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NEW FIXED POINT THEOREMS FOR $(\tau-\psi)-$CONTRACTION MAPPING IN COMPLETE RECTANGULAR M-METRIC SPACE

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Abstract. In this article, we prove a fixed point result for $(\tau-\psi)-$ contraction in rectangular M-metric space. Moreover, we discuss some examples that realized the results. Finally, we investigate the existence and uniqueness of a solution of non-linear matrix equations and integral equations of Fredholm type as well.

Keywords: fixed point; M-metric spaces; generalized $(\tau-\psi)-$contraction.

2020 AMS Subject Classification: 47H10, 47H09.

1. Introduction

Fixed point theory is an important and one of the most useful results in a variety of areas such as non linear analysis, operator theory,differential equation, etc.

In 1922 [1], Stefan Banach first proved formula and proved a theorem regarding a contraction mapping. Because of its application in mathematics, several authors have obtained many interesting extensions and generalization of the Banach contraction principle (see [3, 9, 6, 2, 14, 13, 10, 11, 4, 5]).

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A partial metric space is one of the most influential generalizations, of ordinary metric space. It has a wide range of uses in mathematical research and scientific applications. It was first established by Shukla in [15]. In 2014, Asadi et al [7] generalized the partial metric space to M-metric space and obtained certain theorems related to M-metric space.

Branciari [8] gave a generalization of the notion of metric spaces, which is called Branciari distance spaces, by replacing triangle inequality with trapezoidal inequality, and he gave an extension of Banach contraction principle to Branciari distance spaces.

In 2018, Ozgur et al [12] introduced rectangular M-metric space and obtained certain theorems related to M-metric space.

In this article, we establish the fixed point theorem for $(\tau - \psi)$-contraction in rectangular M-metric space. Also, examples are given to illustrate the obtained results we derive some useful corollaries of these results.

2. PRELIMINARIES

In what follows, we recall basic notions, definitions, examples and results on the topics for the sake of completeness.

Notation We need the following symbols and class of functions to prove certain results of this section:

$\mathbb{R}$ is the set of all real numbers;

N is the set of all natural numbers;

$\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+$, such that, ψ is non-decreasing, continuous, $\sum_{k=1}^{\infty} \psi^k(t) < \infty$, $\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$ if and only if $t = 0$, where ψ^k is the kth iterate of ψ $\}$.

Definition 2.1. Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $z,t \in X$ and for all distinct points $m,n \in X$, each of them different from z and t, on has

(i) $d(z,t) = 0 \Leftrightarrow z = t$;

(ii) $d(z,t) = d(t,z)$;

(iii) $d(z,t) \leq d(z,m) + d(m,n) + d(n,t)$.

Then (X,d) is called a generalized metric space.

After that, Shukla [15] introduced partial rectangular metric space. The definition is as follows:

Definition 2.2. Let X be a non-empty set and $p : X \times X \to \mathbb{R}^+$ be a mapping such that for all $a, b \in X$ and for all distinct points $c, d \in X$, each of them different from a and b, on has
(i) $a = b$ if and only if $p(a,b) = p(a,a) = p(b,b)$;
(ii) $p(a,b) = d(b,a)$;
(iii) $p(a,b) \leq p(a,c) + p(c,d) + p(d,b) - p(c,c) - p(d,d)$.
Then (X, p) is called a partial rectangular metric space.

In 2014, Asadi et al [7] generalized the partial metric space to M-metric space and obtained certain theorems related to M-metric space.
Notation: The following notations are useful in the sequel:
Let $m : X \times X \to \mathbb{R}^+$ be a mapping.
Denote

(i) $m_{ab} = m(a,a) \vee m(b,b) = \min\{m(a,a), m(b,b)\}$ and
(ii) $M_{ab} = m(a,a) \wedge m(b,b) = \max\{m(a,a), m(b,b)\}$

Definition 2.3. Let X be a non-empty set and $m : X \times X \to \mathbb{R}^+$ be a mapping such that for all $a, b, c \in X$,

(i) $a = b$ if and only if $m(a,b) = m(a,a) = m(b,b)$;
(ii) $m(a,b) = m(b,a)$;
(ii) $m_{ab} \leq m(a,b)$;
(iv) $(m(a,b) - m_{ab}) \leq (m(a,c) - m_{ac}) + (m(c,b) - m_{cb})$.

