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G-METRIC SPACES AND THE RELATED APPROXIMATE FIXED POINT RESULTS

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Abstract: In this paper, we investigate the existence and diameter of the approximate fixed point results on G-metric spaces (not necessarily complete) by using various contraction mappings, including $G-B$ contraction, G-Bianchini contraction, and so on. Additionally, we prove the same approximate fixed point results for rational type contraction mappings, which were discussed mainly in [11] and [16], in the setting of G-metric space. Also, a few examples are provided to demonstrate our findings. Finally, we discuss some applications of approximate fixed point results in the field of applied mathematics rigorously.

Keywords: G-metric space; $G-B$ contraction; G-Bianchini contraction; rational contraction; approximate fixed point.

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1. Introduction

Because of its wide range of applications in various fields of mathematics, including differential geometry, numerical analysis, fluid dynamics, and approximation theory, fixed point theory (FPT) serves as one of the most important roles in nonlinear analysis. Also, FPT has been researched in various metric spaces over the past 200 years by several researchers. Originally, the notion of FPT was developed in the early 1900's. The father of FPT, mathematician Brouwer

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[8], proposed FP results for continuous mappings on finite dimensional spaces. In 1922, Banach [2] established and confirmed the renowned Banach contraction principle (*BCP*). Several authors used the *BCP* in numerous ways and presented numerous FP results (see, [7], [9], [10], [12], [20], [21], [22], [41], [45]). In particular, M. Marudai and V. Bright [26], also pointed out many fixed point results by using B-contraction operator on metric spaces. Moreover, an article [25] [W. A. Kirk, P. S. Srinivasan, P. Veeramani, *Fixed point for mappings satisfying cyclical contraction conditions*, Fixed point theory, 4, 2003, 79-89], long-windedly explains about the notion of cyclic mappings and its theorems. In this regard, several generalized metric spaces have been produced over the decades by various researchers. Particularly, G-metric space, which is the most generalized of all extended metric spaces. Mustafa and Sims [38], initially formulated the concept of G-metric spaces and proved many FP results in complete G-metric spaces for contraction mappings, expansive mappings, and so on. For more details, one may refer to ([1], [17], [18], [19], [27], [34], [36], [39]). Later, Obiedat and Mustafa (refer to [35], [37], [40]), studied FP results for Reich-type contraction mapping on G-metric spaces as well as non-symmetric metric spaces.

Definition 1.1. [19][38] *Let L be a nonempty set and the function $d_G : L\times L\times L \to [0,\infty)$ satisfy the following axioms:*
$(G_1) G(q,r,s) = 0$ *if* $q = r = s$ *whenever* $q,r,s \in L$;
$(G_2) G(q,r,s) > 0$ *whenever* $q,r \in L$ *with* $q \neq r$;
$(G_3) G(q,q,r) \leq G(q,r,s)$ *whenever* $q,r,s \in L$ *with* $r \neq s$;
$(G_4) G(q,r,s) = G(q,s,r) = G(r,s,q) = ...,$ *(symmetry in all three variables);*
$(G_5) G(q,r,s) \leq [G(q,t,t) + G(t,r,s)]$, *for every* $q,r,s,t \in L$.
Then (L,d_G) is called a G-metric space.

Proposition 1.2. [19][38] *Let (L,d_G) be a G-metric space, then for any $q,r,s \in L$ such that $G(q,r,s) = 0$, we have that $q = r = s$.*

Definition 1.3. [19][38] *Let (L,d_G) be a G-metric space. A sequence $\{q_n\}$ is said to be a G-cauchy sequence if for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $G(q_n,q_m,q_l) < \varepsilon$, for every $n,m,l \geq N$, that is $G(q_n,q_m,q_l) \to 0$ as $n,m,l \to +\infty$.*

Proposition 1.4. [19][38] *In a G-metric space (L, d_G), the following are equivalent.*

(i) *The sequence $\{q_n\} \subseteq W$ is a G-cauchy.*

(ii) *For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(q_n, q_m, q_m) < \varepsilon$, for all $n, m \geq N$.*

Proposition 1.5. [19][38] *Every G-metric (L, G) defines a G-metric space (L, d_G), by*

(i) $d_G(q, r) = G(q, r, r) + G(r, q, q)$.

if (L, G) is symmetric G-metric space, then

(ii) $d_G(q, r) = 2G(q, r, s)$

On the other hand, the first researchers to investigate a generalization of the BCP while simultaneously using a contraction condition of the rational type were Dass and Gupta [11]. Later, Jaggi [16], used a contraction condition of the rational type to prove a FP reults in complete metric spaces. Moreover, rational contraction conditions have been heavily employed in both the FP and common FP locations. Successively, many researchers carried out many rational contraction mappings in numerous spaces. Among these, notable early generalizations and extensions of the rational type contraction can be found in ([15], [22], [24], [43]).

Let us consider a selfmap $W : L \to L$. A FP is a point (say, q_0) which is equal to Wq_0. That is, $d(Wq_0, q_0) = 0$. Assume that a mapping has a FP, q_0. In which case the point (q_0, q_0) is located on its diagram. Naturally, the conditions for FP existence are very strict. As a result, there is no assurance that fixed points will always exists. In the absence of exact FP, approximate FP may be used because the FP methods have overly strict limitations. This is the primary reason for attempting to locate approximate $FP's$ $(AFP's)$ on metric spaces. Usually, an AFP is also referred to as $\varepsilon - FP$. One can see, the point Wq_0 is "very near" to the point q_0. An AFP is a point that is nearly located at its respective FP. Here, the distance is less than ε, i.e., $d(Wq_0, q_0) < \varepsilon$. Initially, In 2003, Tijs et al. [44] proved the existence of FP results turns out to be still guaranteed under various weakened versions of the well-known FP theorems of Brouwer, Kakutani, and Banach (refer, Theorems 2.1 and 2.2). Moreover, he proved AFP results for contraction maps and nonexpansive maps in Theorems 3.1 and 4.1, respectively. After that, Berinde [3] proved AFP results (qualitative theorems) by using various operators (Kannan, Chatterjea, Zamfirescu, and weak contractions) on metric spaces (not necessarily complete). Further, he found the diameter of the AFP results (quantitative theorems) by using two main lemmas (see also [4], [5]).

Subsequently, Dey and Saha [13] extended these results, and they have shown that the diameter of the AFP for the Reich operator tends to zero when ε approaches zero. In the same manner, S. A. M. Mohsenialhosseini [29] derived some new AFP results for cyclical contraction mappings. Also, he extended these results to a family of contraction mappings and found a common FP for the Mohseni-Saheli contraction mapping (refer to [30], [31]). In addition to that, Mohsenialhosseini and Ahmadi [32] studied some AFP theorems by using various operators on G-metric spaces. Later, Mohsenialhosseini [33] independently showed the same AFP results for another contraction operator on G-metric spaces. Furthermore, the authors in [28] went one step farther and demonstrated the approximate best proximity point outcomes on metric spaces. Even though FP theory has more than 200-years history, recently because of its applications of applied mathematics, AFP theory have received much attention. For example, in 2021, K. Tijani and S. Olayemi [43] proposed some AFP results using rational-type contraction mapping on metric spaces. Inspired by that, we have converted these results into some innovative AFP results on G-metric spaces. On the other hand , which is also recent, the authors R. Theivaraman et al. [42] have proposed many AFP results using various contraction mappings such as the B-contraction, the Bianchini contraction, convex contraction, rational type contraction mappings, and their related consequences. Also, they have pointed out the applications of AFP results in the field of applied mathematics.

The article is organized as follows: Section 1 is introductory. In Section 2, we present the preliminary notions, some notations, essential definitions, and needed lemmas from the previous work, which are used throughout the paper. In Section 3, we present two types of AFP results. Firstly, we prove AFP results for contraction mappings such as the $G-B$ contraction [Theorem 3.2], the G-Bianchini contraction [Theorem 3.3], and their subsequents. Secondly, we prove AFP results for various rational type contraction mappings (see, Theorems 3.6, 3.7, 3.8, 3.9, and 3.10). In Section 4, we present some applications in the field of applied mathematics that support the main findings of this paper. Finally, in Section 5, we present some conclusions.

2. PRELIMINARIES

In this section, some notations and basic notions, such as definitions and lemmas, from earlier research are recalled. These are then employed throughout the remainder of the main findings of this manuscript.

