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BEST PROXIMITY POINT THEOREMS FOR PROXIMAL POINTWISE TRICYCLIC CONTRACTION

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Abstract. This article provides an introduction to the topics of proximal pointwise tricyclic contraction (PPTC) and best proximity point (BPP) existence in a weakly compact convex subset triad.

Keywords: proximal pointwise; best proximity point; weakly compact convex; tricyclic contraction; triad normal structure.

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In the year 1970, Kirk [5] introduced the notion the pointwise contraction mappings and established the existence of a fixed point for a pointwise contraction mapping on a weak* compact convex subset of a conjugate Banach space. Back in 2003 [3] Kirk, W.A., Srinivasan, P.S., and Veeramani, P. introduced the class of cyclical contractive mapping and proved fixed point

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results for this class of mappings. In 2019 J. Anuradha and P. Veeramani. [1] introduced the notion of proximal pointwise contraction and studied the existence of the best proximity point on a pair of weakly compact convex subsets of a Banach space. In [4] introduced the notion of tricyclic contractions mapping and used it to study the existence of the best proximity point for a tricyclic contractions mapping. we introduce a class of mappings called pointwise tricyclic contractions on $X \cup Y \cup Z$

1. Introduction

In this section, we give some basic definitions and concepts that are useful and related to the context of our results.

Let (E,d) be a metric space and let X,Y and Z be nonempty subsets of E.

A mapping $T : X \cup Y \cup Z \to X \cup Y \cup Z$ is said to be a tricyclic mapping provided that

$$T(X) \subseteq Y,\ T(Y) \subseteq Z \text{ and } T(Z) \subseteq X \tag{1}$$

In [4], M.Aamri, T. Sabar, and A.Bassou established new fixed point theorems

Theorem 1. *Suppose that (X,Y,Z) is a nonempty and closed triad of subsets of a complete metric space (E,d) and $T : X \cup Y \cup Z \to X \cup Y \cup Z$ is tricyclic mapping for which there exists $k \in]0,1[$ such that $\Delta(Tx,Ty,Tz) \leq k\Delta(x,y;z)$ for all $(x,y,z) \in X \times Y \times Z$.*

where the mapping

$$\Delta \quad : \quad X \times Y \times Z \to [0,+\infty) \tag{2}$$

$$\Delta(x,y;z) \quad \to \quad d(x,y) + d(y,z) + d(z,x) \tag{3}$$

Then $X \cap Y \cap Z$ is non empty and T has a unique fixed point in $X \cap Y \cap Z$.

Definition 2. *Let (E,d) be a metric space and let X,Y and Z be nonempty subsets of E.*

Let $T : X \cup Y \cup Z \to X \cup Y \cup Z$ be a tricyclic mapping. A point $x \in X \cup Y \cup Z$ is said to be the best proximity point for T if

$$\Delta\left(x,Tx,T^2x\right) = \delta(X,Y,Z) \tag{4}$$

Definition 3. *Let (X,Y,Z) be a nonempty triad subsets of a metric space (E,Δ). Let $T: X \cup Y \cup Z \to X \cup Y \cup Z$ be a tricyclic mapping. Then a sequence (x_n) in $X \cup Y \cup Z$ is said to be an approximate best proximity point sequence for T if*

$$\Delta\left(x_n, Tx_n, T^2x_n\right) \to \delta(X,Y,Z) \text{ as } n \to \infty \tag{5}$$

2. Preliminaries and Main Results

The concept of proximal pointwise tricyclic contraction (PPTC) is an extension of the pointwise cyclic contraction concept introduced in Theorem 2.1 of a paper referenced as [1]. The PPTC concept is used to establish the existence of best proximity points in weakly compact convex triads of a Banach space.

Definition 4. *Let (X,Y,Z) be a nonempty triad of subsets of a metric space (E,d). A mapping $T: X \cup Y \cup Z \to X \cup Y \cup Z$ is said to be a pointwise tricyclic contraction if*

it satisfies

i) $T(X) \subseteq Y$, $T(Y) \subseteq Z$ and $T(Z) \subseteq X$

ii) For each $(x,y,z) \in X \times Y \times C$ there exists $\alpha(x), \alpha(y), \alpha(z)$ in $(0,1)$ such that

$$\begin{aligned}
\Delta(Tx,Ty,Tz) &\leq \alpha(x)\Delta(x,y;z) + (1-\alpha(x))\Delta(x,y;z) \text{ for } (y,z) \in Y \times Z \\
\Delta(Tx,Ty,Tz) &\leq \alpha(y)\Delta(x,y;z) + (1-\alpha(y))\Delta(x,y;z) \text{ for } (x,z) \in X \times Z \\
\Delta(Tx,Ty,Tz) &\leq \alpha(z)\Delta(x,y;z) + (1-\alpha(z))\Delta(x,y;z) \text{ for } (x,y) \in X \times Y
\end{aligned}$$