Then (X, m) is called a M-metric space.

Example 2.4. Let $X = [0, \infty)$.
Then $\mathscr{N} : X^2 \to [0, \infty)$ defined by $\mathscr{N}(x,y) = \dfrac{x+y}{2}$ is a M-metric on X.

In 2018, Ozgur et al. [12] introduced rectangular M-metric space and definition are as follows:
Notation: The following notations are useful in the sequel:

Let $m_r : X \times X \to \mathbb{R}^+$ be a mapping.

Denote

(i) $m_{r_{ab}} = m_r(a,a) \vee m_r(b,b) = \min\{m_r(a,a), m_r(b,b)\}$ and

(ii) $M_{r_{ab}} = m_r(a,a) \wedge m_r(b,b) = \max\{m_r(a,a), m_r(b,b)\}$

Definition 2.5. Let X be a non-empty set and $m_r : X \times X \to \mathbb{R}^+$ be a mapping such that for all $a, b \in X$ and for all distinct points $c, d \in X$, each of them different from a and b, on has

(i) $a = b$ if and only if $m_r(a,b) = m_r(a,a) = m_r(b,b)$;

(ii) $m_r(a,b) = m_r(b,a)$;

(ii) $m_{r_{ab}} \leq m_r(a,b)$;

(iv) $(m_r(a,b) - m_{r_{ab}}) \leq (m_r(a,c) - m_{r_{ac}}) + (m_r(c,d) - m_{r_{cd}}) + (m_r(d,a) - m_{r_{db}})$ (M-rectangular inequality).

Then (X, m_r) is called a rectangular M-metric space.

Example 2.6. Let X be a m_r-metric. Put

(i) $m_r^{\omega}(a,b) = m_r(a,b) - 2m_{r_{ab}} + M_{r_{ab}}$

(ii) $m_r^{s}(a,b) = m_r(a,b) - m_{r_{ab}}$.

Then, m_r^{ω} and m_r^{s} are ordinary metrics.

Definition 2.7. Let (X, m_r) be a rectangular M-metric space.

Then

(i) A sequence $\{a_n\}_{n\in\mathbb{N}}$ in X converges to a point a, if and only if

$$\lim_{n\to+\infty} (m_r(a_n, a) - m_{r_{a_n,a}}) = 0. \tag{2.1}$$

(ii) A sequence $\{a_n\}_{n\in\mathbb{N}}$ in X is said to be m_r-Cauchy sequence, if and only if

$$\lim_{n,m\to+\infty} (m_r(a_n, a_m) - m_{r_{a_n,a_m}}) \text{ and } \lim_{n,m\to+\infty} (M_{r_{a_n,a_m}} - m_{r_{a_n,a_m}}) \tag{2.2}$$

exist and finite.

(iii) A rectangular M-metric space is said to be m_r-complete, if every m_r Cauchy sequence $\{a_n\}$ converges to a point a such that

$$\lim_{n\to+\infty} (m_r(a_n, a) - m_{r_{a_n,a}}) = 0 \text{ and } \lim_{n\to+\infty} (M_{r_{a_n,a}} - m_{r_{a_n,a}}) = 0 \tag{2.3}$$

Lemma 2.8. *Let (X,m_r) be a rectangular M-metric space.*

Then

(1) *$\{a_n\}$ is a m_r-Cauchy sequence in (X,m_r) if and only if it is a Cauchy sequence in the metric space (X,m_r^{ω}).*

(2) *(X,m_r) is m_r-complete if and only if the metric space (X,m_r^{ω}) is complete. Furthermore,*

$$\lim_{n\to+\infty} m_r^{\omega}(a_n,a)=0 \Leftrightarrow \lim_{n\to+\infty}(m_r(a_n,a)-m_{r_{a_n,a}})=0\ ,\ \lim_{n\to+\infty}(M_{r_{a_n,a}}-m_{r_{a_n,a}})=0$$

Likewise the above definition holds also for m_r^s.

