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EXISTENCE AND UNIQUENESS OF FIXED POINT FOR α-CONTRACTIONS IN RECTANGULAR QUASI b-METRIC SPACES

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Abstract. In this manuscript, we give some new examples of rectangular quasi b-metric spaces and it is not rectangular metric space nor metric space. After that we prove existence and uniqueness of new fixed points for some new contractions in rectangular quasi b-metric spaces. Then we validate these findings with appropriate and innovative examples.

Keywords: fixed point; rectangular quasi b-metric space; α-contractions.

2020 AMS Subject Classification: 47H10, 47H09.

1. Introduction

The fundamental result of fixed point theory of Banach contraction principle was given by Banach in 1922 [1], which was extended in many ways. After this fundamental contraction principle, several generalized forms of metric spaces were introduced by various mathematicians (see [2, 3, 9, 8, 11]).

In 1989, Bakhtin [13] introduced b-metric space and Czerwik, S. [5] presented some generalizations of well known Banach's fixed point theorem in so-called b-metric spaces. He proved

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the following result: Let (X; d) be a complete b-metric space and let $T : X \to X$ satisfy

$$d(T\psi, T\phi) < \rho d(\psi, \phi), \quad \psi, \phi \in X$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing function such that $lim_{n\to\infty}\rho^n(t) = 0$ for each $t > 0$. Then T has exactly one fixed point u and $lim_{n\to\infty}d(T^n(\psi), u) = 0$.

A. Branciari in [4] initiated the notions of a generalized metric space as a generalization of a metric space, where the triangular inequality of metric spaces replaced by $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ (quadrilateral inequality). Various fixed point results were established on such spaces, see [1, 3, 8, 6, 7, 11, 12] and references therein.

Combining conditions used for definitions of asymmetric metric and generalized metric spaces, Piri et al [10] announced the notions of generalized quasi metric space.

In this paper, we introduce the notion of new contractions and establish some new fixed point theorems for mappings in the setting of complete rectangular quasi b-metric spaces. Moreover, an illustrative examples is presented to support the obtained results.

2. PRELIMINARIES

Definition 2.1. Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric if and only if for all $x, y, z \in X$,the following conditions are satisfied:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all distinct points $x, y \in X$;

(iii) $d(x,z) \leq s(d(x,y) + d(y,z))$.

Then (X,d) is called a b-metric space.

Definition 2.2. Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, on has

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all distinct points $x, y \in X$;

(iii) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$. (quadrilateral inequality)

Then (X,d) is called a rectangular metric space.

Definition 2.3. Let X be a non-empty set, $s \geq 1$ be a given real number,

and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, on has

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all distinct points $x, y \in X$;

(iii) $d(x,y) \leq s(d(x,u) + d(u,v) + d(v,y))$. (quadrilateral inequality)

Then (X,d) is called a rectangular b-metric space.

Note: It is worth noting that every metric space may be a b-metric space, but its converse is not always true.Also every metric space may be a rectangular metric space and every rectangular metric space may be a rectangular b-metric space (with coefficient $s = 1$).

Example 2.4. Let $X = N, \alpha \geq 0$ and $d : X \times X \to \mathbb{R}^+$ such that:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all distinct points $x, y \in X$;

(iii) $d(x,y) = 4\alpha$.if $x, y \in \{1,2\}$ and $x \neq y$

(iv) $d(x,y) = \alpha$,.if $x, y \notin \{1,2\}$ and $x \neq y$

Then (X,d) is called a rectangular b-metric space with coefficient $s = \dfrac{4}{3} > 1$.but (X,d) is not rectangular metric space, as $d(1,2) = 4\alpha \nleq 3\alpha = d(1,3) + d(3,4) + d(4,2)$.

Example 2.5. Let $X = N$, $\alpha \geq 0$ and $d : X \times X \to \mathbb{R}^+$ such that:

$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y; \\ 8\lambda & \text{if} \quad x = 1, y = 4; \\ 3\lambda & \text{if} \quad (x,y) \in \{1,2,3\} \quad and \quad x \neq y; \\ \lambda & \text{otherwise} \end{cases}$$

where $\lambda > 0$ is a constant

Then (X,d) is called a rectangular b-metric space with coefficient $s = \dfrac{8}{7} > 1$.but (X,d) is not rectangular metric space, as $d(1,4) = 8\lambda \nleq 7\lambda = d(1,2) + d(2,3) + d(3,4)$.

The following is the definition of the notion of rectangular quasi metric space.

Definition 2.6. Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, on has

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$. (quadrilateral inequality)

Then (X,d) is a rectangular quasi metric space.

Note: Any rectangular metric space is a rectangular quasi metric space, but the converse is not true in general.

We give an example to show that not every rectangular quasi metric on a set X is a rectangular metric space on X.

Example 2.7. Let $X = \{t, 2t, 3t, 4t, 5t\}$ with $t > 0$, as a constant$\alpha > 0$ and define
$d : X \times X \to \mathbb{R}^+$ by:
$d(x,x) = 0$ for all $x \in X$;
$d(t,2t) = d(2t,t) = 3\alpha$;
$d(t,3t) = d(2t,3t) = d(3t,t) = d(3t,2t) = \alpha$;
$d(t,4t) = d(2t,4t) = d(3t,4t) = d(4t,t) = d(4t,2t) = d(4t,3t) = 2\alpha$;
$d(t,5t) = d(2t,5t) = d(3t,5t) = d(4t,5t) = \frac{3}{2}\alpha$;
$d(5t,t) = d(5t,2t) = d(5t,3t) = d(5t,4t) = \frac{5}{4}\alpha$;
Then (X,d) is a rectangular quasi metric space, but for the fact that
$d(t,5t) = \frac{3}{2}\alpha \neq \frac{5}{4}\alpha = d(5t,t)$.
but (X,d) is not a rectangular metric space.