Definition 5. *Let X, Y and Z be nonempty subsets of a normed linear space is said to be a proximal triad if for each $(x,y,z) \in X \times Y \times Z$ there exists $(x',y',z') \in X \times Y \times Z$ such that*

$$\Delta\left(x',y,z\right) = \Delta(x,y',z) = \Delta(x,y,z') = \delta(X,Y,Z)$$

In [2], Eldred et al. showed that a metric space with proximal normal structure has the property that any relatively nonexpansive mapping on the space has a best proximity point, which is a point that is closest to the fixed point set of the mapping in a certain sense. This result has important implications for the study of fixed point theory and optimization, as it provides a way to ensure the existence of a "best" solution to certain problems.

Definition 6. *A convex triad $(K_1;K_2;K_3)$ in a Banach space is said to have proximal triad normal structure if for any closed bounded and convex proximal triad $(H_{1;}H_{2;}H_3) \subseteq (K_1;K_2;K_3)$ for which $\Delta(H_{1;}H_{2;}H_3) = \Delta(K_1;K_2;K_3)$ and $\delta(H_{1;}H_{2;}H_3) > \Delta(H_{1;}H_{2;}H_3)$ there exists $(x_1,x_2,x_3) \in H_1 \times H_2 \times H_3$ such that*

$$\delta(x_{1;}H_{2;}H_3) < \delta(H_{1;}H_{2;}H_3),\ \delta(x_{2,}H_{1,}H_3) < \delta(H_{1;}H_{2;}H_3), \delta(x_{3,}H_{1,}H_2) < \delta(H_{1;}H_{2;}H_3)$$

Then $x_{1,}$ is a nondiametral point of $H_{1,}$ and $x_{2,}$ is a nondiametral point of $H_{2,}$ and $x_{3,}$ is a nondiametral point of $H_{3,}$

The triad $(x,y,z) \in X \times Y \times Z$ is said to be proximal in (X,Y,Z) if $\Delta(x,y,z) = \delta(X,Y,Z)$ we set

$$\begin{aligned} X_0 &= \{x_1 \in X : \Delta(x_1,y_2,z_3) = \delta(X,Y,Z), \text{ for some } (y_2,z_3) \in Y \times Z\} \\ Y_0 &= \{y_1 \in Y : \Delta(x_3,y_1,z_2) = \delta(X,Y,Z), \text{ for some } (x_3,z_2) \in X \times Z\} \\ Z_0 &= \{z_1 \in Z : \Delta(x_2,y_3,z_1) = \delta(X,Y,Z), \text{ for some } (x_2,y_3) \in X \times Y\} \end{aligned}$$

Clearly, $\delta(X_0,Y_0,Z_0) = \delta(X,Y,Z)$

Theorem 7. *Let X, Y and Z be nonempty weakly compact convex subsets in a Banach space and T is a pointwise tricyclic contraction mapping . Then T has a best proximity point.*

Proof. We saw that (X_0,Y_0,Z_0) is a nonempty closed convex triad satisfying

$$\delta(X_0,Y_0,Z_0) = \delta(X,Y,Z), TX_0 \subset Y_0, TY_0 \subset Z_0 \text{ and } TZ_0 \subset X_0.$$

Let

$$\Gamma = \left\{ K \subset X \cup Y \cup Z \Big/ \begin{array}{c} K \cap X_0, K \cap Y_0 \text{ and } K \cap Z_0 \text{ are nonempty closed and convex subests of } X \\ T(K \cap X_0) \subset K \cap Y_0,\ T(K \cap Y_0) \subset K \cap Z_0, T(K \cap Z_0) \subset K \cap X_0 \text{ and} \\ \delta(K \cap X_0, K \cap Y_0, K \cap Z_0) = \delta(X,Y,Z) \end{array} \right\}$$

is nonempty as $X_0 \cup Y_0 \cup Z_0 \in \Gamma$.