Lemma 2.9. *Assume that $a_n \to a$ as $n\to\infty$ in a rectangular M- metric space (X,m_r).*

Then

$$\lim_{n\to+\infty} m_r(a_n,y)-m_{r_{a_n,y}} = m_r(a,y)-m_{r_{a,y}} \forall\ y\in X.$$

Lemma 2.10. *Assume that $a_n \to a$ as $n\to\infty$ and $b_n\to b$ as $n\to\infty$ in a rectangular M- metric space (X,m_r).*

Then

$$\lim_{n\to+\infty} m_r(a_n,b_n)-m_{r_{a_n,b_n}} = m_r(a,b)-m_{r_{a,b}}.$$

Lemma 2.11. *Let $\{a_n\}$ be a sequence in a rectangular M-metric space (X,m_r), such that there exists $k\in\,]0,1[$ such that*

$$m_r(a_{n+1},a_n)\leq km_r(a_n,a_{n-1})\ for\ all\ n\in\mathbb{N}.$$

Then,

(A) $\lim_{n\to\infty} m_r(a_n,a_{n-1})=0$,

(B) $\lim_{n\to\infty} m_r(a_n,a_n)=0$,

(C) $\lim_{n,m\to\infty} m_{r_{a_n,a_m}}=0$
and

(D) *$\{a_n\}$ is a m_r-Cauchy sequence.*

Definition 2.12. Let (X,d) be a generalized metric space. m_r is said to be complete if every Cauchy sequence $\{a_n\}_n$ in X converges to an $a\in X$.

3. Main Results

The following definition is new version of the $(\tau-\psi)$-contraction for a rectangular M-metric space.

Definition 3.1. Let (X,m_r) be a rectangular M-metric space and $T:X\to X$ be a mapping. T is said to be a $(\tau-\psi)$-contraction on X, if there exist $\psi\in\Psi$ and $\tau>1$ such that for any $x,y\in X$, $\tau m_r(Tx,Ty)\leq\psi(M(x,y))$,
where

$$M(x,y)=max\{m_r(x,y),m_r(x,Tx),m_r(y,Ty)\}$$

Theorem 3.2. *Let (X,m_r) be a complete rectangular M-metric space and let $T:X\to X$ be a continuous $(\tau-\psi)$-contraction. Then, T has a unique fixed point $x\in X$ and for every $x_0\in X$ a sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is convergent to x.*

Proof. Suppose that there exists $n_0\in\mathbb{N}$ such that $x_{n_0}=Tx_{n_0}$. Then x_{n_0} is a fixed point of T and the prove is finished. Hence, we assume that $x_n\neq Tx_n$, i.e. $m_r(x_{n-1},x_n)-m_{r_{x_n,x_{n+1}}}>0$ for all $n\in\mathbb{N}$.
Denote $D(x_n,x_{n+1})=m_r(x_n,x_{n+1})-m_{r_{x_n,x_{n+1}}}$. Then, (3.1) implies that

$$D(x_n,x_{n+1})\leq m_r(x_n,x_{n+1})=\tau m_r(Tx_{n-1},Tx_n)\leq\psi(M(x_{n-1},x_n))\quad for\ \ all\quad n\leq 1,$$

where

$$\begin{aligned}M(x_{n-1},x_n)&=max\{m_r(x_{n-1},x_n),m_r(x_{n-1},Tx_{n-1}),m_r(x_n,Tx_{n+1})\}\\&=max\{m_r(x_{n-1},x_n),m_r(x_{n-1},x_n),m_r(x_n,x_{n+1})\}\\&=max\{m_r(x_{n-1},x_n),m_r(x_n,x_{n+1})\}\end{aligned}$$

If $M(x_{n-1},x_n)=m_r(x_n,x_{n+1})$, we get

$$m_r(x_n,x_{n+1})\leq\psi(m_r(x_n,x_{n+1}))<m_r(x_n,x_{n+1})$$

which is a contradiction.

