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β-σ-GERAGHTY TYPE CONTRACTION MAPPING AND FIXED POINT RESULTS IN GENERALIZED G_b-METRIC SPACE

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Abstract. This article investigates Fixed Point Results in Generalized G_b-metric space by focusing on the concept of β-σ-Geraghty type contraction mapping in generalized G_b-complete metric space (CMS). We examine the fixed point results for this type of mapping and present several theorems that extend and generalize previous findings in the field. Our study contributes to a deeper understanding of Fixed Point Results in Generalized G_b-metric space.

Keywords: admissible mapping; complete metric space; contraction mapping; fixed point.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Fixed point theory is among the most exciting fields of research in nonlinear analysis. Numerous disciplines use fixed point theory, including algebra, physics, biology, economics, and many other fields. The most famous and ground-breaking discovery in this field is the Banach contraction mapping principle (BCP) [2]. Several researchers have generalized and extended the BCP by defining new contractive conditions and by introducing new abstract spaces (for

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example [3], [4], [5], [7]). Motivated by the results of [3], [4] and [5], we introduce a new contractive mapping named $\beta-\sigma-$*Geraghty type contraction mapping*, which is a generalization of many contractive mappings existing in the literature.

2. Preliminaries

We discuss certain terminology and research in this section that will be utilized to support our main conclusions.

In 1973, Geraghty [3] defined the following set of functions.

Definition 2.1. [3] *Let $\mathscr{P}$ be the set of all functions $\beta:[0,\infty)\to[0,1)$ which satisfies the condition*

$$\lim_{n\to\infty}\beta(m_n)=1 \Rightarrow \lim_{n\to\infty}m_n=0.$$

The following result was demonstrated by Geraghty using similar functions.

Theorem 2.1. [3] *Let $H:R\to R$ be an operator and $(R,\mathfrak{h})$ be a CMS. If H fulfills the inequality shown below:*

$$\mathfrak{h}(H\mathfrak{u}_1,H\mathfrak{u}_2)\leq\beta(\mathfrak{h}(\mathfrak{u}_1,\mathfrak{u}_2))\mathfrak{h}(\mathfrak{u}_1,\mathfrak{u}_2),\ \textit{for any}\ \mathfrak{u}_1,\mathfrak{u}_2\in R,$$

where $\beta\in\mathscr{P}$, then H has a unique fixed point.

The term β-*admissible map* was first introduced by Alghamdi and Karapinar [1] as:

Definition 2.2. [1] *Assume H: $R\to R$ and $\beta:R\times R\times R\to[0,+\infty)$ be two mappings. H is called β-admissible if $\forall\ \mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3\in R$, we have*

$$\beta(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)\geq 1\ \textit{implies}\ \beta(H\mathfrak{u}_1,H\mathfrak{u}_2,H\mathfrak{u}_3)\geq 1.$$

Jain and Kaur [4] in 2019 proposed the concept of Generalized G_b-metric space defined as:

Definition 2.3. [4] *Let $s\geq 1$ be a real number and R be a non-empty set. Let G: $R\times R\times R$ $\to[0,+\infty)$ be a function meeting the requirements listed below:*

(i) $G(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)=0$ *if* $\mathfrak{u}_1=\mathfrak{u}_2=\mathfrak{u}_3$*;*

(ii) $0<G(\mathfrak{u}_1,\mathfrak{u}_1,\mathfrak{u}_2)$*, for all* $\mathfrak{u}_1,\mathfrak{u}_2\in R$ *with* $\mathfrak{u}_1\neq\mathfrak{u}_2$;

(iii) $G(\mathfrak{u}_1,\mathfrak{u}_1,\mathfrak{u}_2) \leq s\, G(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)$, *for all* $\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3 \in R$ *with* $\mathfrak{u}_2 \neq \mathfrak{u}_3$*;*

(iv) $G(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3) = G(\mathfrak{u}_2,\mathfrak{u}_3,\mathfrak{u}_1) = \ldots$ *(symmetric in all three variables);*

(v) $G(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3) \leq s[G(\mathfrak{u}_1,\mathfrak{u},\mathfrak{u}) + G(\mathfrak{u},\mathfrak{u}_2,\mathfrak{u}_3)]$, *for all* $\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3,\mathfrak{u} \in R$.

Then the function G is called a generalized G_b-metric on R, and the pair (R,G) is a generalized G_b-metric space.

3. Main Results

This study aims to propose the idea of a new form of mapping called a $\beta-\sigma-$Geraghty type contraction mapping and demonstrate the fixed point findings for this type of mapping in a generalized G_b- metric space.

Firstly, we recall the following set of functions.

Definition 3.1. [6] *Let* $\mathscr{P}_s$ *represent the collection of all functions* $\beta : [0,\infty) \to [0,\frac{1}{s})$ *that satisfy the condition* $\lim_{n\to\infty} \beta(m_n) = \dfrac{1}{s}$ *implies* $\lim_{n\to\infty} m_n = 0$, *for some* $s \geq 1$.

The following set of functions and the new notion of $\beta-\sigma-$Geraghty type contraction mapping is defined as follows.

Definition 3.2. *Let* Σ^* *be the class of the functions* $\sigma : [0,\infty) \to [0,\infty)$ *be those that meet the following criteria.*

(1) σ *is increasing.*

(2) σ *is linear, that is,* $\sigma(\mathfrak{u}_2+\mathfrak{u}_3) = \sigma(\mathfrak{u}_2)+\sigma(\mathfrak{u}_3)$;

(3) σ *is continuous;*

(4) $\sigma(\mathfrak{u}) = 0 \Leftrightarrow \mathfrak{u} = 0$.