Definition 2.8. Let (X,d) is a rectangular quasi metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X, and $x \in X$. Then

(i) We say that $\{x_n\}_{n\in\mathbb{N}}$ is called convergent to x if and only if

$$\lim_{n\to+\infty} d(x,x_n) = \lim_{n\to+\infty} d(x_n,x) = 0.$$

(ii) We say that $\{x_n\}_{n\in\mathbb{N}}$ is called the Cauchy sequence if

$$\lim_{n,m\to+\infty} d(x_n,x_m) = \lim_{n,m\to+\infty} d(x_m,x_n) = 0.$$

(iii) (X,d) is called complete rectangular quasi metric space if every Cauchy sequence in X converges to some $x \in X$

Lemma 2.9. *Let (X,d) be a rectangular quasi metric space and $\{x_n\}_n$ be a Cauchy sequence with pairwise disjoint elements in X. If $\{x_n\}_n$ forward converges to $x \in X$ and backward converges to $y \in X$, then $x = y$.*

Definition 2.10. Let (X,d) be a rectangular quasi metric space. X is said to be complete if X is forward and backward complete.

3. Main Results

In this section, firstly we prove common fixed point theorem in complete rectangular quasi b-metric space. We start by introducing the notion of a rectangular quasi b-metric space as follows:

Definition 3.1. Let X be a non-empty set, $s \geq 1$ be a given real number, and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, on has

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) \leq s(d(x,u) + d(u,v) + d(v,y))$. (quadrilateral inequality)

Then (X,d) is called a rectangular quasi b-metric space.

Now, we give an example of a rectangular quasi b-metric space.

Example 3.2. Let $X = A \cup B$ where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1,2]$. Define the generalized metric d on X as follows

$d(\frac{1}{2},\frac{1}{3}) = d(\frac{1}{4},\frac{1}{5}) = 0.3; d(\frac{1}{3},\frac{1}{2}) = d(\frac{1}{5},\frac{1}{4}) = d(\frac{1}{3},\frac{1}{4}) = 0.1;$

$d(\frac{1}{2},\frac{1}{4}) = d(\frac{1}{3},\frac{1}{5}) = 0.6; d(\frac{1}{4},\frac{1}{2}) = d(\frac{1}{5},\frac{1}{3}) = 0.4;$

$d(\frac{1}{2},\frac{1}{5}) = 1.05; d(\frac{1}{5},\frac{1}{2}) = d(\frac{1}{4},\frac{1}{3}) = 0.5;$

$d(\frac{1}{2},\frac{1}{2}) = d(\frac{1}{3},\frac{1}{3}) = d(\frac{1}{4},\frac{1}{4}) = d(\frac{1}{5},\frac{1}{5}) = 0;$

and $d(x,y) = |x-y|$ if $x, y \in B$ or $x \in B, y \in A$ Then (X,d) is a rectangular quasi b-metric space with coefficient $s = \frac{3}{2} \geq 1$. Indeed Condition (i) in Definition trivially holds.

Now, we show condition (ii) in Definition holds:

Case (i) If $x, y \in A$, then

$d(x,y)=d(\frac{1}{2},\frac{1}{3})=0.3\leq s[d(\frac{1}{2},u)+d(u,v)+d(v,\frac{1}{3})]$ when $u,v\in\{\frac{1}{4},\frac{1}{5}\}$.
$d(x,y)=d(\frac{1}{3},\frac{1}{2})=0.1\leq s[d(\frac{1}{3},u)+d(u,v)+d(v,\frac{1}{2})]$ when $u,v\in\{\frac{1}{4},\frac{1}{5}\}$.
$d(x,y)=d(\frac{1}{3},\frac{1}{4})=0.1\leq s[d(\frac{1}{3},u)+d(u,v)+d(v,\frac{1}{4})]$ when $u,v\in\{\frac{1}{2},\frac{1}{5}\}$.
$d(x,y)=d(\frac{1}{4},\frac{1}{3})=0.5\leq s[d(\frac{1}{4},u)+d(u,v)+d(v,\frac{1}{3})]$ when $u,v\in\{\frac{1}{2},\frac{1}{5}\}$.
$d(x,y)=d(\frac{1}{4},\frac{1}{5})=0.3\leq s[d(\frac{1}{4},u)+d(u,v)+d(v,\frac{1}{5})]$ when $u,v\in\{\frac{1}{2},\frac{1}{3}\}$.
$d(x,y)=d(\frac{1}{5},\frac{1}{4})=0.1\leq s[d(\frac{1}{5},u)+d(u,v)+d(v,\frac{1}{4})]$ when $u,v\in\{\frac{1}{2},\frac{1}{3}\}$.
$d(x,y)=d(\frac{1}{2},\frac{1}{4})=0.6\leq s[d(\frac{1}{2},u)+d(u,v)+d(v,\frac{1}{4})]$ when $u,v\in\{\frac{1}{3},\frac{1}{5}\}$.
$d(x,y)=d(\frac{1}{4},\frac{1}{2})=0.4\leq s[d(\frac{1}{4},u)+d(u,v)+d(v,\frac{1}{2})]$ when $u,v\in\{\frac{1}{3},\frac{1}{5}\}$.
$d(x,y)=d(\frac{1}{2},\frac{1}{5})=1.05\leq s[d(\frac{1}{2},u)+d(u,v)+d(v,\frac{1}{5})]$ when $u,v\in\{\frac{1}{3},\frac{1}{4}\}$.
$d(x,y)=d(\frac{1}{5},\frac{1}{2})=0.5\leq s[d(\frac{1}{5},u)+d(u,v)+d(v,\frac{1}{2})]$ when $u,v\in\{\frac{1}{3},\frac{1}{4}\}$.
$d(x,y)=d(\frac{1}{3},\frac{1}{5})=0.6\leq s[d(\frac{1}{3},u)+d(u,v)+d(v,\frac{1}{5})]$ when $u,v\in\{\frac{1}{2},\frac{1}{4}\}$.
$d(x,y)=d(\frac{1}{5},\frac{1}{3})=0.6\leq s[d(\frac{1}{5},u)+d(u,v)+d(v,\frac{1}{3})]$ when $u,v\in\{\frac{1}{2},\frac{1}{4}\}$.

Case (ii) If $x,y\in B$ or $x\in A,y\in B$ or $x\in B,y\in A$,

then $d(x,y)=|x-y|\leq s|x-u|+|u-v|+|v-y|$ for all distinct points $u,v\in X\setminus\{x,y\}$.