Hence, $M(x_{n-1},x_n) = m_r(x_{n-1},x_n)$

$$\begin{aligned} m_r(x_n,x_{n+1}) &= m_r(Tx_{n-1},Tx_n) \\ &\le \psi(m_r(x_{n-1},x_n)) \\ &= \psi(m_r(Tx_{n-2},Tx_{n-1})) \\ &\le \psi^2(m_r(x_{n-3},x_{n-2})) \\ &\le .. \\ &.. \le \psi^n(m_r(x_0,x_1)) \longrightarrow 0 \quad as \quad n \longrightarrow \infty, \end{aligned}$$

Then

$$D(x_n,x_{n+1}) \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

Now, we show $\lim_{n\to\infty} D(x_n,x_{n+2}) = 0$

$$D(x_n,x_{n+2}) \le m_r(Tx_{n-1},Tx_{n+1}) \le \tau m_r(Tx_{n-1},Tx_{n+1}) \le \psi(M(x_{n-1},x_{n+1})), \quad forall \quad n \le 1$$

where

$$\begin{aligned} M(x_{n-1},x_{n+1}) &= max\{m_r(x_{n-1},x_{n+1}), m_r(x_{n-1},Tx_{n-1}), m_r(x_{n+1},Tx_{n+1})\} \\ &= max\{m_r(x_{n-1},x_{n+1}), m_r(x_{n-1},x_n), m_r(x_{n+1},x_{n+2})\} \\ &= max\{m_r(x_{n-1},x_{n+1}), m_r(x_{n-1},x_n), m_r(x_{n+1},x_{n+2})\}. \end{aligned}$$

We consider three different cases:

Case (i) if $M(x_{n-1},x_{n+1}) = m_r(x_{n-1},x_{n+1})$, we get

$$D(x_n,x_{n+2}) \le \psi(m_r(x_{n-1},x_{n+1})) \le \psi^{n-1}(m_r(x_0,x_2)) \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

Case (ii) if $M(x_{n-1},x_{n+1}) = m_r(x_{n-1},x_n)$, we get

$$D(x_n,x_{n+2}) \le \psi m_r(x_{n-1},x_n) \le \psi^{n-1} m_r(x_0,x_1) \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

Case (iii) if $M(x_{n-1},x_{n+1}) = m_r(x_{n+1},x_{n+2})$, we get

$$D(x_n,x_{n+2}) \le \psi^{n+1}(m_r(x_0,x_1)) \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

From Case (i)-Case (iii), we get

$$D(x_n, x_{n+2}) \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

We shall prove that $\{x_n\}$ is a rectangular m_r-Cauchy sequence, that is,

$$\lim_{n\to\infty} D(x_n, x_{n+q}) = 0 \quad forall \quad q \in \mathbb{N}$$

Suppose that for some $n, p \in \mathbb{N}$ with $p > n$, we have $x_n = x_p$,

$$\begin{aligned}
m_r(x_n, Tx_n) &= m_r(x_p, Tx_p) \\
&= m_r(x_p, x_{p+1}) \\
&\leq \psi^{p-n}(m_r(x_n, x_{n+1})) \\
&\leq \psi(m_r(x_n, x_{n+1})) < m_r(x_n, x_{n+1}),
\end{aligned}$$

which is a contradiction.

Therefore, $x_n \neq x_p$, for $p \neq n$,

The case $q = 1$ and $q = 2$ is proved. Now we take $q \geq 3$; arbitrary, we distinguish two different cases:

Case (i) Let $q = 2p$, where $p \geq 2$. By the rectangular inequality, we get

$$\begin{aligned}
&D(x_n, x_{n+2p}) \\
&\leq m_r(x_n, x_{n+2p}) \\
&\leq \left[m_r(x_n, x_{n+2}) + m_r(x_{n+2}, x_{n+3}) + m_r(x_{n+3}, x_{n+2p})\right] \\
&\leq m_r(x_n, x_{n+2}) + m_r(x_{n+2}, x_{n+3}) + \left[m_r(x_{n+3}, x_{n+4}) + m_r(x_{n+4}, x_{n+5}) + m_r(x_{n+5}, x_{n+2p})\right] \\
&\leq m_r(x_n, x_{n+2}) + m_r(x_{n+2}, x_{n+3}) + m_r(x_{n+3}, x_{n+4}) + m_r(x_{n+4}, x_{n+5}) + ... + m_r(x_{n+2p-1}, x_{n+2p}) \\
&= m_r(x_n, x_{n+2}) + \sum_{k=n+2}^{n+2p-1} m_r(x_k, x_{k+1}) \\
&\leq m_r(x_n, x_{n+2}) + \sum_{k=n+2}^{n+2m-1} \psi^k(m_r(x_0, x_1)) \\
&\leq m_r(x_n, x_{n+2}) + \sum_{k=n+2}^{\infty} \psi^k(m_r(x_0, x_1)).
\end{aligned}$$