Definition 2.1. [25] *Let W_1 and W_2 be two nonempty subsets of a G-metric space (L, d_G). A mapping $W : W_1 \cup W_2 \to W_1 \cup W_2$ is said to be a cyclic mapping if $W(W_1) \subseteq W_2$ and $W(W_2) \subseteq W_1$.*

Definition 2.2. [32][33] *Let W_1, W_2, W_3 are three nonempty closed subsets of a G-metric space (L, d_G) and $W : W_1 \cup W_2 \cup W_3 \to W_1 \cup W_2 \cup W_3$ be a cyclic map. Let $\varepsilon > 0$ and $q \in W_1 \cup W_2 \cup W_3$. Then q is an $\varepsilon - FP$ of W if*

$$[G(q, Wq, Wq) + G(Wq, q, q)] < \varepsilon.$$

Remark 2.3. [32][33] *In this paper we will denote the set of all $\varepsilon - FP$ of W, for a given ε, by:*

$$F_{G\varepsilon}(W) = \{q \in W_1 \cup W_2 \cup W_3 \big| [G(q, Wq, Wq) + G(Wq, q, q)] < \varepsilon\}.$$

Definition 2.4. [32][33] *Let W_1, W_2, W_3 are three nonempty closed subsets of a G-metric space (L, d_G) and $W : W_1 \cup W_2 \cup W_3 \to W_1 \cup W_2 \cup W_3$ be a cyclic map. Then W has an AFP property $(AFPP)$ if for every $\varepsilon > 0$,*

$$F_{G\varepsilon}(W) \neq 0.$$

Lemma 2.5. [32][33] *Let W_1, W_2, W_3 are three nonempty closed subsets of a G-metric space (L, d_G) and $W : W_1 \cup W_2 \cup W_3 \subseteq L \to W_1 \cup W_2 \cup W_3 \subseteq L$ be a cyclic map. For every $\varepsilon > 0$, the followings hold:*

(i) $F_{G\varepsilon}(W) \neq 0$; and

(ii) for every $\theta > 0$, there exists $\phi(\theta) > 0$ such that

$$[G(q, r, r) + G(r, q, q)] - [G(W_q, W_r, W_r) + G(W_r, W_q, W_q)] < \theta$$

implies that $G(q, r, r) + G(r, q, q) \leq \phi(\theta)$, for all $q, r \in F_{G\varepsilon}(W)$. Then;

$$Diam(F_{G\varepsilon}(W)) \leq \phi(2\varepsilon).$$

Definition 2.6. *Let (L, d_G) be a G-metric space and W_1, W_2, W_3 are three nonempty closed subsets of L. A cyclic mapping $W : W_1 \cup W_2 \cup W_3 \subseteq L \to W_1 \cup W_2 \cup W_3 \subseteq L$ is said to be a $G-B$ contraction mapping if there exists $g_1, g_2, g_3 \in (0,1)$ with $2g_1 + g_2 + 2g_3 < 1$ and for every $q, r \in W_1 \cup W_2 \cup W_3 \subseteq L$ such that*

$$\begin{aligned} G(Wq, Wr, Wr) + G(Wr, Wq, Wq) \leq g_1[&G(q, Wq, Wq) + G(Wq, q, q) \\ &+ G(r, Wr, Wr) + G(Wr, r, r)] \\ &+ g_2[G(q, r, r) + G(r, q, q)] \\ &+ g_3[G(q, Wr, Wr) + G(Wr, q, q) \\ &+ G(r, Wq, Wq) + G(Wq, r, r)] \end{aligned} \tag{2.1}$$

Definition 2.7. *Let (L, d_G) be a G-metric space and W_1, W_2, W_3 are three nonempty closed subsets of L. A cyclic mapping $W : W_1 \cup W_2 \cup W_3 \subseteq L \to W_1 \cup W_2 \cup W_3 \subseteq L$ is said to be a G-Hardy and Rogers contraction mapping if there exists $g_1, g_2, g_3, g_4, g_5 \in (0,1)$ with $g_1 + g_2 + g_3 + g_4 + g_5 < 1$ and for every $q, r \in W_1 \cup W_2 \cup W_3 \subseteq L$ such that*

$$\begin{aligned} G(Wq, Wr, Wr) + G(Wr, Wq, Wq) \leq g_1[&G(q, r, r) + G(r, q, q)] \\ &+ g_2[G(q, Wq, Wq) + G(Wq, q, q)] \\ &+ g_3[G(r, Wr, Wr) + G(Wr, r, r)] \\ &+ g_4[G(q, Wr, Wr) + G(Wr, q, q)] \\ &+ g_5[G(r, Wq, Wq) + G(Wq, r, r)] \end{aligned} \tag{2.2}$$

Definition 2.8. *Let (L, d_G) be a G-metric space and W_1, W_2, W_3 are three non-empty closed subsets of L. A cyclic mapping $W : W_1 \cup W_2 \cup W_3 \subseteq L \to W_1 \cup W_2 \cup W_3 \subseteq L$ is said to be a G-Bianchini contraction operator if there exists $g \in (0,1)$ and for every $p, q \in W_1 \cup W_2 \cup W_3 \subseteq L$ such that*

$$G(Wp, Wq, Wq) + G(Wq, Wp, Wp) \leq gB[G(p, q, q) + G(q, p, p)]$$

where,

$$B[G(p, q, q) + G(q, p, p)] = max\{G(p, Wp, Wp) + G(Wp, p, p), G(q, Wq, Wq) + G(Wq, q, q)\}.$$

Definition 2.9. [43] *A mapping $\varnothing : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if it satisfies the conditions:*
(i) $\varnothing$ is monotone increasing, and
(ii) $\varnothing^n(p)$ converges to 0 as $n \to \infty$, for all $p \in \mathbb{R}_+$.

3. Main Results

In this section, firstly, we demonstrate the AFP results for various contraction mappings such as the $G-B$ contraction mapping, the G-Bianchini contraction mapping, and their related consequences on G-metric spaces.

Theorem 3.1. *Let (L, d_G) be a G-metric space and W_1, W_2 and W_3 are three nonempty closed subsets of L. A cyclic mapping $W : W_1 \cup W_2 \cup W_3 \subseteq L \to W_1 \cup W_2 \cup W_3 \subseteq L$ is a contraction. Then W has an $\varepsilon - FP$.*

Proof. Let $q \in W_1 \cup W_2 \cup W_3 \subseteq L$. Then, a sequence $\{q_n\}$ is defined by

$$q_{n+1} = Wq_n, \text{ for all } n \geq 0.$$

That is, $\{q_n\}$ is a Cauchy sequence. Thus, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ implies that

$$G(q_n, q_m, q_m) + G(q_m, q_n, q_n) < \varepsilon.$$

In particular, if $n \geq n_0$,

$$G(q_n, q_{n+1}, q_{n+1}) + G(q_{n+1}, q_n, q_n) < \varepsilon.$$

That is,

$$G(q_n, Wq_n, Wq_n) + G(Wq_n, q_n, q_n) < \varepsilon.$$

Therefore,

$$q_n \in F_{G\varepsilon}(W) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

Hence, W has an $\varepsilon - FP$. $\square$

Theorem 3.2. *Let a cyclic mapping $W : W_1 \cup W_2 \cup W_3 \subseteq L \to W_1 \cup W_2 \cup W_3 \subseteq L$ be a $G-B$ contraction mapping. Then W has an $\varepsilon - FP$ and*

$$Diam(F_{G\varepsilon}(W)) \leq \frac{2\varepsilon(g_1+g_3+1)}{1-g_2-2g_3}, \textit{ for all } \varepsilon > 0.$$