But (X,d)is neither a metric space, a rectangular metric space nor a rectangular quasi metric space because the triangle inequality, symmetry, and rectangular inequality fail respectively as follows:

$d(\frac{1}{2},\frac{1}{4})=0.6\nleq 0.4=d(\frac{1}{2},\frac{1}{3})+d(\frac{1}{3},\frac{1}{4})=0.3+0.1$, $d(\frac{1}{2},\frac{1}{4})=0.6\neq 0.4=d(\frac{1}{4},\frac{1}{2})$,
and $d(\frac{1}{2},\frac{1}{5})=1.05\nleq 0.7=d(\frac{1}{2},\frac{1}{3})+d(\frac{1}{3},\frac{1}{4})+d(\frac{1}{4},\frac{1}{5})$.

We next give the definitions of rectangular quasi b-convergence of a sequence and completeness of rectangular quasi b-metric spaces.

Definition 3.3. Let (X,d) is a rectangular quasi b-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X, and $x\in X$. Then

(i) We say that $\{x_n\}_{n\in\mathbb{N}}$ is called convergent to x if and only if

$$\lim_{n\to+\infty} d(x,x_n)=\lim_{n\to+\infty} d(x_n,x)=0.$$

(ii) We say that $\{x_n\}_{n\in\mathbb{N}}$ is called Cauchy sequence if

$$\lim_{n,m\to+\infty} d(x_n, x_m) = \lim_{n,m\to+\infty} d(x_m, x_n) = 0.$$

(iii) (X,d) is called complete rectangular quasi b-metric space if every Cauchy sequence in X converges to some $x \in X$.

Now, we state and prove the following fixed point theorem.

Theorem 3.4. *Let (X,d) be a complete rectangular quasi b-metric space with $s \geq 1$, and let f; g be two self maps define onto itself such that*

$$d(fx,gy) \leq \alpha M(x,y) + N(x,y) \tag{3.1}$$

and

$$d(gy,fx) \leq \alpha M'(y,x) + N'(y,x) \tag{3.2}$$

for all $x,y \in X$, where $\alpha \in [0, \frac{1}{s})$, and
$M(x,y) = max\{d(x,y), d(y,fx), d(y,gy)\}$
$M'(y,x) = max\{d(y,x), d(fx,y), d(gy,y)\}$
$N(x,y) = min\{d(x,y), d(x,fx), d(x,gy), d(y,fx), d(y,gy)\}$
$N'(y,x) = min\{d(y,x), d(fx,x), d(gy,x), d(fx,y), d(gy,y)\}$
then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X.Define the sequence x_n in X as $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for $n \geq 1$.
Suppose that there is some$n \geq 1$ such that $x_n = x_{n+1}$.
If $n = 2k$, then $x_{2k} = x_{2k+1}$ and from (3.1)

$$d(x_{2k+1}, x_{2k+2}) = d(fx_{2k}, gx_{2k+1}) \leq \alpha M(x_{2k}, x_{2k+1}) + N(x_{2k}, x_{2k+1}) \tag{3.3}$$

where

$$\begin{aligned} M(x_{2k}, x_{2k+1}) &= max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, fx_{2k}), d(x_{2k+1}, gx_{2k+1})\} \\ &= max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\} \end{aligned}$$

$$= max\{0,0,d(x_{2k+1},x_{2k+2})\}$$

$$= d(x_{2k+1},x_{2k+2})$$

$$N(x_{2k},x_{2k+1}) = min\{d(x_{2k},x_{2k+1}),d(x_{2k},fx_{2k}),d(x_{2k},gx_{2k+1}),d(x_{2k+1},fx_{2k}),d(x_{2k+1},gx_{2k+1})\}$$

$$= min\{d(x_{2k},x_{2k+1}),d(x_{2k+1},x_{2k+1}),d(x_{2k},x_{2k+2}),d(x_{2k+1},x_{2k+1}),d(x_{2k+1},x_{2k+2})\}$$

$$= min\{0,0,d(x_{2k},x_{2k+2}),0,d(x_{2k+1},x_{2k+2})\}$$

$$= 0$$

Thus we have from (3.3)

$$d(x_{2k+1},x_{2k+2}) \leq \alpha d(x_{2k+1},x_{2k+2})$$

$$(1-\alpha)d(x_{2k+1},x_{2k+2}) \leq 0$$

but $1-\alpha > 0$ since $\alpha \in [0,\frac{1}{s})$,
therefore

$$d(x_{2k+1},x_{2k+2}) = 0$$

and

$$d(x_{2k+2},x_{2k+1}) = d(gx_{2k+1},fx_{2k}) \leq \alpha M'(x_{2k+1},x_{2k}) + N'(x_{2k+1},x_{2k}) \tag{3.4}$$

where

$$M'(x_{2k+1},x_{2k}) = max\{d(x_{2k+1},x_{2k}),d(fx_{2k},x_{2k+1}),d(gx_{2k+1},x_{2k+1})\}$$

$$= max\{d(x_{2k+1},x_{2k}),d(x_{2k+1},x_{2k+1}),d(x_{2k+2},x_{2k+1})\}$$

$$= max\{0,0,d(x_{2k+2},x_{2k+1})\}$$

$$= d(x_{2k+2},x_{2k+1})$$

$$N'(x_{2k+1},x_{2k}) = min\{d(x_{2k+1},x_{2k}),d(fx_{2k},x_{2k}),d(gx_{2k+1},x_{2k}),d(fx_{2k},x_{2k+1}),d(gx_{2k+1},x_{2k+1})\}$$

$$= min\{d(x_{2k+1},x_{2k}),d(x_{2k+1},x_{2k}),d(x_{2k+2},x_{2k}),d(x_{2k+1},x_{2k+1}),d(x_{2k+2},x_{2k+1})\}$$

$$= min\{0,0,d(x_{2k+2},x_{2k}),0,d(x_{2k+1},x_{2k+2})\}$$

$$= 0$$

Thus we have from (3.4)