$$\lim_{n\to\infty} m_r(x_n, x_{n+2}) = 0$$

and

$$\sum_{k=n+2}^{\infty} \psi^k(m_r(x_0, x_1)) \to 0 \quad as \quad n \to \infty.$$

Therefore,

$$\lim_{n,p\to\infty} D(x_n, x_{n+2p}) = 0$$

Case (ii) Let $q = 2p+1$, where $p \geq 1$. By the rectangular inequality, we get

$$\begin{aligned}
& D(x_n, x_{n+2p+1}) \\
& \leq m_r(x_n, x_{n+2p+1}) \\
& \leq \left[m_r(x_n, x_{n+1}) + m_r(x_{n+1}, x_{n+2}) + m_r(x_{n+2}, x_{n+2p+1})\right] \\
& \leq m_r(x_n, x_{n+1}) + m_r(x_{n+1}, x_{n+2}) + \left[m_r(x_{n+2}, x_{n+3}) + m_r(x_{n+3}, x_{n+4}) + m_r(x_{n+4}, x_{n+2p+1})\right] \\
& = m_r(x_n, x_{n+1}) + m_r(x_{n+1}, x_{n+2}) + m_r(x_{n+2}, x_{n+3}) + m_r(x_{n+3}, x_{n+4}) + m_r(x_{n+4}, x_{n+2p+1}) \\
& \leq \ldots \\
& \leq m_r(x_n, x_{n+1}) + m_r(x_{n+1}, x_{n+2}) + m_r(x_{n+2}, x_{n+3}) + m_r(x_{n+3}, x_{n+4}) + \ldots \\
& \ldots. + m_r(x_{n+2p}, x_{n+2p+1}) \\
& = \sum_{k=n}^{n+2p} m_r(x_k, x_{k+1}) \\
& \leq \sum_{k=n}^{n+2p} \psi^k(m_r(x_0, x_1)) \\
& \leq \sum_{k=n}^{\infty} \psi^k(m_r(x_0, x_1)) \to 0 \quad as \quad n \to \infty.
\end{aligned}$$

Thus, we obtain

$$\lim_{n,p\to\infty} m_r(x_n, x_{n+2p+1}) = 0.$$

Finally, from Case (i) and case (ii), we get

$$\lim_{n,m\to\infty} D(x_n, x_{n+q}) = 0 \quad for \ \ all \quad q \geq 3$$

Thus, $\{x_n\}$ is rectangular m_r-Cauchy sequence in (X, m_r).

Since (X, m_r) is a complete rectangular M-metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u, \lim_{n\to\infty} D(x_n,u) = 0.$$

Now, we shaw that u is a fixed point of T. Since T is continuous, we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = Tu, \ \ which \ \ gives \ \ \ u = Tu$$

Thus, u is a fixed point of T. Now, we show that the uniqueness of a fixed point of T. Assume that T has two distinct fixed points $z,u \in X$, such that $z = Tz$ *and* $u = Tu$.

From the Condition (3.2), we have

$$m_r(z,u) = m_r(Tz,Tu) \leq \tau m_r(Tz,Tu) \leq \psi(M(z,u))$$

where

$$M(z,u) = max\{m_r(z,u), m_r(u,Tu), m_r(z,Tz)\} = m_r(z,u).$$

Then

$$m_r(z,u) < m_r(z,u)$$

which is contradiction.