Proof. Given that W is a $G-B$ contraction operator on a G-metric space (L, d_G). A sequence $\{q_n\}$ is defined by $q_{n+1} = Wq_n$, for all $n \geq 0$. Consider,

$$\begin{aligned}
& G(W^nq, W^{n+1}q, W^{n+1}q) + G(W^{n+1}q, W^nq, W^nq) \\
&\quad = G(W(W^{n-1}q), W(W^nq), W(W^nq)) + G(W(W^nq), W(W^{n-1}q), W(W^{n-1}q)) \\
&\quad \leq g_1[G(W^{n-1}q, W^nq, W^nq) + G(W^nq, W^{n-1}q, W^{n-1}q) \\
&\qquad + G(W^nq, W^{n+1}q, W^{n+1}q) + G(W^{n+1}q, W^nq, W^nq)] \\
&\qquad + g_2[G(W^{n-1}q, W^nq, W^nq) + G(W^nq, W^{n-1}q, W^{n-1}q)] \\
&\qquad + g_3[G(W^{n-1}q, W^{n+1}q, W^{n+1}q) + G(G^{n+1}q, W^{n-1}q, W^{n-1}q) \\
&\qquad + G(W^nq, W^nq, W^nq) + G(W^nq, W^nq, W^nq)] \\
&\quad = g_1[G(W^{n-1}q, W^nq, W^nq) + G(W^nq, W^{n-1}q, W^{n-1}q)] \\
&\qquad + g_1[G(W^nq, W^{n+1}q, W^{n+1}q) + G(W^{n+1}q, W^nq, W^nq)] \\
&\qquad + g_2[G(W^{n-1}q, W^nq, W^nq) + G(W^nq, W^{n-1}q, W^{n-1}q)] \\
&\qquad + g_3[G(W^{n-1}q, W^nq, W^nq) + G(W^nq, W^{n-1}q, W^{n-1}w)] \\
&\qquad + g_3[G(W^nq, W^{n+1}q, W^{n+1}q) + G(W^{n+1}q, W^nq, W^nq)] \\
&\quad = \left(\frac{g_1+g_2+g_3}{1-g_1-g_3}\right)[G(W^{n-1}q, W^nq, W^nq) + G(W^nq, W^{n-1}q, W^{n-1}q)] \\
&\quad = \lambda[G(W^{n-1}q, W^nq, W^nq) + G(W^nq, W^{n-1}q, W^{n-1}q)], \text{ where } \lambda = \frac{g_1+g_2+g_3}{1-g_1-g_3} \\
&\quad = \lambda[G(W(W^{n-2}q), W(W^{n-1}q), W(W^{n-1}q)) + G(W(W^{n-1}q), W(W^{n-2}q), W(W^{n-2}q))] \\
&\quad \leq \lambda^2[G(W^{n-2}q, W^{n-1}q, W^{n-1}q) + G(W^{n-1}q, W^{n-2}q, W^{n-2}q)] \\
&\qquad \cdots \\
&\quad \leq \lambda^n[G(q, Wq, Wq) + G(Wq, q, q)]
\end{aligned}$$

Since $G(W^nq, W^{n+1}q, W^{n+1}q) + G(W^{n+1}q, W^nq, W^nq) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $q \in W_1 \cup W_2 \cup W_3 \subseteq L$. That is, $\{q_n\}$ is a Cauchy sequence. Then, by Theorem 3.1, for every $\varepsilon > 0, F_{G\varepsilon}(L) \neq \emptyset$. Hence, W has an $\varepsilon - FP$. Clearly, we proved condition (i) of Lemma 2.5. For diameter, use condition (ii) of Lemma 2.5. For that, take $\theta > 0$ and $q, r \in F_{G\varepsilon}(W)$. That is,

$$G(q, Wq, Wq) + G(Wq, q, q) < \varepsilon$$

and

$$G(r, Wr, Wr) + G(Wr, r, r) < \varepsilon$$

Also, assume that

$$[G(q,r,r) + G(r,q,q)] - [G(Wq, Wr, Wr) + G(Wr, Wq, Wq)] < \theta.$$

To show $\phi(\varepsilon) > 0$. Consider,

$$\begin{aligned} G(q,r,r) + G(r,q,q) &< [G(Wq, Wr, Wr) + G(Wr, Wq, Wq)] + \theta \\ &\leq \left(\frac{2g_1\varepsilon + 2g_3\varepsilon + \theta}{1 - g_2 - g_3} \right) \\ &= \gamma \end{aligned}$$

So, for every $\theta > 0$, there exists $\phi(\theta) = \gamma > 0$, such that

$$[G(q,r,r) + G(r,q,q)] - [G(Wq, Wr, Wr) + G(Wr, Wq, Wq)] < \theta$$

Implies that

$$G(q,r,r) + G(r,q,q) \leq \phi(\theta)$$

Therefore, by Lemma 2.5, we get

$$Diam(F_{G\varepsilon}(W)) \leq \phi(2\varepsilon), \text{ for all } \varepsilon > 0.$$

Hence,

$$Diam(F_{G\varepsilon}(W)) \leq \frac{2\varepsilon(g_1 + g_3 + 1)}{1 - g_2 - 2g_3}, \text{ for all } \varepsilon > 0.$$

$\square$

Theorem 3.3. *Let a cyclic mapping $W : W_1 \cup W_2 \cup W_3 \subseteq L \to W_1 \cup W_2 \cup W_3 \subseteq L$ be a G-Bianchini operator. Then W has an $\varepsilon - FP$ and*

$$Diam(F_{G\varepsilon}(W)) \leq \varepsilon(g+2), \textit{ for all } \varepsilon > 0.$$

Proof. Given that W is a G-Bianchini contraction operator on a G-metric space (W, d_G) and a sequence $\{q_n\}$ is defined by $q_{n+1} = Wq_n,$ for all $n \geq 0$. Consider,

Case 1. *Suppose that,*

$$B(G(q,r,r)+G(r,q,q)) = G(q,Wq,Wq)+G(Wq,q,q)$$

Then, the Definition 2.8 *becomes:*

$$G(Wq,Wr,Wr)+G(Wr,Wq,Wq) \leq g[G(q,Wq,Wq)+G(Wq,q,q)]$$

Substituting $r = Wq$ implies

$$G(Wq,W^2q,W^2q)+G(W^2q,Wq,Wq) \leq g[G(q,Wq,Wq)+G(Wq,q,q)]$$

Again substituting $q = Wq$ implies

$$\begin{aligned} G(W^2q,W^3q,W^3q)+G(W^3q,W^2q,W^2q) &\leq g[G(Wq,W^2q,W^2q)+G(W^2q,Wq,Wq)] \\ &\leq g^2[G(q,Wq,Wq)+G(Wq,q,q)] \\ &\cdots \\ G(W^nq,W^{n+1}q,W^{n+1}q)+G(W^{n+1}q,W^nq,W^nq) &\leq g^n[G(q,Wq,Wq)+G(Wq,q,q)] \end{aligned}$$

Case 2. *Suppose that,*

$$B(G(q,r,r)+G(r,q,q)) = G(r,Wr,Wr)+G(Tr,q,q)$$

Then, the Definition 2.8*, becomes:*

$$G(Wq,Wr,Wr)+G(Wr,Wq,Wq) \leq g[G(r,Wr,Wr)+G(Wr,r,r)]$$

Substituting $r = Wq$ implies

$$G(Wq,W^2q,W^2q)+G(W^2q,Wq,Wq) \leq g[G(Wq,W^2q,W^2q)+G(W^2q,Wq,Wq)]$$

Which is impossible because $g \in (0,1)$. Therefore, **Case 2** does not exists. Then, by **Case 1**, $G(W^n q, W^{n+1} q, W^{n+1} q) + G(W^{n+1} q, W^n q, W^n q) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $q \in W_1 \cup W_2 \cup W_3 \subseteq L$. That is, $\{q_n\}$ is a Cauchy sequence. Again, by Theorem 3.1, for every $\varepsilon > 0, F_{G\varepsilon}(W) \neq \emptyset$. Hence, W has an $\varepsilon - FP$. Here, as in the previous Theorem 3.2, we have

$$\begin{aligned} G(q,r,r) + G(r,q,q) &\leq g[G(Wq,Wr,Wr) + G(Wr,Wq,Wq)] + \theta \\ &\leq g\varepsilon + \theta \\ &= \gamma \end{aligned}$$

So, for every $\theta > 0$ there exists $\phi(\theta) = \gamma > 0$ such that

$$[G(q,r,r) + G(r,q,q)] - [G(Wq,Wr,Wr) + G(Wr,Wq,Wq)] \leq \theta$$

Which implies that

$$G(q,r,r) + G(r,q,q) \leq \phi(\theta).$$

By Lemma 2.5,

$$Diam(F_{G\varepsilon}(W)) \leq \phi(2\varepsilon), \text{ for all } \varepsilon > 0.$$

Hence,

$$Diam(F_{G\varepsilon}(W)) \leq \varepsilon(g+2), \text{ for all } \varepsilon > 0.$$

□

Corollary 3.4. *Let (L, d_G) be a G-metric space and W_1, W_2, W_3 be three nonempty closed subsets of L. A cyclic mapping $W : W_1 \cup W_2 \cup W_3 \subseteq L \to W_1 \cup W_2 \cup W_3 \subseteq L$ is defined on a G-metric space (L, d_G) and $g \in (0,1)$ such that*

$$\begin{aligned} &G(Wq,Wr,Wr) + G(Wr,Wq,Wq) \\ &\qquad \leq g[G(q,Wq,Wq) + G(Wq,q,q)], \textit{ for all } q,r \in W_1 \cup W_2 \cup W_3 \subseteq L. \end{aligned}$$

Then W has an $\varepsilon - FP$ and $Diam(F_{G\varepsilon}(W)) \leq \varepsilon(g+2)$, for all $\varepsilon > 0$.