$$d(x_{2k+2},x_{2k+1}) \leq \alpha d(x_{2k+2},x_{2k+1})$$

$$(1-\alpha)d(x_{2k+2},x_{2k+1}) \leq 0$$

but $1-\alpha > 0$ since $\alpha \in [0,\frac{1}{s})$,
therefore

$$d(x_{2k+2},x_{2k+1}) = 0$$

Hence $x_{2k+2} = x_{2k+1} = x_{2k}$

$$x_{2k} = fx_{2k} = gx_{2k}$$

Hence x_{2k} is a common fixed point of f and g . If $n = 2k+1$, then using same argument, it can be shown that x_{2k+1} is a common fixed point of f and g. Now suppose that $x_n \neq x_{n+1}$ for all $n \geq 1$ and from (3.1),

$$d(x_{2n+1},x_{2n+2}) = d(fx_{2n},gx_{2n+1}) \leq \alpha M(x_{2n},x_{2n+1}) + N(x_{2n},x_{2n+1}) \tag{3.5}$$

where

$$\begin{aligned} M(x_{2n},x_{2n+1}) &= max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},fx_{2n}),d(x_{2n+1},gx_{2n+1})\} \\ &= max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+1}),d(x_{2n+1},x_{2n+2})\} \\ &= max\{d(x_{2n},x_{2n+1}),0,d(x_{2n+1},x_{2n+2})\} \\ &= max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\} \end{aligned}$$

$$\begin{aligned} N(x_{2n},x_{2n+1}) &= min\{d(x_{2n},x_{2n+1}),d(x_{2n},fx_{2n}),d(x_{2n},gx_{2n+1}),d(x_{2n+1},fx_{2n}),d(x_{2n+1},gx_{2n+1})\} \\ &= min\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+1}),d(x_{2n},x_{2n+2}),d(x_{2n+1},x_{2n+1}),d(x_{2n+1},x_{2n+2})\} \\ &= min\{d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+2}),0,d(x_{2n+1},x_{2n+2})\} \\ &= 0 \end{aligned}$$

Case -(i): If $M(x_{2n},x_{2n+1}) = d(x_{2n+1},x_{2n+2})$ then from (3.5)

$$d(x_{2n+1},x_{2n+2}) \leq \alpha d(x_{2n+1},x_{2n+2})$$

$$(1-\alpha)d(x_{2n+1},x_{2n+2}) \leq 0$$

but $1-\alpha > 0$. Therefore

$$d(x_{2n+1}, x_{2n+2}) \leq 0$$

Case -(ii): If $M(x_{2n}, x_{2n+1}) = d(x_{2n}, x_{2n+1})$: Then we have from (3.5)

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+1})$$

In this way we extend and get

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+1})$$

So, for all $n \geq 1$

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+1}) \leq \alpha^2 d(x_{2n-1}, x_{2n}) \leq .... \leq \alpha^{n+1} d(x_0, x_1)$$

and

$$d(x_{2n+2}, x_{2n+1}) = d(gx_{2n+1}, fx_{2n}) \leq \alpha M'(x_{2n+1}, x_{2n}) + N'(x_{2n+1}, x_{2n}) \tag{3.6}$$

where

$$\begin{aligned} M'(x_{2n+1}, x_{2n}) &= max\{d(x_{2n+1}, x_{2n}), d(fx_{2n}, x_{2n+1}), d(gx_{2n+1}, x_{2n+1})\} \\ &= max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+1}), d(x_{2n+2}, x_{2n+1})\} \\ &= max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+1}), d(x_{2n+2}, x_{2n+1})\} \\ &= max\{d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1})\} \end{aligned}$$

$$\begin{aligned} N'(x_{2n+1}, x_{2n}) &= min\{d(x_{2n+1}, x_{2n}), d(fx_{2n}, x_{2n}), d(gx_{2n+1}, x_{2n}), d(fx_{2n}, x_{2n+1}), d(gx_{2n+1}, x_{2n+1})\} \\ &= min\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+1}), d(x_{2n+2}, x_{2n}), d(x_{2n+1}, x_{2n+1}), d(x_{2n+2}, x_{2n+1})\} \\ &= min\{d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n}), 0, d(x_{2n+2}, x_{2n+1})\} \\ &= 0 \end{aligned}$$

Case -(i): If $M'(x_{2n+1}, x_{2n}) = d(x_{2n+2}, x_{2n+1})$ then from (3.6)

$$d(x_{2n+2}, x_{2n+1}) \leq \alpha d(x_{2n+2}, x_{2n+1})$$

$$(1-\alpha) d(x_{2n+2}, x_{2n+1}) \leq 0$$

but $1-\alpha > 0$. Therefore

$$d(x_{2n+2}, x_{2n+1}) \leq 0$$

Case -(ii): If $M'(x_{2n+1}, x_{2n}) = d(x_{2n+1}, x_{2n})$: Then we have from (3.6)

$$d(x_{2n+2}, x_{2n+1}) \leq \alpha d(x_{2n+1}, x_{2n})$$

In this way we extend and get

$$d(x_{2n+2}, x_{2n+1}) \leq \alpha d(x_{2n+1}, x_{2n})$$

So, for all $n \geq 1$

$$d(x_{2n+2}, x_{2n+1}) \leq \alpha d(x_{2n+1}, x_{2n}) \leq \alpha^2 d(x_{2n}, x_{2n-1}) \leq .... \leq \alpha^{n+1} d(x_1, x_0)$$

Similarly we can show that

$$d(x_n, x_{n+2}) \leq \alpha^n d(x_0, x_2)$$

and

$$d(x_{n+2}, x_n) \leq \alpha^n d(x_2, x_0)$$

for all $n \geq 1$.

Now, we show that x_n is a rectangular b-Cauchy sequence.