So, applying Zorn's lemma has a minimal element with respect to inclusion order, say K. Let

$$K_1 = K \cap X_0, K_2 = K \cap Y_0 \text{ and } K_3 = K \cap Z_0$$

$Fix\,(x,y,z) \in K_1 \times K_2 \times K_3$ such that $\Delta(x,y,z) = \delta(K_{1,}K_{2,}K_3)$.

If $\delta(x_{,}K_{2,}K_3) = \delta(X,Y,Z)$ then

$$\delta(X,Y,Z) \le \Delta\left(Tx,T^2x,T^3x\right) \le \Delta\left(x,Tx,T^2x\right) \le \delta(x_{,}K_{2,}K_3) = \delta(X,Y,Z)$$

that is, the triad $\left(x,Tx,T^2x\right)$ satisfies the conclusion. Similarly, we can prove the triad $\left(y,Ty,T^2y\right)$ satisfies the conclusion if $\delta(y,K_{1,}K_3) = \delta(X,Y,Z)$ and the triad $\left(z,Tz,T^2z\right)$ satisfies the conclusion if $\delta(z,K_{1,}K_2) = \delta(X,Y,Z)$.

So it suffices to prove if $\delta_1 = \delta(x_{,}K_{2,}K_3) > \delta(X,Y,Z)$, $\delta_2 = \delta(y,K_{1,}K_3) > \delta(X,Y,Z)$ and $\delta_3 = \delta(z,K_{1,}K_2) > \delta(X,Y,Z)$.

Set

$$\begin{aligned}
H_x &= \left\{x_1 \in K_1 : \Delta\left(x_1,Tx,T^2x\right) \le \alpha(x)\,\delta_1 + (1-\alpha(x))\,\delta(X,Y,Z)\right\} \\
H_y &= \left\{y_1 \in K_2 : \Delta\left(y_1,Ty,T^2y\right) \le \alpha(y)\,\delta_2 + (1-\alpha(y))\,\delta(X,Y,Z)\right\} \\
H_z &= \left\{z_1 \in K_3 : \Delta\left(z_1,Tz,T^2z\right) \le \alpha(z)\,\delta_3 + (1-\alpha(z))\,\delta(X,Y,Z)\right\}
\end{aligned}$$

Now let $(Tx,Ty,Tz) \in K_2 \times K_3 \times K_1$ and

$$\delta(X,Y,Z) \le \Delta(Tx,Ty,Tz) \le \Delta(x,y,z) = \delta(X,Y,Z).$$

Hence $(Tx,Ty,Tz) \in H_y \times H_z \times H_x$ for $x_i \in H_x$, $i = 1,2$ and $\lambda \in (0,1)$

$$\begin{aligned}
&\Delta\left(\lambda x_1 + (1-\lambda)x_2,Tx,T^2x\right) \\
= \;& \Delta(\lambda x_1 + (1-\lambda)x_2,Tx) + \Delta\left(Tx,T^2x\right) + \Delta\left(\lambda x_1 + (1-\lambda)x_2,T^2x\right) \\
\le \;& \lambda\Delta(x_1,Tx) + (1-\lambda)\Delta(x_2,Tx) + \Delta\left(Tx,T^2x\right) + \lambda\Delta\left(x_1,T^2x\right) \\
& + (1-\lambda)\Delta\left(x_2,T^2x\right) \\
\le \;& \lambda\left(\Delta(x_1,Tx) + \Delta\left(x_1,T^2x\right) + \Delta\left(Tx,T^2x\right)\right) \\
& + (1-\lambda)\left(\Delta(x_2,Tx) + \Delta\left(x_2,T^2x\right) + \Delta\left(Tx,T^2x\right)\right) \\
= \;& \lambda\Delta\left(x_1,Tx,T^2x\right) + (1-\lambda)\Delta\left(x_2,Tx,T^2x\right) \\
\le \;& \alpha(x)\,\delta_1 + (1-\alpha(x))\,\delta(X,Y,Z)
\end{aligned}$$

Then if $\{x_n\}_{n=1}^{\infty} \subset H_x$, with $x_n \overset{w}{\to} x'$, then $x' \in K_1$.

Now $\Delta\left(x', Tx, T^2x\right) \leq \liminf_n \Delta\left(x_n, Tx, T^2x\right) \leq \alpha(x)\delta_1 + (1-\alpha(x))\delta(X,Y,Z)$.