Proof. Substituting $B[G(q,r,r) + G(r,q,q)] = G(q,Wq,Wq) + G(Wq,q,q)$ in Theorem 3.3 completes this corollary. □

Remark 3.5. *(1) In Definition* 2.6*, subtitute* $g_2 = g_3 = g_4 = g_5 = 0$*, then* W *becomes G-contraction operator.*

(2) In Definition 2.6*, subtitute* $g_2 = g_3 = 0$*, then* W *becomes G-Kannan operator.*

(3) In Definition 2.6*, subtitute* $g_1 = g_2 = 0$*, then* W *becomes G-Chatterjea operator.*

(4) In Definition 2.7*, subtitute* $g_2 = g_3 = 0$*, then* W *becomes G-Reich operator.*

(5) In Definition 2.7*, subtitute* $g_4 = g_5$*, then* W *becomes G-Ćirić operator.*

Secondly, we demonstrate some $\varepsilon - FP$ results for various rational-type contraction mappings on G-metric spaces. These contractions were discussed mainly in [11] and [16].

Theorem 3.6. *Let* (L, d_G) *be a G-metric space and* W_1, W_2 *and* W_3 *are three non-empty subsets of L. Let* $W : W_1 \cup W_2 \cup W_3 \subseteq L \to W_1 \cup W_2 \cup W_3 \subseteq L$ *be a cyclic mapping. Then there exists* $g_1, g_2 \in (0,1)$ *with* $g_1 + g_2 < 1$ *and* $[G(q,Wr,Wr) + G(Wr,q,q)] + [G(r,Wq,Wq) + G(Wq,r,r)] \neq 0$ *such that*

$$\begin{aligned} &G(Wq,Wr,Wr) + G(Wr,Wq,Wq) \leq \\ &g[[G(q,Wq,Wq) + G(Wq,q,q)][G(q,Wr,Wr) + G(Wr,q,q)] \\ &\frac{+[G(r,Wr,Wr) + G(Wr,r,r)][G(r,Wq,Wq) + G(Wq,r,r)]]}{[G(q,Wr,Wr) + G(Wr,q,q)] + [G(r,Wq,Wq) + G(Wq,r,r)]}, \text{ for all } q,r \in L. \end{aligned} \tag{3.1}$$

has an $\varepsilon - FP$ *and*

$$Diam(F_{G\varepsilon}(W)) \leq \frac{\varepsilon^2(g^2 + 6g + 9) + \varepsilon(g+1)}{2}, \text{ for all } \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$ and $q \in W_1 \cup W_2 \cup W_3$. Define a sequence $\{q_n\}$ such that $q_{n+1} = Wq_n$, for all $n \geq 0$. Consider,

$$\begin{aligned} &G(W^n q, W^{n+1} q, W^{n+1} q) + G(W^{n+1} q, W^n q, W^n q) \\ &\qquad = G(W(W^{n-1} q), W(W^n q), W(W^n q)) + G(W(W^n q), W(W^{n-1} q), W(W^{n-1} q)) \end{aligned}$$

$$\leq \frac{g[[G(W^{n-1}q,W^nq,W^nq)+G(W^nq,W^{n-1}q,W^{n-1}q)][G(W^{n-1}q,W^{n+1}q,W^{n+1}q) + G(W^{n+1}q,W^{n-1}q,W^{n-1}q)]]}{[G(W^{n-1}q,W^{n+1}q,W^{n+1}q)+G(W^{n+1}q,W^{n-1}q,W^{n-1}q)]}$$

$$\leq g[G(W^{n-1}q,W^nq,W^nq)+G(W^nq,W^{n-1}q,W^{n-1}q)]$$

$$\leq g[G(W(W^{n-2}q),W(W^{n-1}q),W(W^{n-1}q)+G(W(W^{n-1}q),W(W^{n-2}q),W(W^{n-2}q)]$$

$$\leq g^2[G(W^{n-2}q,W^{n-1}q,W^{n-1}q)+G(W^{n-1}q,W^{n-2}q,W^{n-2}q)]$$

$$\cdots$$

$$\leq g^n[G(q,Wq,Wq)+G(Wq,q,q)]$$

As same as in the previous Theorem 3.2, $F_{G\varepsilon}(W)\neq\emptyset$. That is, W has an $\varepsilon-FP$. It means that, condition (i) of Lemma 2.5 is verified. To prove condition (ii) of Lemma 2.5. For that, fix on $\theta>0$ and $q,r\in F_{G\varepsilon}(W)$. Also,

$$\begin{aligned}
[G(q,r,r)+G(r,q,q)] &\leq [G(Wq,Wr,Wr)+G(Wr,Wq,Wq)]+\theta \\
&= g\left[\frac{\varepsilon[[G(q,r,r)+G(r,q,q)]+\varepsilon]+\varepsilon[[G(q,r,r)+G(r,q,q)]+\varepsilon]}{2[G(q,r,r)+G(r,q,q)]+2\varepsilon}\right]+2\varepsilon \\
&= \frac{2g\varepsilon[G(q,r,r)+G_m(r,q,q)]+2g\varepsilon^2+4\varepsilon[G(q,r,r)+G(r,q,q)]+4\varepsilon^2}{2[G(q,r,r)+G(r,q,q)]+2\varepsilon}
\end{aligned}$$

On simplyfying, we have

$$2[G(q,r,r)+G(r,q,q)]^2-2\varepsilon(1+g)[G(q,r,r)+G(r,q,q)]\leq 2\varepsilon^2(g+2)$$

$$2[G(q,r,r)+G(r,q,q)]^2\leq 2\varepsilon(1+g)[G(q,r,r)+G(r,q,q)]+2\varepsilon^2(g+2)$$

Which implies that $a=2,b=-2\varepsilon(1+g)$ and $c=-2\varepsilon^2(g+2)$. Therefore,

$$\begin{aligned}
[G(q,r,r)+G(r,q,q)] &\leq \frac{2\varepsilon(1+g)+\sqrt{4\varepsilon^2(1+g)^2+16\varepsilon^2(g+2)}}{4} \\
&= \frac{2\varepsilon(1+g)+\sqrt{4\varepsilon^2(1+2g+g^2)+16\varepsilon^2g+32\varepsilon^2}}{4} \\
&= \frac{2\varepsilon(1+g)+\sqrt{4\varepsilon^2+8\varepsilon^2g+4g^2\varepsilon^2g+16\varepsilon^2g+32\varepsilon^2}}{4}
\end{aligned}$$

$$= \frac{\varepsilon(1+g)+\sqrt{9\varepsilon^2+6\varepsilon^2 g+g^2\varepsilon^2}}{2}$$
$$< \frac{\varepsilon+\varepsilon g+9\varepsilon^2+6\varepsilon^2 g+g^2\varepsilon^2}{2}$$

Hence,

$$Diam(F_{G\varepsilon}(W)) < \frac{\varepsilon^2(g^2+6g+9)+\varepsilon(g+1)}{2}, \text{ for all } \varepsilon > 0.$$

□

Theorem 3.7. *Let (L, d_G) be a G-metric space and W_1, W_2, W_3 are three nonempty subsets of L. Let $W : W_1 \cup W_2 \cup W_3 \to W_1 \cup W_2 \cup W_3$ be a cyclic mapping. Then there exists $g_1, g_2 \in (0,1)$ with $g_1 + g_2 < 1$ and $G(q,r,r)+G(r,q,q) > 0$ such that*

$$\begin{aligned} &G(Wq,Wr,Wr)+G(Wr,Wq,Wq) \\ &\qquad \le \frac{g_1[G(r,Wr,Wr)+G(Wr,r,r)][1+G(q,Wq,Wq)+G(Wq,q,q)]}{1+G(q,r,r)+G(r,q,q)} \\ &\qquad + g_2[G(q,r,r)+G(r,q,q)], \text{ for all } q,r \in L. \end{aligned} \tag{3.2}$$

has an ε-fixed point and

$$Diam(F_{G\varepsilon}(W)) < \frac{g_2^2-g_2+6\varepsilon+4\varepsilon^2(1+g_1-g_1g_2)+4\varepsilon(g_1-g_2-g_1g_2)}{2(1-g_2)}, \text{ for all } \varepsilon > 0.$$