Applying rectangular b-inequality and $x_n \neq x_{n+1}$ for all $n \leq 1$ and $d_n = d(x_n, x_{n+1})$, $d'_n = d(x_{n+1}, x_n)$, $d*_n = d(x_n, x_{n+2})$, $d*'_n = d(x_{n+2}, x_n)$ For the sequence x_n, we consider $d(x_n, x_{n+p})$ in two cases. If p is odd say $p = 2m+1$ then

$$\begin{aligned}
d(x_n, x_{n+2m+1}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \\
&\leq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + ... + s^m d_{n+2m} \\
&\leq s[\alpha^n d_0 + \alpha^{n+1} d_0] + s^2[\alpha^{n+2} d_0 + \alpha^{n+3} d_0] + ... + s^m \alpha^{n+2m} d_0 \\
&\leq s\alpha^n (1 + s\alpha^2 + s^2\alpha^4 + ...) d_0 + s\alpha^{n+1}(1 + s\alpha^2 + s^2\alpha^4 + ...) d_0 \\
&= \frac{1+\alpha}{1-s\alpha^2} s\alpha^n d_0,
\end{aligned}$$

Hence

$$d(x_n, x_{n+2m+1}) \leq \frac{1+\alpha}{1-s\alpha^2} s\alpha^n d_0 \tag{3.7}$$

and

$$d(x_{n+2m+1},x_n) \leq s[d(x_{n+2m+1},x_{n+2})+d(x_{n+2},x_{n+1})+d(x_{n+1},x_n)]$$

$$\begin{aligned}
&\leq s[d'_n+d'_{n+1}]+s^2[d'_{n+2}+d'_{n+3}]+...+s^m d'_{n+2m} \\
&\leq s[\alpha^n d'_0+\alpha^{n+1}d'_0]+s^2[\alpha^{n+2}d'_0+\alpha^{n+3}d'_0]+...+s^m\alpha^{n+2m}d'_0 \\
&\leq s\alpha^n(1+s\alpha^2+s^2\alpha^4+...)d'_0+s\alpha^{n+1}(1+s\alpha^2+s^2\alpha^4+...)d'_0 \\
&= \frac{1+\alpha}{1-s\alpha^2}s\alpha^n d'_0,
\end{aligned}$$

Hence

$$d(x_{n+2m+1},x_n) \leq \frac{1+\alpha}{1-s\alpha^2}s\alpha^n d'_0 \tag{3.8}$$

In the same manner, we can show that when p is even

$$d(x_n,x_{n+2m}) \leq \frac{1+\alpha}{1-s\alpha^2}s\alpha^n d_0+\alpha^{n-2}d*_0 \tag{3.9}$$

and

$$d(x_{n+2m},x_n) \leq \frac{1+\alpha}{1-s\alpha^2}s\alpha^n d'_0+\alpha^{n-2}d*'_0 \tag{3.10}$$

When n tend to infinity of (3.7),(3.8),(3.9) and (3.10) we get $d(x_n,x_{n+p})=d(x_{n+p},x_n)=0$ for all $p=1,2,3,....$ Hence x_n is a rectangular b-Cauchy sequence in (X,d).

By completeness of (X,d), there exists $r\in X$ such that $x_n=fx_{n-1}\to r$ as $n\to\infty$.

Now we show that $fr=r$.

By rectangular b-inequality

$$d(fr,r) \leq s[d(fr,gx_n)+d(gx_n,x_n)+d(x_n,r)]$$

$$\frac{1}{s}d(fr,r) \leq \alpha M(r,x_n)+N(r,x_n)+d(x_{n+1},x_n)+d(x_n,r) \tag{3.11}$$

where

$$\begin{aligned}
M(r,x_n) &= max\{d(r,x_n),d(x_n,fr),d(x_n,gx_n)\} \\
&= max\{d(r,x_n),d(x_n,fr),d(x_n,x_{n+2})\}
\end{aligned}$$

$$= max\{0, d(x_n, fr), 0\}$$

$$= d(x_n, fr)$$

$$N(r,x_n) = min\{d(r,x_n), d(r,fr), d(x_n,gx_n), d(r,gx_n), d(x_n,fr)\}$$

$$= min\{d(r,r), d(r,fr), d(x_n,gr), d(r,gr), d(x_n,fr)\}$$

$$= 0.$$

Putting these values in (3.11) we get

$$\frac{1}{s}d(fr,r) \leq \alpha d(r,fr)$$

$$d(fr,r) \leq s\alpha d(r,fr) \leq d(r,fr)$$

and

$$d(r,fr) \leq s[d(gx_n,fr) + d(x_n,gx_n) + d(r,x_n)]$$

$$\frac{1}{s}d(r,fr) \leq \alpha M'(x_n,r) + N'(x_n,r) \tag{3.12}$$

where

$$M'(x_n,r) = max\{d(x_n,r), d(fr,x_n), d(gx_n,x_n)\}$$

$$= max\{d(x_n,r), d(fr,x_n), d(x_{n+2},x_n)\}$$

$$= max\{0, d(fr,x_n), 0\}$$

$$= d(fr,x_n)$$

$$N'(x_n,r) = min\{d(x_n,r), d(fr,r), d(gx_n,x_n), d(gx_n,r), d(fr,x_n)\}$$

$$= min\{d(r,r), d(fr,r), d(gr,x_n), d(gr,g), d(fr,x_n)\}$$

$$= 0.$$

Putting these values in (3.12) we get

$$\frac{1}{s}d(r,fr) \leq \alpha d(fr,r)$$

$$d(fr,r) \leq d(r,fr) \leq s\alpha d(fr,r)$$

$$(1-s\alpha)d(fr,r) \leq 0$$

Since $1-s\alpha > 0$ therefore $fr = r$

Hence r is a fixed point of f.

Now we show that r is a fixed point of g. Suppose $gr \neq r$.

$$d(r,gr) = d(fr,gr) \leq \alpha M(r,r) + N(r,r) \tag{3.13}$$

where

$$M(r,r) = max\{d(r,r), d(r,fr), d(r,gr)\}$$

$$= max\{0,0,d(r,gr)\}$$

$$= d(r,gr)$$

$$N(r,r) = min\{d(r,r), d(r,fr), d(r,gr), d(r,fr), d(r,gr)\}$$

$$= min\{0,0,d(r,gr),0,d(r,gr)\}$$

$$= 0.$$

Now putting these values in (3.13) we have

$$d(r,gr) \leq \alpha d(r,gr)$$

$$(1-\alpha)d(r,gr) \leq 0$$

Similarly we can show that

$$(1-\alpha)d(gr,r) \leq 0$$

But $1-\alpha \nleq 0$ since $\alpha \in [0, \frac{1}{s})$. Thus $gr = r$. Hence

$$fr = gr = r.$$

Therefore f and g have a common fixed point of X.