Definition 3.3. *Let* $\beta : R\times R\times R \to [0,+\infty)$ *represent a function and* $H : R \to R$ *represent a mapping. We say that* H *is Rectangular* β*-admissible mapping of type-I if* H *is* β*-admissible and*

$$\beta(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_2) \geq 1 \textit{ and } \beta(\mathfrak{u}_2,\mathfrak{u}_3,\mathfrak{u}_4) \geq 1 \;\Rightarrow\; \beta(\mathfrak{u}_1,\mathfrak{u}_3,\mathfrak{u}_4) \geq 1.$$

Definition 3.4. *Let* $\beta : R\times R\times R \to [0,+\infty)$ *represent a function and* $H : R \to R$ *represent a mapping. We say that* H *is Rectangular* β*-admissible mapping of type-II if* H *is* β*-admissible* *and*

$$\beta(u_1,u_1,u_2) \geq 1 \text{ and } \beta(u_2,u_3,u_4) \geq 1 \;\Rightarrow\; \beta(u_1,u_3,u_4) \geq 1.$$

Remark 3.1. *If* $u_1 = u_2$*, then Rectangular* β*-admissible mapping of type-I and Rectangular* β*-admissible mapping of type-II become identical.*

Example 3.1. *Let* $R = \{1,2,3,4\}$, $H : R \to R$ *such that* $H1 = 1$, $H2 = 3$, $H3 = 2$, $H4 = 4$ *and* $\beta : R \times R \times R \to [0,\infty)$,

$$\beta(u_1,u_2,u_3) = \begin{cases} 1 & \text{if } (u_1,u_2,u_3) \in \{(1,2,2),(2,3,4),(1,3,4),(1,3,3),(3,2,4),(1,2,4), \\ & (2,2,3),(1,2,3),(1,3,2),(3,3,2)\}, \\ 0 & \text{otherwise}. \end{cases}$$

Obviously, H is $\beta-$ *admissible and Rectangular* β*-admissible map of type-I.*
As $\beta(1,2,2) = 1$ *and* $\beta(2,3,4) = 1 \Rightarrow \beta(1,3,4) = 1$,
$\beta(1,2,2) = 1$ *and* $\beta(2,2,3) = 1 \Rightarrow \beta(1,2,3) = 1$,
$\beta(1,3,3) = 1$ *and* $\beta(3,2,4) = 1 \Rightarrow \beta(1,2,4) = 1$ *and*
$\beta(1,3,3) = 1$ *and* $\beta(3,3,2) = 1 \Rightarrow \beta(1,3,2) = 1$,
but H is not Rectangular β*-admissible of type-II because* $\beta(3,3,2) = 1$ *and* $\beta(2,3,4) = 1$ *but* $\beta(3,3,4) \neq 1$.

Example 3.2. *Let* $R = \{1,2,3,4\}$, $H : R \to R$ *such that* $H1 = 1$, $H2 = 3$, $H3 = 2$, $H4 = 4$ *and* $\beta : R \times R \times R \to [0,\infty)$,

$$\beta(u_1,u_2,u_3) = \begin{cases} 1 & \text{if } (u_1,u_2,u_3) \in \{(1,1,2),(2,3,4),(1,3,4),(1,1,3),(3,2,4),(1,2,4), \\ & (1,1,1),(3,1,1),(2,1,1),(3,3,2)\}, \\ 0 & \text{otherwise}. \end{cases}$$

Clearly, H is $\beta-$ *admissible and Rectangular* β*-admissible map of type-II.*
As $\beta(1,1,2) = 1$ *and* $\beta(2,3,4) = 1 \Rightarrow \beta(1,3,4) = 1$,
$\beta(1,1,2) = 1$ *and* $\beta(2,1,1) = 1 \Rightarrow \beta(1,1,1) = 1$,
$\beta(1,3,3) = 1$ *and* $\beta(3,2,4) = 1 \Rightarrow \beta(1,2,4) = 1$ *and*

$\beta(1,3,3)=1$ *and* $\beta(3,1,1)=1 \Rightarrow \beta(1,1,1)=1$,

but H is not Rectangular β*-admissible of type-I because* $\beta(3,1,1)=1$ *and* $\beta(1,1,2)=1$ *but* $\beta(3,1,2)\neq 1$.

Definition 3.5. *Let* $H:R\to R$ *be a mapping and* $\beta:R\times R\times R\to[0,+\infty)$ *be a function. We say that H is* β*-orbital admissible mapping of type-I if*

$$\beta(\mathfrak{u}_1,H\mathfrak{u}_1,H\mathfrak{u}_1)\geq 1 \;\Rightarrow\; \beta(H\mathfrak{u}_1,H^2\mathfrak{u}_1,H^2\mathfrak{u}_1)\geq 1.$$

Definition 3.6. *Let* $H:R\to R$ *be a mapping and* $\beta:R\times R\times R\to[0,+\infty)$ *be a function. We say that H is* β*-orbital admissible mapping of type-II if*

$$\beta(\mathfrak{u}_1,\mathfrak{u}_1,H\mathfrak{u}_1)\geq 1 \;\Rightarrow\; \beta(H\mathfrak{u}_1,H\mathfrak{u}_1,H^2\mathfrak{u}_1)\geq 1.$$

Remark 3.2. *Every* β*-admissible mapping is* β*-orbital admissible mapping of type-I and* β*-orbital admissible mapping of type-II also but the converse is not true.*

Example 3.3. *Let* $R=\{1,2,3\}$, $H:R\to R$ *such that* $H1=1$, $H2=3$, $H3=2$ *and* $\beta:R\times R\times R\to[0,\infty)$,

$$\beta(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)=\begin{cases}1 & \text{if } (\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)\in\{(1,1,1),(2,3,3),(3,2,2),(1,2,3)\},\\ 0 & \text{else}.\end{cases}$$

As $\beta(1,H1,H1)=1\Rightarrow\beta(H1,H^21,H^21)=1$,

$\beta(2,H2,H2)=1\Rightarrow\beta(H2,H^22,H^22)=1$ *and*

$\beta(3,H3,H3)=1\Rightarrow\beta(H3,H^23,H^23)=1$.