Proof. As same as in the previous Theorem 3.6, $F_{G\varepsilon}(W) \neq \emptyset$. That is, W has an ε-fixed point. It means that, condition (i) of Lemma 2.5 is verified. To prove condition (ii) of Lemma 2.5. For that, fix on $\theta > 0$ and $q, r \in F_{G\varepsilon}(W)$. Also,

$$\begin{aligned} [G(q,r,r)+G(r,q,q)] &\le [G(Wq,Wr,Wr)+G(Wr,Wq,Wq)]+\theta \\ &\le \frac{g_1[G(r,Wr,Wr)+G(Wr,r,r)][1+G(q,Wq,Wq)+G(Wq,q,q)]}{1+G(q,r,r)+G(r,q,q)} \\ &\quad + g_2[G(q,r,r)+G(r,q,q)]+2\varepsilon \\ &\le \frac{g_1\varepsilon[1+\varepsilon]}{1+G(q,r,r)+G(r,q,q)} + g_2[G(q,r,r)+G(r,q,q)]+2\varepsilon \end{aligned}$$

On simplifying, we have

$$
\begin{aligned}
&[G(q,r,r)+G(r,q,q)] \\
&\leq \frac{1}{2(1-g_2)}\sqrt{g_2^2-2g_2+4\varepsilon^2(1+g_1-g_1g_2)+4\varepsilon(1+g_1-g_2-g_1g_2)+1}+\left(\frac{g_2+2\varepsilon-1}{2(1-g_2)}\right) \\
&< \frac{1}{2(1-g_2)}\left(g_2^2-2g_2+4\varepsilon^2(1+g_1-g_1g_2)+4\varepsilon(1+g_1-g_2-g_1g_2)+1+g_2+2\varepsilon-1\right)
\end{aligned}
$$

That is,

$$
\begin{aligned}
Diam(F_{G\varepsilon}(W)) &< \frac{g_2^2-2g_2+g_2+2\varepsilon+4\varepsilon^2(1+g_1-g_1g_2)+4\varepsilon(1+g_1-g_2-g_1g_2)}{2(1-g_2)} \\
&= \frac{g_2^2-g_2+6\varepsilon+4\varepsilon^2(1+g_1-g_1g_2)+4\varepsilon(g_1-g_2-g_1g_2)}{2(1-g_2)}
\end{aligned}
$$

Hence,

$$
Diam(F_{G\varepsilon}(W)) < \frac{g_2^2-g_2+6\varepsilon+4\varepsilon^2(1+g_1-g_1g_2)+4\varepsilon(g_1-g_2-g_1g_2)}{2(1-g_2)}, \text{ for all } \varepsilon>0.
$$

□

Theorem 3.8. *Let (L,d_G) be a G-metric space and W_1,W_2,W_3 are three nonempty subsets of L. Let $W: W_1\cup W_2\cup W_3 \to W_1\cup W_2\cup W_3$ be a cyclic mapping. Then there exists $g_1,g_2\in(0,1)$ with $g_1+g_2<1$ and $G(q,r,r)+G(r,q,q)>0$ such that*

$$
\begin{aligned}
&G(Wq,Wr,Wr)+G(Wr,Wq,Wq) \\
&\qquad\leq \frac{g_1[[G(q,Wq,Wq)+G(Wq,q,q)][G(r,Wr,Wr)+G(Wr,r,r)]]}{G(q,r,r)+G(r,q,q)} \\
&\qquad\quad +g_2[G(q,r,r)+G(r,q,q)], \textit{ for all } q,r\in L
\end{aligned}
\tag{3.3}
$$

has an $\varepsilon-FP$ and

$$
Diam(F_{G\varepsilon}(W)) < \varepsilon\left(\frac{2}{1-g_2}+g_1\right), \textit{ for all } \varepsilon>0.
$$

Proof. As same as in the previous Theorem 3.6, $F_{G\varepsilon}(W)\neq\emptyset$. That is, W has an $\varepsilon-FP$. It means that, condition (i) of Lemma 2.5 is verified. To prove condition (ii) of Lemma 2.5. For that, fix on $\theta > 0$ and $q, r \in F_{G\varepsilon}(W)$. Also,

$$\begin{aligned}
&[G(q,r,r)+G(r,q,q)] \\
&\qquad \le [G(Wq,Wr,Wr)+G(Wr,Wq,Wq)]+\theta \\
&\qquad \le \frac{g_1[[G(q,Wq,Wq)+G(Wq,q,q)][G(r,Wr,Wr)+G(Wr,r,r)]]}{G(q,r,r)+G(r,q,q)} \\
&\qquad\quad + g_2[G(q,r,r)+G(r,q,q)]+2\varepsilon.
\end{aligned}$$

Substituting the ε value, we have

$$\begin{aligned}
(1-g_2)[G(q,r,r)+G(r,q,q)]^2 &\le g_1\varepsilon^2+2\varepsilon[G(q,r,r)+G(r,q,q)] \\
\left[[G(q,r,r)+G(r,q,q)]-\left(\frac{\varepsilon}{1-g_2}\right)\right]^2 &\le \frac{g_1\varepsilon^2}{1-g_2}+\left(\frac{\varepsilon}{1-g_2}\right)^2 \\
G(q,r,r)+G(r,q,q)-\left(\frac{\varepsilon}{1-g_2}\right) &\le \sqrt{\frac{(1-g_2)g_1\varepsilon^2+\varepsilon^2}{(1-g_2)^2}} \\
G(q,r,r)+G(r,q,q) &= \sqrt{\frac{g_1\varepsilon^2-g_1g_2\varepsilon^2+\varepsilon^2}{(1-g_2)^2}}+\left(\frac{\varepsilon}{1-g_2}\right) \\
&= \sqrt{\frac{g_1\varepsilon^2+(1-g_1g_2)\varepsilon^2}{(1-g_2)^2}}+\left(\frac{\varepsilon}{1-g_2}\right) \\
&= \left(\frac{\varepsilon}{1-g_2}\right)\sqrt{g_1+1-g_1g_2}+\left(\frac{\varepsilon}{1-g_2}\right) \\
&< \left(\frac{\varepsilon}{1-g_2}\right)(1+g_1+1-g_1g_2) \\
&< \left(\frac{\varepsilon}{1-g_2}\right)(2+g_1(1-g_2))
\end{aligned}$$

Hence,

$$Diam(F_{G\varepsilon}(W)) < \varepsilon\left(\frac{2}{1-g_2}+g_1\right), \text{ for all } \varepsilon > 0.$$

$\square$

Theorem 3.9. *Let (L, d_G) be a G-metric space and W_1, W_2, W_3 are three nonempty subsets of L. Let $W : W_1 \cup W_2 \cup W_3 \to W_1 \cup W_2 \cup W_3$ be a cyclic mapping. Then there exists $g_1, g_2 \in (0,1)$*

with $g_1+g_2<1$ *and* $G(r,Wr,Wr)+G(Wr,r,r)+G(q,r,r)+G(r,q,q)>0$ *such that*

(3.4)

$$G(Wq,Wr,Wr)+G(Wr,Wq,Wq)\leq \frac{g_1[[G(q,Wq,Wq)+G(Wq,q,q)][G(q,Wr,Wr)+G(Wr,q,q)][G(r,Wr,Wr)+G(Wr,r,r)]]}{G(r,Wr,Wr)+G(Wr,r,r)+G(q,r,r)+G(r,q,q)} + g_2[G(q,r,r)+G(r,q,q)], \text{ for all } q,r\in L.$$

has an $\varepsilon-FP$ *and*

$$Diam(F_{G\varepsilon}(W))<\frac{1}{2(1-g_2)}[\varepsilon^4 g_1^2+\varepsilon^3(6g_1-2g_1g_2)+\varepsilon^2(g_1+10)+\varepsilon(1+g_2), \text{ for all } \varepsilon>0.$$

Proof. Let $\varepsilon>0$ and $q\in W_1\cup W_2\cup W_3$. Define a sequence $\{q_n\}$ such that $q_{n+1}=Wq_n,$ for all $n\geq 0$. Consider,

$$\begin{aligned}&G(W^{n+1}q,W^nq,W^nq)+G_m(W^nq,W^{n+1}q,W^{n+1}q)\\&=G(W(W^nq),W(W^{n-1}q),W(W^{n-1}q))+G(W(W^{n-1}q),W(W^nq),W(W^nq))\end{aligned}$$