Hence, T has a unique fixed point. □

Definition 3.3. Let (X,m_r) be a rectangular M-metric space and $T : X \to X$ be a mapping. T is said to be a $(\tau-\psi)$-contraction on X, if there exist $\psi \in \Psi$ and $\tau > 1$ such that for any $x,y \in X$, $\tau m_r(Tx,Ty) \leq \psi(M(x,y))$ for all $x,y \in X$,

where

$$M(x,y) = max\{m_r(x,y), m_r(x,Tx), \frac{m_r(x,Tx)m_r(x,Ty)}{1+m_r(x,Ty)+m_r(y,Tx)}\}$$

Theorem 3.4. *Let (X,m_r) be a complete rectangular M-metric space and let $T : X \to X$ be a continuous $(\tau-\psi)$-contraction. Then, T has a unique fixed point $x \in X$ and for every $x_0 \in X$ a sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is convergent to x.*

Proof. Suppose that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = Tx_{n_0}$. Then x_{n_0} is a fixed point of T and the prove is finished. Hence, we assume that $x_n \neq Tx_n$, i.e. $m_r(x_{n-1},x_n) - m_{r_{x_n,x_{n+1}}} > 0$ for all $n \in \mathbb{N}$. Denote $D(x_n,x_{n+1}) = m_r(x_n,x_{n+1}) - m_{r_{x_n,x_{n+1}}}$.

Then, (3.1) implies that

$D(x_n,x_{n+1}) \leq m_r\,(x_n,x_{n+1}) = \tau m_r\,(Tx_{n-1},Tx_n) \leq \psi(M(x_{n-1},x_n))$ for all $n \leq 1$,

where

$$
\begin{aligned}
M(x_{n-1},x_n) &= \max\{m_r(x_{n-1},x_n), m_r(x_{n-1},Tx_{n-1})\frac{d(x_{n-1},Tx_{n-1})m_r(x_{n-1},Tx_n)}{1+m_r(x_{n-1},Tx_n)+m_r(x_n,Tx_{n-1})}\} \\
&= \max\{m_r(x_{n-1},x_n), m_r(x_{n-1},x_n), \frac{m_r(x_{n-1},x_n)m_r(x_{n-1},x_{n-1})}{1+m_r(x_{n-1},x_{n-1})+m_r(x_n,x_{n-1})}\} \\
&= m_r(x_{n-1},x_n)
\end{aligned}
$$

$M(x_{n-1},x_n) = m_r(x_{n-1},x_n)$

$$
\begin{aligned}
m_r(x_n,x_{n+1}) &= m_r(Tx_{n-1},Tx_n) \\
&\leq \psi(m_r(x_{n-1},x_n)) \\
&= \psi(m_r(Tx_{n-2},Tx_{n-1})) \\
&\leq \psi^2(m_r(x_{n-3},x_{n-2})) \\
&\leq .. \\
&.. \leq \psi^n(m_r(x_0,x_1)) \longrightarrow 0 \quad as \quad n \longrightarrow \infty,
\end{aligned}
$$

Then

$$D(x_n,x_{n+1}) \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

similarly we show that

$$\lim_{n\to\infty} m_r\,(x_n,x_{n+2}) = 0.$$

Similar to the proof of Theorem 3.2, we get

$$\lim_{n\to\infty} D(x_n,x_{n+q}) = 0 \quad for\;\; all \quad q \in \mathbb{N}$$

$$\lim_{n,m\to\infty} D(x_n,x_{n+q}) = 0 \quad for\;\; all \quad q \geq 3$$

Thus, $\{x_n\}$ is rectangular m_r-Cauchy sequence in (X,m_r).

Since (X,m_r) is a complete rectangular M-metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u, \lim_{n\to\infty} D(x_n,u) = 0.$$

Now, we shaw that u is a fixed point of T. Since T is continuous, we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = Tu, \quad whichgives \quad u = Tu$$

Thus, u is a fixed point of T. Now, we show that the uniqueness of a fixed point of T. Assume that T has two distinct fixed points $z, u \in X$, such that $z = Tz \; and \; u = Tu$.