So, H is β*-orbital admissible mapping of type-I but H is not* β*-admissible map because* $\beta(1,2,3)=1$ *but* $\beta(H1,H2,H3)\neq 1$.

Example 3.4. *Let* $R=\{1,2,3\}$, $H:R\to R$ *such that* $H1=1$, $H2=3$, $H3=2$ *and* $\beta:R\times R\times R\to[0,\infty)$,

$$\beta(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)=\begin{cases}1 & \text{if } (\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)\in\{(2,2,3),(3,3,2),(2,3,3)\},\\ 0 & \text{else}.\end{cases}$$

As $\beta(2,2,H2) = 1 \Rightarrow \beta(H2,H2,H^2 2) = 1$ *and*

$\beta(3,3,H3) = 1 \Rightarrow \beta(H3,H3,H^2 3) = 1.$

So, H is β*-orbital admissible mapping of type-II but H is not* β*-admissible map because* $\beta(2,3,3) = 1$ *but* $\beta(H2,H3,H3) \neq 1.$

Definition 3.7. *Let* $H : R \to R$ *be a mapping and* $\beta : R\times R\times R \to [0,+\infty)$ *be a function. We say that H is Rectangular* β*-orbital admissible mapping of type-I if H is* β*-orbital admissible mapping of type-I and*

$$\beta(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_2) \geq 1 \text{ and } \beta(\mathfrak{u}_2,H\mathfrak{u}_2,H\mathfrak{u}_2) \geq 1 \;\Rightarrow\; \beta(\mathfrak{u}_1,H\mathfrak{u}_2,H\mathfrak{u}_2) \geq 1.$$

Definition 3.8. *Let* $H : R \to R$ *be a mapping and* $\beta : R\times R\times R \to [0,+\infty)$ *be a function. We say that H is Rectangular* β*-orbital admissible mapping of type-II if H is* β*-orbital admissible mapping of type-II and*

$$\beta(\mathfrak{u}_1,\mathfrak{u}_1,\mathfrak{u}_2) \geq 1 \text{ and } \beta(\mathfrak{u}_2,\mathfrak{u}_2,H\mathfrak{u}_2) \geq 1 \;\Rightarrow\; \beta(\mathfrak{u}_1,\mathfrak{u}_1,H\mathfrak{u}_2) \geq 1.$$

Remark 3.3. *Every Rectangular* β*-admissible mapping of type-I is Rectangular* β*-orbital admissible mapping of type-I and every Rectangular* β*-admissible mapping of type-II is Rectangular* β*-orbital admissible mapping of type-II but converse is not true.*

Example 3.5. *Let* $R = \{1,2,3,4\}$, $H : R \to R$ *such that* $H1 = 1$, $H2 = 3$, $H3 = 2$, $H4 = 4$ *and* $\beta : R\times R\times R \to [0,\infty)$,

$$\beta(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3) = \begin{cases} 1 & \text{if } (\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3) \in \{(2,2,2),(3,3,3),(1,2,2),(1,3,3),(2,3,3),(3,2,2), \\ & (2,4,4),(3,4,4)\}, \\ 0 & \text{else}. \end{cases}$$

Clearly, H is β*-orbital admissible mapping of type-I and rectangular* β*-orbital admissible mapping of type-I.*

As $\beta(1,2,2) = 1,\ \beta(2,H2,H2) = 1 \Rightarrow \beta(1,H2,H2) = 1,$

$\beta(1,3,3) = 1,\ \beta(3,H3,H3) = 1 \Rightarrow \beta(1,H3,H3) = 1,$

$\beta(2,3,3)=1$, $\beta(3,H3,H3)=1 \Rightarrow \beta(2,H3,H3)=1$ *and*
$\beta(3,2,2)=1$, $\beta(2,H2,H2)=1 \Rightarrow \beta(3,H2,H2)=1$, *but H is not rectangular β-admissible map of type-I because* $\beta(1,3,3)=1$, *and* $\beta(3,4,4)=1$, *but* $\beta(1,4,4)\neq 1$.

Example 3.6. *Let $R=\{1,2,3,4\}$, $H:R\to R$ such that $H1=1$, $H2=3$, $H3=2$, $H4=4$ and $\beta : R\times R\times R\to[0,\infty)$,*

$$\beta(u_1,u_2,u_3) = \begin{cases} 1 & \text{if } (u_1,u_2,u_3)\in\{(1,1,2),(1,1,3),(2,2,3),(3,3,2),(2,2,4),(3,3,4),(2,2,2),(3,3,3)\}, \\ 0 & \text{else}. \end{cases}$$

Clearly, H is β-orbital admissible mapping of type-II and rectangular β-orbital admissible mapping of type-II.
As $\beta(1,1,2)=1$, $\beta(2,2,H2)=1 \Rightarrow \beta(1,1,H2)=1$,
$\beta(1,1,3)=1$, $\beta(3,3,H3)=1 \Rightarrow \beta(1,1,H3)=1$,
$\beta(2,2,3)=1$, $\beta(3,3,H3)=1 \Rightarrow \beta(2,2,H3)=1$ *and*
$\beta(3,3,2)=1$, $\beta(2,2,H2)=1 \Rightarrow \beta(3,3,H2)=1$, *but H is not rectangular β-admissible map of type-II because* $\beta(1,1,2)=1$, *and* $\beta(2,2,4)=1$, *but* $\beta(1,2,4)\neq 1$.