Also for $x_0 \in H_x, \Delta(x_0, y, Ty) \leq \delta_2$ and hence $\Delta\left(Tx_0, Ty, T^2y\right) \leq \alpha(y)\delta_2 + (1-\alpha(y))\delta(X,Y,Z)$.

Therefore H_x is a nonempty closed and convex subset of X with $TH_x \subset H_y$,Similarly, one can verify that H_y and H_z is a nonempty closed convex subset of X and $TH_y \subset H_z$ and $TH_z \subset H_x$.

By the minimality of K then $H_x = K_1$, $H_y = K_2$ and $H_z = K_3$.

Now we show that $\delta(Ty, K_1, K_2) < \delta_2$ and $\delta(Tx, K_1, K_3) < \delta_1$ and $\delta(Tz, K_2, K_3) < \delta_3$ for $x_1 \in K_1$, $\Delta\left(x_1, Tx, T^2x\right) \leq \alpha(x)\delta_1 + (1-\alpha(x))\delta(X,Y,Z) < \delta_1$ as $\delta(X,Y,Z) < \delta_1$ therefore $\delta(Tx, K_1, K_3) < \delta_1$. Similarly, $\delta(Ty, K_1, K_2) < \delta_2$ and $\delta(Tz, K_2, K_3) < \delta_3$.If

$\delta(Ty, K_1, K_2) = \delta(X,Y,Z)$ and $\delta(X,Y,Z) < \Delta\left(y, T^2y, T^3y\right) \leq \Delta\left(y, Ty, T^2y\right) \leq \delta(Ty, K_1, K_2) = \delta(X,Y,Z)$, that is, the triad $\left(y, T^2y, T^3y\right)$ satisfies the conclusion. In a similar fashion, one can prove that the triad $\left(x, T^2x, T^3x\right)$ and $\left(z, T^2z, T^3z\right)$ satisfies the conclusion, if $\delta(Tx, K_1, K_3) = \delta(X,Y,Z)$ and $\delta(Tz, K_2, K_3) = \delta(X,Y,Z)$.

Suppose

$$\delta(Ty, K_1, K_2) > \delta(X,Y,Z) \text{ and } \delta(Tz, K_2, K_3) > \delta(X,Y,Z) \text{ and } \delta(Tx, K_1, K_3) > \delta(X,Y,Z)$$

then by the similar fashion one can show that:

$$\delta\left(T^2y, K_2, K_3\right) < \delta_1 \text{ and } \delta\left(T^2x, K_1, K_2\right) < \delta_3 \text{ and } \delta\left(T^2z, K_1, K_3\right) < \delta_2.$$

That is (K_1, K_2, K_3) has the proximal normal structure.

Since (K_1, K_2, K_3) is a proximal triad in (X_0, Y_0, Z_0).

By proximal normal structure there exist $(x_1, x_2, x_3) \in K_1 \times K_2 \times K_3$ and $\beta \in (0,1)$ such that

$$\begin{cases} \delta(x_1, K_2, K_3) \leq \beta\delta(K_1, K_2, K_3) \\ \delta(x_2, K_1, K_3) \leq \beta\delta(K_1, K_2, K_3) \\ \delta(x_3, K_1, K_2) \leq \beta\delta(K_1, K_2, K_3) \end{cases}$$

Since (K_1, K_2, K_3) is a proximal triad there exists $\left(x_1', x_2', x_3'\right) \in K_1 \times K_2 \times K_3$ such that

$$\Delta\left(x_1', x_2, x_3\right) = \Delta(x_1, x_2', x_3) = \Delta(x_1, x_2, x_3') = \delta(K_1, K_2, K_3)$$