As same as in the previous Theorem 3.6, $F_{G\varepsilon}(W)\neq\emptyset$. That is, W has an $\varepsilon-FP$. It means that, condition (i) of Lemma 2.5 is verified. To prove condition (ii) of Lemma 2.5. For that, fix on $\theta>0$ and $q,r\in F_{G\varepsilon}(W)$. Also,

$$\begin{aligned}&[G(q,r,r)+G(r,q,q)]\leq[G(Tq,Tr,Tr)+G(Tr,Tq,Tq)]+\theta\\&\leq\frac{g_1[[G(q,Wq,Wq)+G(Wq,q,q)][G(q,Wr,Wr)+G(Wr,q,q)][G(r,Wr,Wr)+G(Wr,r,r)]]}{G(r,Wr,Wr)+G(Wr,r,r)+G(q,r,r)+G(r,q,q)}\\&\quad+g_2[G(q,r,r)+G(r,q,q)]+2\varepsilon\end{aligned}$$

Substituting the ε value, we have

$$[G(q,r,r)+G(r,q,q)]\leq\frac{[\varepsilon g_1[G(q,r,r)+G(r,q,q)]+g_1\varepsilon^2]\varepsilon}{\varepsilon+[G(q,r,r)+G(r,q,q)]}+g_2[G(q,r,r)+G(r,q,q)]+2\varepsilon$$

That is,

$$(1-g_2)[G(q,r,r)+G(r,q,q)]\leq\frac{[\varepsilon^2 g_1[G(q,r,r)+G(r,q,q)]+g_1\varepsilon^3]}{\varepsilon+[G(q,r,r)+G(r,q,q)]}+2\varepsilon$$

On simplification, we get

$$[G(q,r,r)+G(r,q,q)]^2+\frac{G(q,r,r)+G(r,q,q)[-g_2\varepsilon-\varepsilon-\varepsilon^2 g_1]}{1-g_2}\leq\frac{2\varepsilon^2+k_1\varepsilon^3}{1-k_2}$$

Taking square

$$\left[G(q,r,r)+G(r,q,q)+\frac{-g_2\varepsilon-\varepsilon-\varepsilon^2 g_1}{2(1-g_2)}\right]^2\leq\frac{2\varepsilon^2+g_1\varepsilon^3}{1-g_2}+\left[\frac{-g_2\varepsilon-\varepsilon-\varepsilon^2 g_1}{2(1-g_2)}\right]^2$$

Implies that

$$\begin{aligned}
&G(q,r,r)+G(r,q,q)\\
&\qquad\leq\frac{g_2\varepsilon+\varepsilon+\varepsilon^2 g_1}{2(1-g_2)}+\sqrt{\frac{2\varepsilon^2+g_1\varepsilon^3}{1-g_2}+\left[\frac{-g_2\varepsilon-\varepsilon-\varepsilon^2 g_1}{2(1-g_2)}\right]^2}\\
&\qquad=\frac{g_2\varepsilon+\varepsilon+\varepsilon^2 g_1}{2(1-g_2)}+\sqrt{\frac{4(1-g_2)(2\varepsilon^2+g_1\varepsilon^3)+(-g_2\varepsilon-\varepsilon-\varepsilon^2 g_1)^2}{4(1-g_2)^2}}\\
&\qquad=\frac{1}{2(1-g_2)}\left[g_2\varepsilon+\varepsilon+\varepsilon^2 g_1+\sqrt{4(1-g_2)(2\varepsilon^2+g_1\varepsilon^3)+(-g_2\varepsilon-\varepsilon-\varepsilon^2 g_1)^2}\right]\\
&\qquad<\frac{1}{2(1-g_2)}\left[g_2\varepsilon+\varepsilon+\varepsilon^2 g_1+4(1-g_2)(2\varepsilon^2+g_1\varepsilon^3)+(-g_2\varepsilon-\varepsilon-\varepsilon^2 g_1)^2\right]
\end{aligned}$$

Hence,

$$Diam(F_{G\varepsilon}(W))\leq\frac{1}{2(1-g_2)}[\varepsilon^4 g_1^2+\varepsilon^3(6g_1-2g_1g_2)+\varepsilon^2(g_1+10)+\varepsilon(1+g_2)],\ \text{for all }\varepsilon>0.$$

□

Theorem 3.10. *Let (L,d_G) be a G-metric space and W_1,W_2,W_3 are three nonempty subsets of L. Let $W:W_1\cup W_2\cup W_3\to W_1\cup W_2\cup W_3$ be a cyclic mapping. Then there exists $g\in(0,1)$ and $G(r,Wr,Wr)+G(Wr,r,r)+G(q,r,r)+G(r,q,q)>0$ such that*

$$\begin{aligned}
&G(Wq,Wr,Wr)+G(Wr,Wq,Wq)\\
&\quad\leq\frac{g[[G(q,Wq,Wq)+G(Wq,q,q)][G(q,Wr,Wr)+G(Wr,q,q)][G(r,Wr,Wr)+G(Wr,r,r)]]}{[G(r,Wr,Wr)+G(Wr,r,r)]+[G(q,r,r)+G(r,q,q)]}
\end{aligned}$$

(3.5)

$$+\varnothing[G(q,r,r)+G(r,q,q)],\ \textit{for all } q,r\in L$$

has an $\varepsilon-FP$ and the diameter is imperfect.

Proof. As same as in the previous Theorem 3.6, $F_{G\varepsilon}(W) \neq \emptyset$. That is, W has an $\varepsilon - FP$. It means that, condition (i) of Lemma 2.5 is verified. To prove condition (ii) of Lemma 2.5. For that, fix on $\theta > 0$ and $q, r \in F_{G\varepsilon}(W)$. Also,

$$\begin{aligned}
&[G(q,r,r)+G(r,q,q)] \leq [G(Wq,Wr,Wr)+G(Wr,Wq,Wq)]+\theta \\
&\leq \frac{g[[G(q,Wq,Wq)+G(Wq,q,q)][G(q,Wr,Wr)+G(Wr,q,q)][G(r,Wr,Wr)+G(Wr,r,r)]]}{[G(r,Wr,Wr)+G(Wr,r,r)]+[G(q,r,r)+G(r,q,q)]} \\
&\quad +\varnothing[G(q,r,r)+G(r,q,q)]+2\varepsilon
\end{aligned}$$

On applying the ε value, we get

$$[G(q,r,r)+G(r,q,q)] \leq \frac{g\varepsilon[[G(q,r,r)+G(r,q,q)]+\varepsilon]\varepsilon}{\varepsilon+[G(q,r,r)+G(r,q,q)]}+\varnothing[G(q,r,r)+G(r,q,q)]+2\varepsilon$$

Since $\varnothing(p) < p$, for every $p > 0$, we have

$$[G(q,r,r)+G(r,q,q)] \leq \frac{g\varepsilon[[G(q,r,r)+G(r,q,q)]+\varepsilon]\varepsilon}{\varepsilon+[G(q,r,r)+G(r,q,q)]}+[G(q,r,r)+G(r,q,q)]+2\varepsilon$$

It gives that

$$\frac{-g\varepsilon[[G(q,r,r)+G(r,q,q)]+\varepsilon]\varepsilon}{\varepsilon+[G(q,r,r)+G(r,q,q)]} \leq 2\varepsilon$$

So,

$$\begin{aligned}
-g\varepsilon[[G(q,r,r)+G(r,q,q)]+\varepsilon]\varepsilon &\leq 2\varepsilon^2+2\varepsilon[G(q,r,r)+G(r,q,q)] \\
-g\varepsilon^2[G(q,r,r)+G(r,q,q)]-g\varepsilon^3 &\leq 2\varepsilon^2+2\varepsilon[G(q,r,r)+G(r,q,q)]
\end{aligned}$$

Therefore,

$$\begin{aligned}
(-g\varepsilon^2-2\varepsilon)[G(q,r,r)+G(r,q,q)] &\leq 2\varepsilon^2+g\varepsilon^3 \\
(-g\varepsilon-2)[G(q,r,r)+G(r,q,q)] &\leq 2\varepsilon+g\varepsilon^2 \\
(g\varepsilon+2)[G(q,r,r)+G(r,q,q)] &\geq -2\varepsilon^2-g\varepsilon^2 \\
[G(q,r,r)+G(r,q,q)] &\geq \frac{-2\varepsilon-g\varepsilon^2}{g\varepsilon+2}
\end{aligned}$$

That is,

$$Diam(F_{G\varepsilon}(T)) \geq \frac{-(2+g\varepsilon)\varepsilon}{g\varepsilon+2}, \text{ for all } \varepsilon > 0.$$

Hence, the diameter is imperfect. □

Example 3.11. *Let $L = \{0, 1, 2, ..., 18\}$ and $G : L \times L \times L \to \mathbb{R}^+$ be defined by:*

$$G(q,r,s) = \begin{cases} q+r+s & \text{when } q \neq r \neq s \neq 0 \\ q+r & \text{when } q = r \neq s; q,r,s \neq 0 \\ r+s+1 & \text{when } q = 0, r \neq s; r,s \neq 0 \\ r+2 & \text{when } q = 0, r = s \neq 0 \\ s+1 & \text{when } q = 0, r = 0, s \neq 0 \\ 0 & \text{when } q = r = s \end{cases}$$