Definition 3.9. *Let $H:R\to R$ be a defined mapping and (R,G) be generalized G_b-metric space. We say that H is β-σ-Geraghty type contraction mapping if there exist three functions $\sigma\in\Sigma^*$, $\gamma\in\mathscr{P}_s$ and $\beta : R\times R\times R\to[0,+\infty)$ such that*

$$\beta(u_1,u_2,u_3)\sigma(s^2G(Hu_1,Hu_2,Hu_3))\leq\gamma(\sigma(M(u_1,u_2,u_3)))\sigma(M(u_1,u_2,u_3)) \tag{3.1}$$

for all $u_1,u_2,u_3\in R$, where,

$$M(u_1,u_2,u_3)=\max\Big\{G(u_1,u_2,u_3),G(u_1,Hu_1,Hu_1),G(u_2,Hu_2,Hu_2),G(u_3,Hu_3,Hu_3), \\ \frac{G(u_1,Hu_2,Hu_3)+G(Hu_1,u_2,Hu_3)+G(Hu_1,Hu_2,u_3)}{6s}\Big\}.$$

To support our main conclusions, the following lemma is required:

Lemma 3.1. *Let $H : R \to R$ be a rectangular β- orbital admissible mapping of type-I. Let's assume that there exists $u_0 \in R$ such that $\beta(u_0, Hu_0, Hu_0) \geq 1$. For each $n \in \mathbb{N} \bigcup \{0\}$, define a sequence $\{u_n\}$ by $Hu_n = u_{n+1}$. Then we get $\beta(u_n, u_m, u_m) \geq 1$, $\forall m, n \in \mathbb{N}$ with $m > n$.*

Proof. Since there exist $u_0 \in R$ such that $\beta(u_0, Hu_0, Hu_0) \geq 1$ and by using the definition of rectangular β-orbital admissible mapping of type-I, we have

$$\beta(u_0, Hu_0, Hu_0) \geq 1$$

$$\beta(Hu_0, H^2u_0, H^2u_0) \geq 1$$

$$\beta(u_1, u_2, u_2) \geq 1.$$

If we keep doing this, we get

$$\beta(u_n, u_{n+1}, u_{n+1}) \geq 1 \tag{3.2}$$

On using (3.2), we have,

$$\beta(u_{n+1}, u_{n+2}, u_{n+2}) \geq 1$$

$$\beta(u_{n+1}, Hu_{n+1}, Hu_{n+1}) \geq 1 \tag{3.3}$$

Combining (3.2) and (3.3) we get, $\beta(u_n, Hu_{n+1}, Hu_{n+1}) \geq 1$, that is $\beta(u_n, u_{n+2}, u_{n+2}) \geq 1$. Continuing in this manner, we obtain $\beta(u_n, u_m, u_m) \geq 1$, $\forall m, n \in \mathbb{N}$ with $m > n$. □

Theorem 3.1. *Let (R, G) be a generalized G_b-complete metric space with constant $s \geq 1$ and let $H : R \to R$ be β-σ-Geraghty type contraction mapping satisfying the following conditions:*
(i)H is rectangular β-orbital admissible mapping of type-I;
(ii) there exists $u_0 \in R$ such that $\beta(u_0, Hu_0, Hu_0) \geq 1$;
(iii)H is continuous.
Then H has a fixed point, $u \in R$ with $G(u, u, u) = 0$.

Proof. Define the sequence $\{u_n\}$ in R by $u_{n+1} = Hu_n$, for all $n \in N$,
where $u_0 \in R$ such that $\beta(u_0, Hu_0, Hu_0) \geq 1$. Now, if $u_n = u_{n+1}$ for any $n \in N$, then u_n is a fixed point of H from the definition of $\{u_n\}$. Suppose that $u_n \neq u_{n+1}$ for each $n \in N$, without losing generality.

As, H is rectangular β-orbital admissible mapping of type-I, so by Lemma 3.1, we have $\beta(u_n, u_{n+1}, u_{n+1}) \geq 1$, for all $n \in \mathbb{N}$.
By, (3.1) we get

$$\begin{aligned}
\sigma(s^2 G(u_{n+1}, u_{n+2}, u_{n+2})) &\leq \beta(u_n, u_{n+1}, u_{n+1})\sigma(s^2 G(u_{n+1}, u_{n+2}, u_{n+2})) \\
&= \beta(u_n, u_{n+1}, u_{n+1})\sigma(s^2 G(Hu_n, Hu_{n+1}, Hu_{n+1})) \\
&\leq \gamma(\sigma(M(u_n, u_{n+1}, u_{n+1})))\sigma(M(u_n, u_{n+1}, u_{n+1})). \qquad (3.4)
\end{aligned}$$