So for any $(y,z) \in K_2 \times K_3$

$$\begin{aligned}
\Delta\left(\frac{x_1+x_1'}{2},y,z\right) &= \Delta\left(\frac{x_1+x_1'}{2},y\right)+\Delta(y,z)+\Delta\left(z,\frac{x_1+x_1'}{2}\right) \\
&\leq \frac{1}{2}\Delta(x_1,y)+\frac{1}{2}\Delta\left(x_1',y\right)+\Delta(y,z)+\frac{1}{2}\Delta(z,x_1)+\frac{1}{2}\Delta\left(z,x_1'\right) \\
&= \frac{1}{2}\left(\Delta(x_1,y)+\Delta(z,x_1)+\Delta(y,z)\right)+\frac{1}{2}\left(\Delta\left(z,x_1'\right)+\Delta\left(x_1',y\right)+\Delta(y,z)\right) \\
&\leq \frac{1}{2}\Delta(x_1,y,z)+\frac{1}{2}\Delta\left(x_1',y,z\right) \\
&\leq \beta\delta(K_1,K_2,K_3)/2+\delta(K_1,K_2,K_3)/2=\alpha\delta(K_1,K_2,K_3)
\end{aligned}$$

where $\alpha = \frac{1+\beta}{2} \in (0,1)$

Let $y_1 = \frac{x_1+x_1'}{2}$, and similarly $y_2 = \frac{x_2+x_2'}{2}$, $y_3 = \frac{x_3+x_3'}{2}$.

Then

$$\begin{cases}
\delta(y_1,K_2,K_3) \leq \alpha\delta(K_1,K_2,K_3) \\
\delta(y_2,K_1,K_3) \leq \alpha\delta(K_1,K_2,K_3) \\
\delta(y_3,K_1,K_2) \leq \alpha\delta(K_1,K_2,K_3)
\end{cases} \tag{6}$$

and

$$\Delta(y_1,y_2,y_3) = \delta(K_1,K_2,K_3).$$

Define

$$\begin{aligned}
M_1 &= \left\{x \in K_1 : \delta(x,K_2,K_3) \leq \alpha\delta(K_1,K_2,K_3)\right\} \\
M_2 &= \left\{y \in K_2 : \delta(y,K_1K_3) \leq \alpha\delta(K_1,K_2,K_3)\right\} \\
M_3 &= \left\{z \in K_3 : \delta(x,K_1,K_2) \leq \alpha\delta(K_1,K_2,K_3)\right\}
\end{aligned}$$

Since $(y_1,y_2,y_3) \in M_1 \times M_2 \times M_3$, M_i is a nonempty closed and convex subset of K_i and $\Delta(M_1,M_2,M_3) = \Delta(K_1,K_2,K_3)$

Now let $x \in M_1$, $y \in M_2$, $z \in K_3$,then

$$\begin{aligned}
\Delta(Tx,Ty,Tz) &\leq \alpha(x)\Delta(x,y;z)+(1-\alpha(x))\Delta(K_1,K_2,K_3) \\
&\leq \Delta(x,y;z) \\
&\leq \delta(x,K_2,K_3) \leq \alpha\delta(K_1,K_2,K_3)
\end{aligned}$$

Thus, we get

$$T\left(K_3\right) \in B\left(Tx, Ty, \alpha\delta\left(K_{1,}K_{2,}K_3\right)\right) \cap K_1 = K_1^{'}$$

Clearly $K_1^{'}$ is closed and convex. Since $M_1 \subseteq K_1$,implies $x \in M_1$ and there exists $y \in M_2$ and $z \in M_3$ satisfies $\Delta(x, y; z) = \Delta(x, y; z)$.

Hence

$$\Delta(Tx, Ty, Tz) = \Delta(x, y, z), \;\; z \in K_3$$

implies $Tz \in M_1$ and thus

$$\Delta\left(K_1', K_2, K_3\right) = \Delta(K_1, K_2, K_3)$$

Therefore $K_1' \cup K_2 \cup K_3 \in \Gamma$. Hence by minimality, $K_1' = K_1$.

Thus

$$K_1 \subseteq B\left(Tx, Ty, \alpha\delta\left(K_{1,}K_{2,}K_3\right)\right)$$

So for any $u \in M_1$

$$\Delta(u, Tx, Ty) \leq \alpha\delta\left(K_{1,}K_{2,}K_3\right) \text{ implies } \delta\left(Tu, K_{1,}K_3\right) \leq \alpha\delta\left(K_{1,}K_{2,}K_3\right)$$

which shows $T\left(M_1\right) \subseteq M_2$.

In a similar manner we can see $T\left(M_2\right) \subseteq M_3$ and $T\left(M_3\right) \subseteq M_1$.

Hence $K_1' \cup K_2 \cup K_3 \in \Gamma$. But $\delta\left(M_{1,}M_{2,}M_3\right) \leq \alpha\delta\left(K_{1,}K_{2,}K_3\right)$and this contradicts the minimality of K.

□

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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