Consider a closed subsets $W_1 = \{4, 18\}, W_2 = \{3, 7, 17\}$ and $W_3 = \{0\}$ of a metric space (L, d_G) and $W : W_1 \cup W_2 \cup W_3 \to W_1 \cup W_2 \cup W_3$ is defined by:

$$Wq = \begin{cases} q-1 & \text{when } q \in \{4,8\} \\ 0 & \text{when } q \in \{3,7,17\} \\ 4 & \text{when } q = 0 \end{cases}$$

This clearly shows that $W(W_1) \subseteq W_2, W(W_2) \subseteq W_3$ and $W(W_3) \subseteq W_1$. Also for every $W_1, W_2 \in W_1 \cup W_2 \cup W_3 \subseteq L$ satisfies the equation $(2.1), (2.2)$ and (2.3). Thus, W satisfies Theorems 3.2 and 3.3. Therefore,

$$Diam(F_{G\varepsilon}(W)) \leq \frac{2\varepsilon(g_1+g_3+1)}{1-g_2-2g_3}, \text{ for all } \varepsilon > 0$$

and

$$Diam(F_{G\varepsilon}(W)) \leq \varepsilon(g+2), \text{ for all } \varepsilon > 0$$

satisfies respectively.

Remark 3.12. *We have proved many AFP results by using various operators on G-metric spaces (not necessarily complete). The diameters of several contraction operators and a few rational type contraction operators are shown in the table below.*

S. No	*Operator(s)*	*Diameter, for every* $\varepsilon > 0$, *Diam*$(F_{G\varepsilon}(W))$
1	*Contraction* [20]	$\leq \frac{2\varepsilon}{1-g_2}$
2	*Kannan* [20]	$\leq (l+g_1)2\varepsilon$
3	*Chatterjea* [9]	$\leq \frac{(g_3+1)2\varepsilon}{1-2g_3}$
4	*B-contraction* [26]	$\leq \frac{(g_1+g_3+1)2\varepsilon}{1-g_2-2g_3}$
5	*Bianchini* [7]	$\leq (g+2)\varepsilon$
6	*Hardy-Rogers* [14]	$\leq \frac{(g_2+g_3+g_4+g_5+2)\varepsilon}{1-g_1-g_4-g_5}$
7	*Ćirić* [10]	$\leq \frac{(g_2+g_3+2g_4+2)\varepsilon}{1-g_1-2g_4}$
8	*Ćirić-Reich-Rus* [6]	$\leq \frac{(1+g_1)2\varepsilon}{1-g_1}$
9	*Reich* [41]	$\leq \frac{(g_2+g_3+2)\varepsilon}{1-g_1}$
10	*Zamfirescu* [45]	$\leq \frac{(1+\delta)2\varepsilon}{1-\delta}$
11	*Mohseni-saheli* [29]	$\leq \frac{(1+g)2\varepsilon}{1-2g}$
12	*Mohseni-semi* [29]	$\leq \frac{(g+2)\varepsilon}{1-g}$
13	*Weak contraction* [3]	$\leq \frac{(2+W)\varepsilon}{1-g-W}$
14	*Contraction* (3.1)[42]	$< \frac{(g^2+6g+9)\varepsilon^2+(g+1)\varepsilon}{2}$
15	*Contraction* (3.2)[43]	$< \left(\frac{2}{1-g_2}+g_1\right)\varepsilon$
16	*Contraction* (3.3) [43]	$< \frac{g_2^2-g_2+6\varepsilon+4\varepsilon^2(1+g_1-g_1g_2)+4\varepsilon(g_1-g_2-g_1g_2)}{2(1-g_2)}$
17	*Contraction* (3.4)[43]	$< \frac{\varepsilon^4 g_1^2+\varepsilon^3(6g_1-2g_1g_2)+\varepsilon^2(g_1+10)+\varepsilon(1+g_2)}{2(1-g_2)}$
18	*Contraction* (3.5)[43]	$Imperfect$

Example 3.13. *Let $L = [0,1]$ and consider the closed subsets $W_1 = [0,3/6], W_2 = [2/6,3/6]$ and $W_3 = [5/6,1]$ of a metric space (L,d_G) and $W : W_1 \cup W_2 \cup W_3 \to W_1 \cup W_2 \cup W_3$ is defined by:*

$$Wq = \begin{cases} \frac{2}{6} + q & \text{when } q \in \left[0, \frac{3}{6}\right] \\ \\ \frac{3}{6} + q & \text{when } q \in \left[\frac{2}{6}, \frac{3}{6}\right] \\ \\ 1 - \frac{3}{6} & \text{when } q \in \left[\frac{5}{6}, 1\right] \end{cases}$$

This clearly shows that $W(W_1) \subseteq W_2, W(W_2) \subseteq W_3$ and $W(W_3) \subseteq W_1$. Also for every $q, r \in W_1 \cup W_2 \cup W_3 \subseteq L$ satisfies the Definition 2.6 and Definition 2.8. Thus, W satisfies all the conditions of the Theorems 3.2, 3.3 and Corollary 3.4 .

4. APPLICATIONS

The AFP theory covers a wide range of applications in the field of applied mathematics, particularly Fourier series, numerical analysis, and so on. By reading [23] and references therein, one can find a variety of applications involving AFP results in the field of applied mathematics. The examples below demonstrate how to apply AFP results in differential equations.

Example 4.1. *Let $T = C([0,1],\mathbb{R})$ and T is G-metric space defined by $d(p,q) = \sup_{t\in[0,1]} |p - q|^2$. Also, consider $y''(t) = 3y^2(t)/2$, $0 \le t \le 1$ and the initial conditions $y(0) = 4$, $y(1) = 1$. Here, the exact solution is $y(t) = 4/(1+t)^2$. We have, $y_0(t) = c_1 t + c_2$. By using the initial conditions, we get $y_0(t) = 4 - 3t$. Now, define the integral operator,*

$$A(y) = y + \int_0^1 G(t,s)[y'' - f(s,y,y')]ds \tag{4.1}$$

where

$$G(t,s) = \begin{cases} s(1-t) & 0 \le s \le t \\ t(1-s) & t \le s \le 1 \end{cases}$$

Then, the equation (5.1) *becomes*

$$\begin{aligned}
A(y) &= y(t) + \int_0^1 G(t,s)y''(s)ds - \int_0^1 G(t,s)f(s,y,y')ds \\
&= (4-3t) - \int_0^1 G(t,s)\big[-\frac{3y^2(s)}{2}\big]ds \\
&= 4-3t+\frac{3}{2}\left\{\int_0^1 G(t,s)y^2(s)ds\right\}
\end{aligned}$$

Let us take $G(Ap,Aq,Aq)+G(Aq,Ap,Ap)=d(Ap,Aq)$*. So, we have*

$$\begin{aligned}
d(Ap,Aq) &= \sup_{t\in[0,1]} |Ap-Aq|^2 \\
&= \sup_{t\in[0,1]} \left|\frac{3}{2}\int_0^1 G(t,s)p^2(s)ds - \frac{3}{2}\int_0^1 G(t,s)q^2(s)ds\right|^2 \\
&\le \frac{9}{4}\left(\int_0^1 |G(t,s)|^2 ds\right)\left(\int_0^1 |p^2(s)-q^2(s)|^2 ds\right) \\
&\le \frac{3}{4}\frac{t^2(1-t)^2}{3}\int_0^1 \left|p^2(s)-q^2(s)\right|^2 ds \\
&\le \frac{3}{4}(\frac{1}{4})(\frac{1}{4})\int_0^1 \left|p^2(s)-q^2(s)\right|^2 ds \\
&\le \frac{3}{64}\sup_{t\in[0,1]} |p(s)-q(s)|^2 \\
&\le \frac{3}{64}d(p,q)
\end{aligned}$$

That is,

$$G(Ap,Aq,Aq)+G(Aq,Ap,Ap) \le \frac{3}{64}[G(p,q,q)+G(q,p,p)]$$

From this, we get $g_2 = 3/64$ *and* $g_1 = g_3 = 0$*. Hence, it satisfies all the conditions of Theorem* 3.2*. Also, by Theorem* 3.1*,* A *has* $\varepsilon - FP$ *in* $T = C([0,1],\mathbb{R})$*. Therefore, the given bounded value problem has* $\varepsilon - FP$ *in* T*.*

5. Conclusion

In this paper, some AFP theorems are established on G-metric spaces by utilizing various types of contraction mappings. Further, some AFP theorems are newly developed for rational type contraction mappings in the setting of G-metric spaces. It is worth observing that in the limiting case $\varepsilon \longrightarrow 0$, all the results established in the present paper produces more restricted

$AFP's$. Also, $AFP's$ are consequently not less important than $FP's$. As various future results can be demonstrated in a smaller setting to ensure the existence of the $AFP's$.