Thus, we have

$$\sigma(s^2 G(u_{n+1}, u_{n+2}, u_{n+2})) < \frac{1}{s}\sigma(M(u_n, u_{n+1}, u_{n+1})), \qquad (3.5)$$

where,

$$\begin{aligned}
&M(u_n, u_{n+1}, u_{n+1}) \\
&\quad = max\Big\{G(u_n, u_{n+1}, u_{n+1}), G(u_n, u_{n+1}, u_{n+1}), G(u_{n+1}, u_{n+2}, u_{n+2}), G(u_{n+1}, u_{n+2}, u_{n+2}), \\
&\qquad\qquad \frac{G(u_n, u_{n+2}, u_{n+2}) + 2G(u_{n+1}, u_{n+1}, u_{n+2})}{6s}\Big\}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{G(u_n, u_{n+2}, u_{n+2}) + 2G(u_{n+1}, u_{n+1}, u_{n+2})}{6s} \\
&\leq \frac{s[G(u_n, u_{n+1}, u_{n+1}) + G(u_{n+1}, u_{n+2}, u_{n+2})]}{6s} \\
&\qquad + \frac{2s[G(u_{n+1}, u_{n+2}, u_{n+2}) + G(u_{n+2}, u_{n+1}, u_{n+2})]}{6s} \\
&= \frac{G(u_n, u_{n+1}, u_{n+1}) + 5G(u_{n+1}, u_{n+2}, u_{n+2})}{6} \\
&\leq max\{G(u_n, u_{n+1}, u_{n+1}), G(u_{n+1}, u_{n+2}, u_{n+2})\}.
\end{aligned}$$

So,

$$M(u_n, u_{n+1}, u_{n+1}) = max\{G(u_n, u_{n+1}, u_{n+1}), G(u_{n+1}, u_{n+2}, u_{n+2})\}.$$

If $M(u_n, u_{n+1}, u_{n+1}) = G(u_{n+1}, u_{n+2}, u_{n+2})$, then (3.5) becomes

$$\sigma(s^2 G(u_{n+1}, u_{n+2}, u_{n+2})) < \tfrac{1}{s}\sigma(G(u_{n+1}, u_{n+2}, u_{n+2})) \leq \sigma(G(u_{n+1}, u_{n+2}, u_{n+2})).$$

Since σ is increasing, so $s^2 G(u_{n+1}, u_{n+2}, u_{n+2}) < G(u_{n+1}, u_{n+2}, u_{n+2})$, which is a contradiction as $s \geq 1$.

Thus, $M(u_n, u_{n+1}, u_{n+1}) = G(u_n, u_{n+1}, u_{n+1})$. So, (3.5) reduces to

$$\sigma(s^2 G(u_{n+1}, u_{n+2}, u_{n+2})) < \frac{1}{s}\sigma(G(u_n, u_{n+1}, u_{n+1})) \leq \sigma(G(u_n, u_{n+1}, u_{n+1})).$$

By using increasing property of function σ, this equation implies

$$s^2 G(u_{n+1}, u_{n+2}, u_{n+2}) < G(u_n, u_{n+1}, u_{n+1})$$

$$G(u_{n+1}, u_{n+2}, u_{n+2}) < \tfrac{1}{s^2} G(u_n, u_{n+1}, u_{n+1}). \tag{3.6}$$

Case (i) If $s > 1$. Then $\frac{1}{s^2} < 1$, then the sequence $\{u_n\}$ is Cauchy and

$$\lim_{n,m\to\infty} G(u_n, u_m, u_m) = 0,$$

for $n, m \in \mathbb{N}$ and $m > n$.

Case (ii) If $s = 1$. From (3.6), we have $G(u_{n+1}, u_{n+2}, u_{n+2}) < G(u_n, u_{n+1}, u_{n+1})$ for all n which implies that the sequence $\{G(u_n, u_{n+1}, u_{n+1}\}$ is decreasing, so it converges to some $r \geq 0$, that is

$$\lim_{n\to\infty} G(u_n, u_{n+1}, u_{n+1}) = r. \tag{3.7}$$

Suppose $r > 0$. Then, for $s = 1$, the inequality (3.4) turns into

$$\sigma(G(u_{n+1}, u_{n+2}, u_{n+2})) \leq \gamma(\sigma(M(u_n, u_{n+1}, u_{n+1})))\sigma(M(u_n, u_{n+1}, u_{n+1})), \tag{3.8}$$

where $M(u_n, u_{n+1}, u_{n+1}) = G(u_n, u_{n+1}, u_{n+1})$ as evaluated above. Thus, (3.8) yields

$$\frac{\sigma(G(u_{n+1}, u_{n+2}, u_{n+2}))}{\sigma(G(u_n, u_{n+1}, u_{n+1}))} \leq \gamma(\sigma(G(u_n, u_{n+1}, u_{n+1}))) < 1. \tag{3.9}$$

Using continuity of σ and by taking limit as $n \to \infty$, (3.9) reduces to

$$\lim_{n\to\infty} \gamma(\sigma(G(u_n, u_{n+1}, u_{n+1}))) = 1.$$

The above equation suggests

$$\lim_{n\to\infty} \sigma(G(u_n, u_{n+1}, u_{n+1})) = 0 \text{ and so } \lim_{n\to\infty} G(u_n, u_{n+1}, u_{n+1}) = 0.$$

Consequently $r = 0$.