Now suppose p and q are two common fixed points.

Then clearly $p = fp$ and $q = gq$, then

$$d(p,q) = d(fp,gq) \leq \alpha M(p,q) + N(p,q) \tag{3.14}$$

where

$$
\begin{aligned}
M(p,q) &= max\{d(p,q), d(q,fp), d(q,gq)\} \\
&= max\{d(p,q), d(q,p), d(q,q)\} \\
&= max\{d(p,q), d(q,p)\}
\end{aligned}
$$

$$
\begin{aligned}
N(p,q) &= min\{d(p,q), d(p,fp), d(p,gq), d(q,fp), d(q,gq)\} \\
&= min\{d(p,q), d(p,p), d(p,q), d(q,p), d(q,q)\} \\
&= 0.
\end{aligned}
$$

If $M(p,q) = d(p,q)$ On putting these values in (3.14) we get

$$d(p,q) \leq \alpha d(p,q)$$

$$(1-\alpha)d(p,q) \leq 0$$

but $1-\alpha > 0$ thus $d(p,q) = 0$
If $M(p,q) = d(q,p)$ On putting these values in (3.14) we get

$$d(p,q) \leq \alpha d(q,p)$$

and

$$d(q,p) \leq \alpha max\{d(p,q), d(q,p)\}$$

Therefore

$$d(p,q) \leq \alpha d(q,p) \leq \alpha^2 d(p,q)$$

or

$$d(p,q) \leq \alpha d(q,p) \leq \alpha^2 d(q,p)$$

Thus

$$(1-\alpha)d(p,q) \leq 0$$

and

$$(1-\alpha)d(q,p) \leq 0$$

But $1-\alpha > 0$ thus $d(p,q) = 0$ Hence $p = q$. □

Corollary 3.5. *Let (X,d) be a complete rectangular quasi b-metric space with $s \geq 1$, and let $f : X \to X$; $g : X \to X$ be two self maps define onto itself such that*

$$d(fx,gy) \leq \alpha M(x,y) + N(x,y)$$

and

$$d(gy,fx) \leq \alpha M'(y,x) + N'(y,x)$$

for all $x,y \in X$, where $\alpha \in [0,\frac{1}{s})$, and
$M(x,y) = max\{d(x,y), d(y,gy)\}$
$M'(y,x) = max\{d(y,x), d(gy,y)\}$
$N(x,y) = min\{d(x,y), d(x,fx), d(x,gy)\}$
$N'(y,x) = min\{d(y,x), d(fx,x), d(gy,x)\}$ *then f and g have a unique common fixed point.*

Corollary 3.6. *Let (X,d) be a complete rectangular quasi b-metric space with $s \geq 1$, and let $f : X \to X$; $g : X \to X$ be two self maps define onto itself such that*

$$d(fx,gy) \leq \alpha M(x,y) + N(x,y)$$

and

$$d(gy,fx) \leq \alpha M'(y,x) + N'(y,x)$$

for all $x,y \in X$, where $\alpha \in [0,\frac{1}{s})$, and
$M(x,y) = max\{d(x,y), d(y,gy)\}$
$M'(y,x) = max\{d(y,x), d(gy,y)\}$
$N(x,y) = min\{d(x,fx), d(x,gy)\}$
$N'(y,x) = min\{d(fx,x), d(gy,x)\}$
then f and g have a unique common fixed point.

Now, we prove another common fixed point theorem with new contraction map- ping in complete rectangular quasi b-metric spaces.

Theorem 3.7. *Let (X,d) be a complete rectangular quasi b-metric space with $s \geq 1$, and let $f : X \to X$; $g : X \to X$ be two self maps satisfying*

$$d(fx,gy) \leq \alpha M(x,y) + N(x,y) \tag{3.15}$$

and

$$d(gy,fx) \leq \alpha M'(y,x) + N'(y,x) \tag{3.16}$$

for all $x,y \in X$, *where* $\alpha \in [0,\frac{1}{s})$, *and*

$M(x,y) = max\left\{d(x,y), \dfrac{d(x,gy)d(y,fx)}{1+d(fx,gy)}\right\}$

$M'(y,x) = max\left\{d(y,x), \dfrac{d(gy,x)d(fx,y)}{1+d(gy,fx)}\right\}$

$N(x,y) = min\{d(x,y), d(y,fx), d(x,gy)\}$

$N'(y,x) = min\{d(y,x), d(fx,y), d(gy,x)\}$

then f *and* g *have a common fixed point.*

Proof. Taking x_0 be an arbitrary point in X.define the sequence x_n in X as $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for $n \geq 1$.

Assume that there is some $n \geq 1$ such that $x_n = x_{n+1}$.

If $n = 2k$, then $x_{2k} = x_{2k+1}$ and from (3.15)

$$d(x_{2k+1},x_{2k+2}) = d(fx_{2k},gx_{2k+1}) \leq \alpha M(x_{2k},x_{2k+1}) + N(x_{2k},x_{2k+1}) \tag{3.17}$$

where

$$\begin{aligned}
M(x_{2k},x_{2k+1}) &= max\left\{d(x_{2k},x_{2k+1}), \frac{d(x_{2k},gx_{2k+1})d(x_{2k+1},fx_{2k})}{1+d(fx_{2k},gx_{2k+1})}\right\} \\
&= max\left\{d(x_{2k},x_{2k+1}), \frac{d(x_{2k},x_{2k+2})d(x_{2k+1},x_{2k+1})}{1+d(x_{2k+1},x_{2k+2})}\right\} \\
&= max\left\{0, \frac{0}{1+d(x_{2k+1},x_{2k+2})}\right\} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
N(x_{2k},x_{2k+1}) &= min\{d(x_{2k},x_{2k+1}), d(x_{2k+1},fx_{2k}), d(x_{2k},gx_{2k+1})\} \\
&= min\{d(x_{2k},x_{2k+1}), d(x_{2k+1},x_{2k+1}), d(x_{2k},x_{2k+2})\} \\
&= min\{0,0,d(x_{2k},x_{2k+2})\} \\
&= 0.
\end{aligned}$$

