).

Case 4:: If $m,r<0$, then $L<0$. So, similarly, we can let $L^*=-L$, $m^*=-m$ and $r^*=-r$ and then we can apply the same previous procedure that has been done in case 1 on the expression

$$L\frac{|z_2|^{r*}}{|z_1|^{m*}}+\frac{m^*}{|z_1|^{L*}}.$$

to also obtain inequality (1), which completes the proof.

□

Corollary 1. *In Theorem 1, If $mr\leq 0$, then the inequality reverses.*

Proof. **Case 1::** If $r\leq 0\leq m$, then $L=m-r>0$. Now, we take the expression $m|z_1|^L+(-r)|z_2|^L$ as follows:

$$m|z_1|^L+(-r)|z_2|^L=\underbrace{\left(|z_1|^L+|z_1|^L+\cdots+|z_1|^L\right)}_{m\text{ times}}+\underbrace{\left(|z_2|^L+|z_2|^L+\cdots+|z_2|^L\right)}_{(-r)\text{ times}}.$$

By using Arithmetic geometric mean inequality, we can get:

$$m|z_1|^L+(-r)|z_2|^L\geq(m-r)\sqrt[m-r]{|z_1|^{mL}.|z_2|^{(-r)L}}=L\sqrt[L]{|z_1|^{mL}.|z_2|^{(-r)L}}=L|z_1|^m.|z_2|^{(-r)},$$

or

$$L|z_1|^m.|z_2|^{(-r)}\leq m|z_1|^L+(-r)|z_2|^L. \tag{8}$$

Similarly, we can obtain the following inequalities:

$$L|z_2|^m.|z_3|^{(-r)} \leq m|z_2|^L + (-r)|z_3|^L, \tag{9}$$

$$L|z_3|^m.|z_4|^{(-r)} \leq m|z_3|^L + (-r)|z_4|^L, \tag{10}$$

$$\vdots$$

$$L|z_{n-1}|^m.|z_n|^{(-r)} \leq m|z_{n-1}|^L + (-r)|z_n|^L, \tag{11}$$

$$L|z_n|^m.|z_1|^{(-r)} \leq m|z_n|^L + (-r)|z_1|^L. \tag{12}$$

Now, by adding (8), (9), (10), (11) and (12), we will obtain:

$$\begin{aligned} &L\left(|z_1|^m.|z_2|^{(-r)} + |z_2|^m.|z_3|^{(-r)} + \cdots + |z_{n-1}|^m.|z_n|^{(-r)} + |z_n|^m.|z_1|^{(-r)}\right) \\ &\leq m\left(|z_1|^L + |z_2|^L + \cdots + |z_n|^L\right) + (-r)\left(|z_1|^L + |z_2|^L + \cdots_| z_n|^L\right), \end{aligned}$$

i.e.,

$$L\left(\frac{|z_1|^m}{|z_2|^r} + \frac{|z_2|^m}{|z_3|^r} + \cdots + \frac{|z_{n-1}|^m}{|z_n|^r} + \frac{|z_n|^m}{|z_1|^r}\right) \leq (m-r)\left(|z_1|^L + |z_2|^L + \cdots + |z_n|^L\right).$$

Since $L = m - r$, we have

$$\frac{|z_1|^m}{|z_2|^r} + \frac{|z_2|^m}{|z_3|^r} + \cdots + \frac{|z_{n-1}|^m}{|z_n|^r} + \frac{|z_n|^m}{|z_1|^r} \leq |z_1|^L + |z_2|^L + \cdots + |z_n|^L.$$

So, the reverse of inequality (1) is held.

Case 2:: If $m \leq 0 \leq r$, then $L < 0$. We take the expression $(-m)|z_1|^L + |z_2|^L$ as follows:

$$(-m)|z_1|^L + |z_2|^L = \underbrace{\left(|z_1|^L + |z_1|^L + \cdots + |z_1|^L\right)}_{(-m) \text{ times}} + \underbrace{\left(|z_2|^L + |z_2|^L + \cdots + |z_2|^L\right)}_{r \text{ times}}.$$

With the aid of Arithmetic geometric mean inequality, we can have:

$$-m|z_1|^L + |z_2|^L \geq (r-m) \sqrt[r-m]{|z_1|^{(-m)L}|z_2|^{rL}} = -L \sqrt[-L]{|z_1|^{m(-L)}|z_2|^{(-r)(-L)}} = -L|z_1|^m|z_2|^{(-r)},$$

i.e.,

$$(-L)|z_1|^m.|z_2|^{(-r)} \leq (-m)|z_1|^L + r|z_2|^L. \tag{13}$$

If we continue in this manner, we can get the following inequalities:

$$(-L)|z_2|^m.|z_3|^{(-r)} \leq (-m)|z_2|^L + r|z_3|^L, \tag{14}$$

$$(-L)|z_3|^m.|z_4|^{(-r)} \leq (-m)|z_3|^L + r|z_4|^L, \tag{15}$$

$$\vdots$$

$$(-L)|z_{n-1}|^m.|z_n|^{(-r)} \leq (-m)|z_{n-1}|^L + r|z_n|^L, \tag{16}$$

$$(-L)|z_n|^m.|z_1|^{(-r)} \leq (-m)|z_n|^L + r|z_1|^L. \tag{17}$$

Adding equations (13), (14), (15), (16), and (17) gives

$$\begin{aligned}
&(-L)\left(|z_1|^m.|z_2|^{(-r)} + |z_2|^m.|z_3|^{(-r)} + \cdots + |z_{n-1}|^m.|z_n|^{(-r)} + |z_n|^m.|z_1|^{(-r)}\right) \\
&\quad \leq (-m)\left(|z_1|^L + |z_2|^L + \cdots + |z_n|^L\right) + r\left(|z_1|^L + |z_2|^L + \cdots + |z_n|^L\right),
\end{aligned}$$

i.e.,

$$(-L)\left(\frac{|z_1|^m}{|z_2|^r} + \frac{|z_2|^m}{|z_3|^r} + \cdots + \frac{|z_{n-1}|^m}{|z_n|^r} + \frac{|z_n|^m}{|z_1|^r}\right) \leq (r-m)\left(|z_1|^L + |z_2|^L + \cdots + |z_n|^L\right).$$

Since $(-L = r - m)$, we have,

$$\frac{|z_1|^m}{|z_2|^r} + \frac{|z_2|^m}{|z_3|^r} + \cdots + \frac{|z_{n-1}|^m}{|z_n|^r} + \frac{|z_n|^m}{|z_1|^r} \leq |z_1|^L + |z_2|^L + \cdots + |z_n|^L.$$

□

Corollary 2. *The equality of the previous inequality is hold if* $|z_1| = |z_2| = \cdots = |z_n|$ *with* $m = 0$ *or* $r = 0$.

Proof. If $|z_1| = |z_2| = \cdots = |z_n| = k$, and $m = 0$, then by inequality (1) we have:

$$k^L + k^L + \cdots + k^L = \frac{1}{k^r} + \frac{1}{k^r} + \cdots + \frac{1}{k^r},$$

which implies the result. On the other hand, if we suppose $|z_1| = |z_2| = \cdots = |z_n|$ with $r = 0$, we also get similarly the desired result. □

3. Conclusion

In this work, some results connected with the zeros of the monic polynomial of the Frobenius companion matrix are presented. We think that such results would certainly contribute to obtaining some new upper bounds for the zeros of such polynomials.

Conflict of Interests

The authors declare that there is no conflict of interests.

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