In the next steps, we will prove that $\{u_n\}$ is a Cauchy sequence. Suppose, on the contrary, that there exist $\varepsilon > 0$ and corresponding subsequences $\{n_k\}$ and $\{m_k\}$ of $\mathbb{N}$ satisfying $n_k > m_k > k$ for which

$$G(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k}) \geq \varepsilon, \tag{3.10}$$

where n_k, m_k are chosen as the smallest integers satisfying (3.10), that is

$$G(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k-1}, \mathfrak{u}_{n_k-1}) < \varepsilon. \tag{3.11}$$

By (3.10), (3.11) and the rectangular inequality, it is easily observed that

$$\varepsilon \leq G(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k}) \leq G(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k-1}, \mathfrak{u}_{n_k-1}) + G(\mathfrak{u}_{n_k-1}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k}) < \varepsilon + G(\mathfrak{u}_{n_k-1}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k}). \tag{3.12}$$

On using squeeze theorem in (3.12)

$$\lim_{k\to\infty} G(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k}) = \varepsilon. \tag{3.13}$$

Similarly, $\lim\limits_{k\to\infty} G(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k+1}, \mathfrak{u}_{n_k+1}) = \varepsilon$ and $\lim\limits_{k\to\infty} G(\mathfrak{u}_{m_k+1}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k+1}) = \varepsilon$.
Because H is $\beta - \sigma$-Geraghty type contraction mapping,

$$\begin{aligned} \sigma(G(\mathfrak{u}_{m_k+1}, \mathfrak{u}_{n_k+1}, \mathfrak{u}_{n_k+1}) &\leq \beta(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k})\sigma(G(H\mathfrak{u}_{m_k}, H\mathfrak{u}_{n_k}, H\mathfrak{u}_{n_k})) \\ &\leq \gamma(\sigma(M(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k})))\sigma(M(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k})), \end{aligned} \tag{3.14}$$

where,

$$\begin{aligned} M(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k}) = max\Big\{ & G(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k}), G(\mathfrak{u}_{m_k}, \mathfrak{u}_{m_k+1}, \mathfrak{u}_{m_k+1}), G(\mathfrak{u}_{n_k}, \mathfrak{u}_{n_k+1}, \mathfrak{u}_{n_k+1}), \\ & \frac{G(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k+1}, \mathfrak{u}_{n_k+1}) + 2G(\mathfrak{u}_{m_k+1}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k+1})}{6} \Big\}. \end{aligned} \tag{3.15}$$

By taking the limit as $k \to \infty$ in (3.14) and using (3.15), we have

$$\sigma(\varepsilon) \leq \lim_{k\to\infty} \gamma(\sigma(M(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k})))\sigma(\varepsilon). \tag{3.16}$$

Since γ is a Geraghty function, so $\sigma(M(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k})) \to 0$. Consequently, $G(\mathfrak{u}_{m_k}, \mathfrak{u}_{n_k}, \mathfrak{u}_{n_k}) \to 0$, which is a contradiction. Hence, it is concluded that $\lim\limits_{n,m\to\infty} G(\mathfrak{u}_m, \mathfrak{u}_n, \mathfrak{u}_n) = 0$, and the sequence

$\{\mathfrak{u}_n\}$ is Cauchy for any $s \geq 1$.

As (R, G) is a complete space so there exists $\mathfrak{u} \in R$ such that

$$\lim_{n\to\infty} G(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}) = \lim_{n,m\to\infty} G(\mathfrak{u}_n, \mathfrak{u}_m, \mathfrak{u}) = G(\mathfrak{u}, \mathfrak{u}, \mathfrak{u}) = 0.$$

Since H is continuous,

$$H\mathfrak{u} = H(\lim_{n\to\infty} \mathfrak{u}_n) = \lim_{n\to\infty} H\mathfrak{u}_n = \lim_{n\to\infty} \mathfrak{u}_{n+1} = \mathfrak{u},$$

and $\mathfrak{u}$ is a fixed point for H. □

By introducing the following condition, we can remove the condition of continuity of H.

(H) For every sequence $\{\mathfrak{u}_n\}$ in R such that $\beta(\mathfrak{u}_n, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}) \geq 1$ for all n and $\mathfrak{u}_n \to \mathfrak{u} \in R$ as $n \to \infty$, there exist a subsequence $\{\mathfrak{u}_{n_k}\}$ of $\{\mathfrak{u}_n\}$ such that $\beta(\mathfrak{u}_{n_k}, \mathfrak{u}, \mathfrak{u}) \geq 1$ for all k.

Theorem 3.2. *Let (R, G) be a generalized G_b-complete metric space with constant $s \geq 1$ and let $H : R \to R$ be β-σ-Geraghty type contraction mapping satisfying the following conditions:*

(i) H is rectangular β-orbital admissible mapping of type-I ;

(ii) there exists $\mathfrak{u}_0 \in R$ such that $\beta(\mathfrak{u}_0, H\mathfrak{u}_0, H\mathfrak{u}_0) \geq 1$;

(iii) Condition (H) is satisfied and G is continuous.

Then H has a fixed point, $\mathfrak{u} \in R$ with $G(\mathfrak{u}, \mathfrak{u}, \mathfrak{u}) = 0$.

Proof Using the proof of Theorem 3.2, it is clear that there exists $\mathfrak{u} \in R$ such that

$$\lim_{n\to\infty} G(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}) = \lim_{n,m\to\infty} G(\mathfrak{u}_n, \mathfrak{u}_m, \mathfrak{u}) = G(\mathfrak{u}, \mathfrak{u}, \mathfrak{u}) = 0.$$

Since $\beta(\mathfrak{u}_n, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}) \geq 1$ for all n. Due to the fact that $\lim_{n\to\infty} \mathfrak{u}_n = \mathfrak{u}$, there exists a subsequence $\{\mathfrak{u}_{n_k}\}$ of $\{\mathfrak{u}_n\}$ such that $\beta(\mathfrak{u}_{n_k}, \mathfrak{u}, \mathfrak{u}) \geq 1$ for all k. To prove that $\mathfrak{u}$ is a fixed point for H, suppose on the contrary that $G(\mathfrak{u}, H\mathfrak{u}, H\mathfrak{u}) > 0$.