Now putting these values in (3.17)

$$d(x_{2k+1},x_{2k+2}) \leq 0$$

and

$$d(x_{2k+2},x_{2k+1}) = d(gx_{2k+1},fx_{2k}) \leq \alpha M'(x_{2k+1},x_{2k}) + N'(x_{2k+1},x_{2k}) \tag{3.18}$$

where

$$\begin{aligned}
M'(x_{2k+1},x_{2k}) &= max\left\{d(x_{2k+1},x_{2k}), \frac{d(gx_{2k+1},x_{2k})d(fx_{2k},x_{2k+1})}{1+d(gx_{2k+1},fx_{2k})}\right\} \\
&= max\left\{d(x_{2k+1},x_{2k}), \frac{d(x_{2k+2},x_{2k})d(x_{2k+1},x_{2k+1})}{1+d(x_{2k+2},x_{2k+1})}\right\} \\
&= max\left\{0, \frac{0}{1+d(x_{2k+2},x_{2k+1})}\right\} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
N'(x_{2k+1},x_{2k}) &= min\{d(x_{2k+1},x_{2k}), d(fx_{2k},x_{2k+1}), d(gx_{2k+1},x_{2k})\} \\
&= min\{0, d(x_{2k+1},x_{2k+1}), d(x_{2k+2},x_{2k})\} \\
&= 0
\end{aligned}$$

Thus we have from (3.18)

$$d(x_{2k+2},x_{2k+1}) \leq 0$$

Therefore

$$x_{2k+1} = x_{2k+2}$$

Thus

$$x_{2k+2} = x_{2k+1} = x_{2k}$$

$$x_{2k} = fx_{2k} = gx_{2k}$$

Hence x_{2k} is a common fixed point of f and g .

If $n = 2k+1$, then using same argument, it can be shown that x_{2k+1} is a common fixed point of

f and g.

Now suppose that $x_n \neq x_{n+1}$ for all $n \geq 1$ and from (3.15),

$$d(x_{2n+1}, x_{2n+2}) = d(fx_{2n}, gx_{2n+1}) \leq \alpha M(x_{2n}, x_{2n+1}) + N(x_{2n}, x_{2n+1}) \tag{3.19}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= max\left\{d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, gx_{2n+1})d(x_{2n+1}, fx_{2n})}{1 + d(fx_{2n}, gx_{2n+1})}\right\} \\ &= max\left\{d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, x_{2n+2})d(x_{2n+1}, x_{2n+1})}{1 + d(x_{2n+1}, x_{2n+2})}\right\} \\ &= max\left\{d(x_{2n}, x_{2n+1}), \frac{0}{1 + d(x_{2n+1}, x_{2n+2})}\right\} \\ &= d(x_{2n}, x_{2n+1}) \end{aligned}$$

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, fx_{2n}), d(x_{2n}, gx_{2n+1})\} \\ &= min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+1}), d(x_{2n}, x_{2n+2})\} \\ &= min\{d(x_{2n}, x_{2n+1}), 0, d(x_{2k}, x_{2k+2})\} \\ &= 0. \end{aligned}$$

Putting these values in (3.19)

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+1})$$

So, for all $n \geq 1$

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+1}) \leq \alpha^2 d(x_{2n-1}, x_{2n}) \leq .... \leq \alpha^{n+1} d(x_0, x_1)$$

Similarly

$$d(x_{2n+2}, x_{2n+1}) \leq \alpha d(x_{2n+1}, x_{2n}) \leq \alpha^2 d(x_{2n}, x_{2n-1}) \leq .... \leq \alpha^{n+1} d(x_1, x_0)$$

Similarly we can show that

$$d(x_n, x_{n+2}) \leq \alpha^n d(x_0, x_2)$$

and

$$d(x_{n+2}, x_n) \leq \alpha^n d(x_2, x_0)$$

for all $n \geq 1$.

Now, we show that x_n is a rectangular b-Cauchy sequence. Applying rectangular b-inequality and $x_n \neq x_{n+1}$ for all $n \leq 1$ and $d_n = d(x_n, x_{n+1})$, $d'_n = d(x_{n+1}, x_n)$, $d*_n = d(x_n, x_{n+2})$, $d*'_n = d(x_{n+2}, x_n)$ For the sequence x_n, we consider $d(x_n, x_{n+p})$ in two cases. If p is odd say $p = 2m+1$ then

$$\begin{aligned}
d(x_n, x_{n+2m+1}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \\
&\leq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + .. + s^m d_{n+2m} \\
&\leq s[\alpha^n d_0 + \alpha^{n+1} d_0] + s^2[\alpha^{n+2} d_0 + \alpha^{n+3} d_0] + .. + s^m \alpha^{n+2m} d_0 \\
&\leq s\alpha^n (1 + s\alpha^2 + s^2\alpha^4 + ...) d_0 + s\alpha^{n+1}(1 + s\alpha^2 + s^2\alpha^4 + ...) d_0 \\
&= \frac{1+\alpha}{1-s\alpha^2} s\alpha^n d_0,
\end{aligned}$$

Hence

$$d(x_n, x_{n+2m+1}) \leq \frac{1+\alpha}{1-s\alpha^2} s\alpha^n d_0 \tag{3.20}$$

and

$$\begin{aligned}
d(x_{n+2m+1}, x_n) &\leq s[d(x_{n+2m+1}, x_{n+2}) + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq s[d'_n + d'_{n+1}] + s^2[d'_{n+2} + d'_{n+3}] + ... + s^m d'_{n+2m} \\
&\leq s[\alpha^n d'_0 + \alpha^{n+1} d'_0] + s^2[\alpha^{n+2} d'_0 + \alpha^{n+3} d'_0] + ... + s^m \alpha^{n+2m} d'_0 \\
&\leq s\alpha^n (1 + s\alpha^2 + s^2\alpha^4 + ...) d'_0 + s\alpha^{n+1}(1 + s\alpha^2 + s^2\alpha^4 + ...) d'_0 \\
&= \frac{1+\alpha}{1-s\alpha^2} s\alpha^n d'_0,
\end{aligned}$$

Hence

$$d(x_{n+2m+1}, x_n) \leq \frac{1+\alpha}{1-s\alpha^2} s\alpha^n d'_0 \tag{3.21}$$

In the same manner, we can show that when p is even we can show

$$d(x_n, x_{n+2m}) \leq \frac{1+\alpha}{1-s\alpha^2} s\alpha^n d_0 + \alpha^{n-2} d*_0 \tag{3.22}$$

and

$$d(x_{n+2m}, x_n) \leq \frac{1+\alpha}{1-s\alpha^2} s\alpha^n d_0' + \alpha^{n-2} d*_0' \tag{3.23}$$

When n tend to infinity of (3.20),(3.21),(3.22) and (3.23) we get $d(x_n, x_{n+p}) = d(x_{n+p}, x_n) = 0$ for all $p = 1,2,3,....$ Hence x_n is a rectangular b-Cauchy sequence in (X,d). By completeness of (X,d), there exists $r \in X$ such that $x_n = fx_{n-1} \to r$ as $n \to \infty$.