From the Condition (3.4), we have

$$m_r(z,u) = m_r(Tz,Tu)) \leq \tau m_r(Tz,Tu) \leq \psi(M(z,u))$$

where

$$M(z,u) = \max\{m_r(z,u), m_r(u,Tu), \frac{m_r(z,Tz)m_r(z,tu)}{1+m_r(z,Tu)+m_r(u,Tz)}\} = m_r(z,u).$$

Then

$$m_r(z,u) < m_r(z,u)$$

which is contradiction. Hence, T has a unique fixed point. □

Example 3.5. Let $X = \left[1, \frac{4}{3}\right]$. Define $m_r : X \times X \to [0,\infty)$ by

$$m_r(a,b) = \frac{|a-b|}{2}$$

and

$$\psi(t) = \frac{3t}{4}, \tau = \frac{3}{2}.$$

Then (X, m_r) is complete rectangular M-metric space, $\tau > 1$ and $\psi \in \Psi$.

Define $T : X \to X$ by

$$T(t) = \frac{1+t}{2},$$

$1 \leq b \leq a.$

$$m_r(Ta,Tb) = \frac{\frac{1+a}{2} - \frac{1+b}{2}}{2} = \frac{a-b}{4}.$$

Since $a, b \in \left[1, \frac{4}{3}\right]$, then

$$\frac{3(a-b)}{8} \leq \psi(m_r(a,b))$$

Thus

$$\tau m_r(Ta,Tb) \le \psi(m_r(a,b)).$$

Hence, the condition (3.2) and (3.4) is satisfied. Therefore, T has a unique fixed point $z = 1$.

4. Application to Nonlinear Integral Equations

In this section, we apply Theorems 3.2 and 3.4 to prove the existence and uniqueness of the integral equation of Fredholm type:

$$u(t) = \nu \int_m^n H(t,r,u(t))dr, \tag{4.1}$$

where $m,n \in \mathbb{R}$, $u \in C([m,n],\mathbb{R})$ and $H : [m,n]^2 \times \mathbb{R} \to \mathbb{R}$ is continuous function and $\nu \in]0,1[$.

Theorem 4.1. *Suppose the function h such that* $|H(t,r,u(t)) - H(t,r,v(t))| \le \frac{1}{n-m}(|u(t)-v(t)|)$ $\forall\ t,r \in [m,n]$ *and* $u,v \in \mathbb{R}$. *Then the equation (4.1) has a unique solution* $u \in C([m,n],\mathbb{R})$.

Proof. Let $X = C([m,n],\mathbb{R})$ and $T : X \to X$ defined by

$$T(u)(t) = \nu \int_m^n H(t,r,u(t))dr,$$

$\forall\ u \in X$. Clearly, T is a complete M-rectangular metric space.

Let $m_r : X \times X \to [0,+\infty[$ given by

$$m_r(u,v) = \left(\sup_{t\in[m,n]} \frac{|u(t)-v(t)|}{2}\right).$$

Then (X,m_r) is a complete generalized metric space. Assume that, $u,v \in X$ and $t,r \in [m,n]$. Then we get

$$\begin{aligned}|Tu(t) - Tv(t)| &= \frac{|\nu|\left(|\int_m^n H(t,r,u(t))dr - \int_m^n H(t,r,v(t))dr|\right)}{2}\\ &= |\nu| \frac{|\int_m^n (H(t,r,u(t)) - H(t,r,v(t)))dr|}{2}\\ &\le |\nu| \int_m^n \frac{|H(t,r,u(r)) - H(t,r,v(r))|}{2} dr\end{aligned}$$

$$\leq |\nu| \int_m^n \frac{((|u(t)-v(t)|))}{2(n-m)} dr$$
$$\leq |\nu| \frac{|u(t)-v(t)|}{2}$$

Thus

$$\sup_{t\in[m,n]} \frac{|Tu(t)-Tv(t)|}{2} \leq \sup_{t\in[m,n]} |\nu| \frac{|u(t)-v(t)|}{2}$$

Hence

$$\tau\left(m_r(Tu,Tv)\right) \leq \left[\psi\left(M(u,v)\right)\right], \tag{4.2}$$

for all $u,v \in X$ with $\psi(t) = \dfrac{3t}{4}$ and $\nu = |\frac{1}{\tau}|$.
Then T satisfies the condition (3.2) and (3.4) is hold. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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