As H is $\beta - \sigma$-Geraghty type contraction mapping,

$$\begin{aligned} \sigma(G(\mathfrak{u}_{n_k+1}, H\mathfrak{u}, H\mathfrak{u}) &\leq \beta(\mathfrak{u}_{n_k}, \mathfrak{u}, \mathfrak{u})\sigma(s^2 G(H\mathfrak{u}_{n_k}, H\mathfrak{u}, H\mathfrak{u}) \\ &\leq \gamma(\sigma(M(\mathfrak{u}_{n_k}, \mathfrak{u}, \mathfrak{u})))\sigma(M(\mathfrak{u}_{n_k}, \mathfrak{u}, \mathfrak{u})) \\ &< \tfrac{1}{s}\sigma(M(\mathfrak{u}_{n_k}, \mathfrak{u}, \mathfrak{u})), \end{aligned} \tag{3.17}$$

where,

$$M(\mathfrak{u}_{n_k},\mathfrak{u},\mathfrak{u})$$
$$= max\left\{G(\mathfrak{u}_{n_k},\mathfrak{u},\mathfrak{u}), G(\mathfrak{u}_{n_k},\mathfrak{u}_{n_k+1},\mathfrak{u}_{n_k+1}), G(\mathfrak{u},H\mathfrak{u},H\mathfrak{u}), \frac{G(\mathfrak{u}_{n_k},H\mathfrak{u},H\mathfrak{u})+2G(\mathfrak{u}_{n_k+1},H\mathfrak{u},\mathfrak{u})}{6s}\right\}.$$

Note that

$$\begin{aligned}
&\frac{G(\mathfrak{u}_{n_k},H\mathfrak{u},H\mathfrak{u})+2G(\mathfrak{u}_{n_k+1},H\mathfrak{u},\mathfrak{u})}{6s}\\
&\leq \frac{s[G(\mathfrak{u}_{n_k},\mathfrak{u},\mathfrak{u})+G(\mathfrak{u},H\mathfrak{u},H\mathfrak{u})]+2s[G(\mathfrak{u}_{n_k+1},H\mathfrak{u},H\mathfrak{u})+G(H\mathfrak{u},H\mathfrak{u},\mathfrak{u})]}{6s}\\
&= \frac{G(\mathfrak{u}_{n_k},\mathfrak{u},\mathfrak{u})+3G(H\mathfrak{u},H\mathfrak{u},\mathfrak{u})+2G(\mathfrak{u}_{n_k+1},H\mathfrak{u},H\mathfrak{u})}{6}.
\end{aligned}$$

Hence

$$\lim_{k\to\infty} M(\mathfrak{u}_{n_k},\mathfrak{u},\mathfrak{u}) \leq max\left\{0, G(\mathfrak{u},H\mathfrak{u},H\mathfrak{u}), \frac{5G(\mathfrak{u},H\mathfrak{u},H\mathfrak{u})}{6}\right\}$$

and by the definition of $M(\mathfrak{u}_{n_k},\mathfrak{u},\mathfrak{u})$, we have $\lim_{k\to\infty} M(\mathfrak{u}_{n_k},\mathfrak{u},\mathfrak{u}) = G(\mathfrak{u},H\mathfrak{u},H\mathfrak{u})$.

By the continuity of σ and G, taking the limit as $k\to\infty$ on both sides of (3.17) we have

$$\sigma(G(\mathfrak{u},H\mathfrak{u},H\mathfrak{u})) < \frac{1}{s}\sigma(G(\mathfrak{u},H\mathfrak{u},H\mathfrak{u})).$$

Thus $1 = \frac{\sigma(G(\mathfrak{u},H\mathfrak{u},H\mathfrak{u}))}{\sigma(G(\mathfrak{u},H\mathfrak{u},H\mathfrak{u}))} < \frac{1}{s}$, which is a contradiction. Hence $G(\mathfrak{u},H\mathfrak{u},H\mathfrak{u}) = 0$.

Therefore $H\mathfrak{u} = \mathfrak{u}$. □

To determine the uniqueness of a fixed point in β-σ-Geraghty type contraction mapping, we will use the following hypothesis.

(A) For all $\mathfrak{u}_1,\mathfrak{u}_2 \in$ Fix(H), either $\beta(\mathfrak{u}_1,\mathfrak{u}_1,\mathfrak{u}_2) \geq 1$ or $\beta(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_2) \geq 1$.

The collection of fixed points for H is shown here by Fix(H).

Theorem 3.3. *The uniqueness of the fixed point of H is obtained by adding condition (A) to the hypotheses of Theorem 3.2 (or Theorem 3.3).*

Proof. Assume that H has two fixed points, $\mathfrak{u}_1$ and $\mathfrak{u}_2$. Then $M(\mathfrak{u}_1,\mathfrak{u}_1,\mathfrak{u}_2) = G(\mathfrak{u}_1,\mathfrak{u}_1,\mathfrak{u}_2)$. So, we have

$$\begin{aligned}\sigma(G(u_1,u_1,u_2)) &\leq \sigma(s^2G(Hu_1,Hu_1,Hu_2))\\ &\leq \beta(u_1,u_1,u_2)\sigma(s^2G(Hu_1,Hu_1,Hu_2))\\ &\leq \gamma(\sigma(G(u_1,u_1,u_2)))\sigma(G(u_1,u_1,u_2))\\ &< \frac{1}{s}\sigma(G(u_1,u_1,u_2)),\end{aligned}$$

which is a contradiction. □

Example 3.7. *Let* $R=[0,+\infty)$ *with metric* $G(u_1,u_2,u_3)=|u_1-u_2|^2+|u_2-u_3|^2+|u_3-u_1|^2$ *for all* $u_1,u_2,u_3\in R$.