Now we show that $fr = r$.

By rectangular b-inequality

$$d(fr,r) \leq s[d(fr,gx_n) + d(gx_n,x_n) + d(x_n,r)]$$

$$\frac{1}{s} d(fr,r) \leq \alpha M(r,x_n) + N(r,x_n) + d(x_{n+1},x_n) + d(x_n,r) \tag{3.24}$$

where

$$\begin{aligned} M(r,x_n) &= max\{d(r,x_n), \frac{d(r,gx_n)d(x_n,fr}{1+d(fr,gx_n)}\} \\ &= max\{d(r,x_n), \frac{d(r,x_{n+1})d(x_n,fr}{1+d(fr,x_{n+1})}\} \\ &= max\{0,0\} \\ &= 0 \end{aligned}$$

$$\begin{aligned} N(r,x_n) &= min\{d(r,x_n), d(x_n,fr), d(r,gx_n)\} \\ &= min\{d(r,r), d(r,fr), d(r,gr)\} \\ &= 0. \end{aligned}$$

Putting these values in (3.24) we get

$$\frac{1}{s} d(fr,r) \leq 0$$

$$d(fr,r) \leq 0$$

and

$$d(r,fr) \leq s[d(gx_n,fr) + d(x_n,gx_n) + d(r,x_n)]$$

$$\frac{1}{s}d(r,fr) \leq \alpha M'(x_n,r) + N'(x_n,r) + d(x_n,gx_n) + d(r,x_n) \tag{3.25}$$

where

$$\begin{aligned} M'(x_n,r) &= max\{d(x_n,r), \frac{d(gx_n,r)d(fr,x_n}{1+d(gx_n,fr)}\} \\ &= max\{d(x_n,r), \frac{d(x_{n+1},r)d(fr,x_n}{1+d(x_{n+1},fr)}\} \\ &= max\{0,0\} \\ &= 0 \end{aligned}$$

$$\begin{aligned} N'(x_n,r) &= min\{d(x_n,r), d(fr,x_n), d(gx_n,r)\} \\ &= min\{d(r,r), d(fr,r), d(gr,r)\} \\ &= 0. \end{aligned}$$

Putting these values in (3.25) we get

$$\frac{1}{s}d(r,fr) \leq 0$$

$$d(r,fr) \leq 0$$

therefore $fr = r$ Hence r is a fixed point of f.

Now we show that r is a fixed point of g. Suppose $gr \neq r$.

$$d(r,gr) = d(fr,gr) \leq \alpha M(r,r) + N(r,r) \tag{3.26}$$

where

$$\begin{aligned} M(r,r) &= max\{d(r,r), \frac{d(r,gr)d(r,fr)}{1+d(fr,gr)}\} \\ &= max\{0, \frac{d(r,gr)d(r,r)}{1+d(fr,gr)}\} \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
N(r,r) &= min\{d(r,r),d(r,fr),d(r,gr)\} \\
&= min\{0,0,d(r,gr)\} \\
&= 0.
\end{aligned}
$$

Thus

$$d(r,gr) \leq 0$$

Similarly we can show that

$$d(gr,r) \leq 0$$

Therefore $gr = r$.

Hence

$$fr = gr = r.$$

Therefore f and g have a common fixed point of X.

$\square$

Example 3.8. Let $X = A \cup B$ where $A = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1,2]$.

We define d on X as follows

$d(0,\frac{1}{2}) = d(\frac{1}{2},\frac{1}{3}) = d(\frac{1}{4},\frac{1}{5}) = 0.3; d(0,\frac{1}{3}) = d(\frac{1}{3},\frac{1}{2}) = d(\frac{1}{5},\frac{1}{4}) = d(\frac{1}{3},\frac{1}{4}) = 0.1;$

$d(0,\frac{1}{4}) = d(\frac{1}{4},\frac{1}{2}) = d(\frac{1}{3},\frac{1}{5}) = 0.6; d(0,\frac{1}{5}) = d(\frac{1}{2},\frac{1}{4}) = d(\frac{1}{5},\frac{1}{3}) = 0.4;$

$d(\frac{1}{2},0) = d(\frac{1}{4},0) = d(\frac{1}{2},\frac{1}{5}) = 1.05; d(\frac{1}{3},0) = d(\frac{1}{5},0) = d(\frac{1}{5},\frac{1}{2}) = d(\frac{1}{4},\frac{1}{3}) = 0.5;$

$d(0,0) = d(\frac{1}{2},\frac{1}{2}) = d(\frac{1}{3},\frac{1}{3}) = d(\frac{1}{4},\frac{1}{4}) = d(\frac{1}{5},\frac{1}{5}) = 0;$

and $d(x,y) = |x-y|$ if $x,y \in B$ or $x \in B, y \in A$

Then (X,d) is a rectangular quasi b-metric space with coefficient $s = \frac{3}{2} \geq 1$.

Let $f : X \to X$ and $g : X \to X$ be defined as

$$f(x) = \begin{cases} \frac{1}{3}, & \text{if } x \in A; \\ \frac{1}{4}, & \text{if } x \in B; \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{3}, & \text{if } x \in A; \\ \frac{1}{5}, & \text{if } x \in B; \end{cases}$$

Then f and g satisfy all conditions of Theorem 3.4 with $\alpha \in [0, \frac{1}{s})$ and has a unique fixed point $x = \frac{1}{3}$.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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