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AUTHOR CONTRIBUTIONS

All authors contributed equally, read and approved the final manuscript.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Azam, N. Mehmood, Fixed point theorems for multivalued mappings in G-cone metric spaces, J. Inequal. Appl. 2013 (2013), 354. https://doi.org/10.1186/1029-242x-2013-354.

[2] S. Banach, Sur les operations dans les ensembles abstract et leur application aux equation integrals, Fund. Math. 3 (1922), 133–181.

[3] M. Berinde, Approximate fixed point theorems, Stud. Univ. Babeş–Bolyai, Math. 51 (2006), 11–25.

[4] V. Berinde, Iterative approximation of fixed points, Editura Efemeride, Baia Mare, (2002).

[5] V. Berinde, On the approximation of fixed points of weak contractive mappings, Carpathian J. Math. 19 (2003), 7–22. https://www.jstor.org/stable/43996763.

[6] V. Berinde, M. Pacurar, Fixed point theorems for enriched Ćirić-Reich-Rus contraction in Banach spaces and convex metric spaces, Carpathian J. Math. 37 (2021), 173–184. https://www.jstor.org/stable/27082095.

[7] R.M.T. Bianchini, Su un problema di S. Reich riguardonte la teoria dei punti fissi, Boll. Un. Mat. Ital. 5 (1972), 103–108. https://cir.nii.ac.jp/crid/1571417124680382336.

[8] L.E. Brouwer, Úber abbildung von mannigfaltigkeiten, Math. Ann. 71 (1911), 97–115.

[9] S.K. Chatterjea, Fixed point theorems, C. R. Acad. Bulg. Sci. 25 (1972), 727–730.

[10] L.B. Ćirić, Generalized contractions and fixed point theorems, Publ. Inst. Math. (Bulgr). 12 (1971), 19–26.

[11] B.K. Dass, S. Gupta, An extension of Banach contraction principle through rational expressions, Indian J. Pure Appl. Math. 6 (1975), 1455–1458.

[12] P. Debnath, N. Konwar, S. Radenović, eds., Metric fixed point theory: applications in science, engineering and behavioural sciences, Springer Nature Singapore, Singapore, 2021. https://doi.org/10.1007/978-981-16-4896-0.

[13] D. Dey, M. Saha, Approximate fixed point of Reich operator, Acta Math. Univ. Comen. 82 (2013), 119–123.

[14] G.E. Hardy, T.D. Rogers, A generalization of a fixed point theorem of Reich, Can. Math. Bull. 16 (1973), 201–206. https://doi.org/10.4153/cmb-1973-036-0.

[15] J. Harjani, B. López, K. Sadarangani, A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space, Abstr. Appl. Anal. 2010 (2010), 190701. https://doi.org/10.1155/2010/190701.

[16] D.S. Jaggi, Some unique fixed point theorems, Indian J. Pure Appl. Math. 8 (1977), 223–230.

[17] Y.U. Gaba, λ-sequences and fixed point theorems in G-metric spaces, J. Nigerian Math. Soc. 35 (2016), 303-311.

[18] Y.U. Gaba, New contractive conditions for maps in G-metric type spaces, Adv. Anal. 1 (2016), 61–67. https://doi.org/10.22606/aan.2016.12001.

[19] Y.U. Gaba, Fixed point theorems in G-metric spaces, J. Math. Anal. Appl. 455 (2017), 528-537. https://doi.org/10.1016/j.jmaa.2017.05.062.

[20] R. Kannan, Some results on fixed points, Bull. Cal. Math. Soc. 60 (1968), 71–76. https://cir.nii.ac.jp/crid/1572543024587220992.

[21] R. Kannan, Some results on fixed points—II, Amer. Math. Mon. 76 (1969), 405–408. https://doi.org/10.1080/00029890.1969.12000228.

[22] M. S. Khan, A fixed point theorems for metric spaces, Rend. Ist. Mat. Univ. Trieste. 8 (1976), 69–72.

[23] S.A. Khuri, I. Louhichi, A novel Ishikawa-Green's fixed point scheme for the solution of BVPs, Appl. Math. Lett. 82 (2018), 50–57. https://doi.org/10.1016/j.aml.2018.02.016.

[24] Komal Goyal, Bhagwati Prasad, Approximate fixed points and summable almost stability, Glob. J. Pure Appl. Math. 13 (2017), 6029–6040.

[25] W. A. Kirk, P. S. Srinivasan, P. Veeramani, Fixed point for mappings satisfying cyclical contraction conditions, Fixed point theory, 4 (2003), 79–89.

[26] M. Marudai, V. S. Bright, Unique fixed point theorem weakly B-contractive mappings, Far East J. Math. Sci. 98 (2015), 897–914.

[27] S.K. Mohanta, Some fixed point theorems in G-metric spaces, An. Stiint. "Ovidius" Constanta Ser. Mat. 20 (2012), 285-306.

[28] S.A.M. Mohsenalhosseini, H. Mazaheri, M.A. Dehghan, Approximate best proximity pairs in metric space, Abstr. Appl. Anal. 2011 (2011), 596971. https://doi.org/10.1155/2011/596971.

[29] S. A. M. Mohsenialhosseini, Approximate fixed point theorems of cyclical contraction mapping, Math. Morav. 21 (2017), 125–137.

[30] S.A.M. Mohsenialhosseini, M. Saheli, Some of a family of contractive maps and approximate common fixed point, J. Fixed Point Theory, 2021 (2021), 2.

[31] S.A.M. Mohsenalhosseini, H. Mazaheri, Fixed points for completely norm space and map T_α, Math. Morov. 16 (2012), 25–35.

[32] S.A.M. Mohsenalhosseini, T. Ahmadi, Approximate fixed point in G-metric spaces for various types of operators, J. Math. Comput. Sci. 6 (2016), 767–782.

[33] S.A.M. Mohsenalhosseini, Approximate fixed points of operators on G-metric spaces, U.P.B. Sci. Bull., Ser. A, 79 (2017), 85-96.

[34] Z. Mustafa, M. Khandagji, W. Shatanawi, Fixed point results on complete G-metric spaces, Stud. Sci. Math. Hung. 48 (2011), 304–319. https://doi.org/10.1556/sscmath.48.2011.3.1170.

[35] Z. Mustafa, H. Obiedat, A fixed point theorems of Reich in G-metric spaces, Cubo. 12 (2010), 83–93. https://doi.org/10.4067/s0719-06462010000100008.

[36] Z. Mustafa, H. Obiedat, F. Awawdeh, Some fixed point theorems for mappings on complete G-metric space, Fixed Point Theory Appl. 2008 (2008), 189870. https://doi.org/10.1155/2008/189870.

[37] Z. Mustafa, H. Obiedat, F. Awawdeh, Fixed point theorems for expansive mappings in G-metric spaces, Int. J. Contemp. Math. Sci. 5 (2010), 2463–2472.

[38] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289–297.

[39] Z. Mustafa, B. Sims, Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory Appl. 2009 (2009), 917175. https://doi.org/10.1155/2009/917175.

[40] H. Obiedat, Z. Mustafa, Fixed point results on a nonsymmetric G-metric spaces, Jordan J. Math. Stat. 3 (2010), 65–79.

[41] S. Reich, Some remarks concerning contraction mappings, Can. Math. Bull. 14 (1971), 121–124. https://doi.org/10.4153/cmb-1971-024-9.

[42] R. Theivaraman, P. S. Srinivasan, S. Thenmozhi, et al. Some approximate fixed point results for various contraction type mappings, Adv. Fixed Point Theory, 13 (2023), 9.

[43] K. Tijani, S. Olayemi, Approximate fixed point results for rational-type contraction mappings, Int. J. Anal. Optim.: Theory Appl. 7 (2021), 76–86.

[44] S.H. Tijs, A. Torre, R. Branzei, Approximate fixed point theorems, Libertas Math. 23 (2003), 35–39.

[45] T. Zamfirescu, Fix point theorems in metric spaces, Arch. Math. 23 (1972), 292–298. https://doi.org/10.1007/bf01304884.