Define the mappings $H:R\to R$ and $\beta:R\times R\times R\to[0,+\infty)$ by

$$H(u_1)=\begin{cases}4u_1-\frac{15}{4}, & \text{if } u_1 \in (1,+\infty),\\ \frac{u_1}{4}, & \text{if } u_1 \in [0,1]\end{cases}$$

and

$$\beta(u_1,u_2,u_3)=\begin{cases}1, & \text{if } u_1,u_2,u_3 \in [0,1],\\ 0, & \text{otherwise.}\end{cases}$$

Clearly, H is continuous and rectangular β-orbital admissible mapping of type-I and $\beta(0,H0,H0)\geq 1$. Let $\sigma(q)=\frac{q}{4}$, $\gamma(q)=\frac{1}{4}$ then clearly $\sigma\in\Sigma^*$ and $\gamma\in\mathscr{P}_2$. Moreover, V satisfies (3.4) for the following reason:

if $u_1,u_2,u_3\in[0,1]$, then

$$\begin{aligned}\beta(u_1,u_2,u_3)\sigma(2^2G(Hu_1,Hu_2,Hu_3)) &= \frac{|u_1-u_2|^2+|u_2-u_3|^2+|u_1-u_3|^2}{16}\\ &= \gamma(\sigma(G(u_1,u_2,u_3)))\sigma(G(u_1,u_2,u_3)).\end{aligned}$$

Otherwise,

$$\beta(u_1,u_2,u_3)\sigma(2^2G(Hu_1,Hu_2,Hu_3))=0\leq\gamma(\sigma(G(u_1,u_2,u_3)))\sigma(G(u_1,u_2,u_3)).$$

Therefore, by Theorem 3.2, H has atleast one fixed point. In this example 0 and $\frac{5}{4}$ are fixed points of H.

4. CONSEQUENCES

Corollary 4.1. *Let (R,G) be a generalized G_b-complete metric space with constant $s \geq 1$, $\beta : R \times R \times R \to [0,\infty)$, $H : R \to R$ be a mapping, $\gamma \in \mathscr{P}_s$ and $\sigma \in \Sigma^*$ satisfying the following conditions:*

(i) $\beta(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)\sigma(s^2G(H\mathfrak{u}_1,H\mathfrak{u}_2,H\mathfrak{u}_3)) \leq \gamma(\sigma(G(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)))\sigma(G(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3))$;

(ii)H is rectangular β-orbital admissible mapping of type-I;

(iii) there exists $\mathfrak{u}_0 \in R$ such that $\beta(\mathfrak{u}_0,H\mathfrak{u}_0,H\mathfrak{u}_0) \geq 1$;

(iv)H is continuous.

Then H has a fixed point, $\mathfrak{u} \in R$ with $G(\mathfrak{u},\mathfrak{u},\mathfrak{u}) = 0$.

Proof Proof of this result follows the similar steps by considering $M(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3) = G(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)$ in Theorem 3.2. □

Corollary 4.2. *Let (R,G) be a generalized G_b-complete metric space with constant $s \geq 1$ and let $H : R \to R$ be a map and $\beta : R \times R \times R \to [0,\infty)$ satisfying the following conditions:*

(i)$\beta(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)G(H\mathfrak{u}_1,H\mathfrak{u}_2,H\mathfrak{u}_3) \leq \frac{1}{3s^3}M(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)$;

(ii) H is rectangular β-orbital admissible mapping of type-I;

(iii) there exists $\mathfrak{u}_0 \in R$ such that $\beta(\mathfrak{u}_0,H\mathfrak{u}_0,H\mathfrak{u}_0) \geq 1$;

(iv)H is continuous.

Then H has a fixed point, $\mathfrak{u} \in R$ with $G(\mathfrak{u},\mathfrak{u},\mathfrak{u}) = 0$.

Proof This can easily be proved by taking $\sigma(q) = q$, $\gamma(q) = \frac{1}{3s}$ in Theorem 3.2. □

Corollary 4.3. *Let (R,G) be a generalized G_b-complete metric space with constant $s \geq 1$, $H : R \to R$ be a mapping, $\gamma \in \mathscr{P}_s$ and $\sigma \in \Sigma^*$ such that*

$$\sigma(s^2G(H\mathfrak{u}_1,H\mathfrak{u}_2,H\mathfrak{u}_3)) \leq \gamma(\sigma(G(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)))\sigma(G(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)),$$

for all $\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3 \in R$, where,

$M(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3) = max\Big\{G(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3), G(\mathfrak{u}_1,H\mathfrak{u}_1,H\mathfrak{u}_1), G(\mathfrak{u}_2,H\mathfrak{u}_2,H\mathfrak{u}_2), G(\mathfrak{u}_3,H\mathfrak{u}_3,H\mathfrak{u}_3),$
$\frac{G(\mathfrak{u}_1,H\mathfrak{u}_2,H\mathfrak{u}_3)+G(H\mathfrak{u}_1,\mathfrak{u}_2,H\mathfrak{u}_3)+G(H\mathfrak{u}_1,H\mathfrak{u}_2,\mathfrak{u}_3)}{6s}\Big\}.$

Then H has a unique fixed point, $\mathfrak{u} \in R$ with $G(\mathfrak{u},\mathfrak{u},\mathfrak{u}) = 0$.

Proof Consider $\beta(u_1, u_2, u_3) = 1$ in Theorem 3.2 and apply Theorem 3.4. $\square$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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