SOME FIXED POINTS RESULTS USING $(\psi, \phi)$ GENERALIZED WEAKLY CONTRACTIVE MAP ON A GENERALIZED 2-METRIC SPACE

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Abstract. The main purpose of this paper is to define a generalized 2-metric and prove the existence and uniqueness of fixed points for $(\psi, \phi)$ generalized weakly contractive mappings in a generalized 2-metric space.

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1. INTRODUCTION

The study of fixed point theorems for self maps satisfying different contraction conditions is the center of rigorous research activities. Dutta et al. introduced $(\psi, \phi)$-weakly contractive maps in 2008 and obtained some fixed point results for such contractions, [4]. Later, G. V. R. Babu et al. introduced $(\psi, \phi)$-almost weakly contractive maps in $G$-metric space, [1]. Fixed points of contractive maps on $S$-metric spaces were studied by several authors and since then, several contractions have been considered for proving fixed point theorems, [6, 2, 3, 10]. The authors D. Venkatesh et al. further proved some fixed point outcomes in $S_h$-metric spaces using $(\psi, \phi)$-generalized weakly contractive maps in $S_h$-metric spaces, [7].
The concept of an area of a triangle in $\mathbb{R}^2$ inspired, Gähler to introduced the concept of a 2-metric, as a generalization of the metric, [5].

**Definition 1.1.** [5] Let $X$ be a non-empty set and $d : X \times X \times X \to [0, \infty)$ be a map satisfying the following properties

(i) If $x, y, z \in X$ then $d(x, y, z) = 0$ only if at least two of the three points are the same.

(ii) For $x, y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

(iii) symmetry property: for $x, y, z \in X$,

$$d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x).$$

(iv) rectangle inequality:

$$d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t)$$

for $x, y, z, t \in X$.

Then $d$ is a 2-metric and $(X, d)$ is a 2-metric space.

**Definition 1.2.** Let $X$ be a non-empty set and $d : X \times X \times X \to [0, \infty)$ be a map satisfying the following properties:

(i) If $x, y, z \in X$ then $d(x, y, z) = 0$ only if at least two of the three points are the same.

(ii) For $x, y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

(iii) symmetry property: for $x, y, z \in X$,

$$d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x).$$

(iv) modified rectangle inequality: there exists $\alpha, \beta, \gamma \geq 1$ such that

$$d(x, y, z) \leq \alpha d(x, y, t) + \beta d(y, z, t) + \gamma d(z, x, t)$$

for $x, y, z, t \in X$.

Then $d$ is a generalized 2-metric and $(X, d)$ is a generalized 2-metric space.

If $\alpha = \beta = \gamma = 1$ then a generalized 2-metric is a 2-metric.
**Definition 1.3.** Let \((X, d)\) be a generalized 2-metric space. Let \(x, y \in X\) and \(\varepsilon > 0\). Then the subset

\[ B_{\varepsilon}(x, y) = \{ z \in X ; d(x, y, z) < \varepsilon \} \]

of \(X\) is called a generalized 2-ball centered at \(x, y\) with radius \(\varepsilon\). A topology can be generated in \(X\) by taking the collection of all generalized 2-balls as a subbasis, which we call the generalized 2-metric topology and is denoted by \(\tau\). Thus \((X, \tau)\) is a generalized 2-metric topological space. Members of \(\tau\) are called 2-open sets. From the property of the metric is can easily be seen that \(B_{\varepsilon}(x, y) = B_{\varepsilon}(y, x)\) for \(\varepsilon > 0\).

**Example 1.4.** Let \(X = [0, 1]\) and define

(1) \[ d(x, y, z) = \begin{cases} 0 & \text{, only if at least two of the three points are the same} \\ e^{\frac{1}{2}|x-y| + \frac{1}{3}|y-z| + \frac{1}{6}|z-x|} & \text{, otherwise} \end{cases} \]

For \(x, y, z \in X\) and using Jensens’ inequality, we get

\[
 d(x, y, z) \\
 = e^{\frac{1}{2}|x-y| + \frac{1}{3}|y-z| + \frac{1}{6}|z-x|} \\
 = e^{\frac{1}{2}|x-y| + \frac{1}{3}|y-z| + \frac{1}{6}|z-x|} e^{\frac{1}{2}|x-y| + \frac{2}{3}|y-z| + \frac{5}{6}|z-x|} \\
 \leq e^{2e^{\frac{1}{2}|x-y| + \frac{1}{3}|y-z| + \frac{1}{6}|z-x|}} \\
 \leq e^{2\left\{ \frac{1}{2}e^{|x-y|} + \frac{1}{3}e^{y-z} + \frac{1}{6}e^{z-x} \right\}} \\
 \leq e^{2\left\{ \frac{1}{2}e^{|x-y| + |y-t| + |t-x|} + \frac{1}{3}e^{|y-z| + |y-t| + |t-z|} + \frac{1}{6}e^{z-x + |x-t| + |t-z|} \right\}} \\
 = \alpha d(x, y, t) + \beta d(z, y, t) + \gamma d(z, x, t)
\]

where \(\alpha = \frac{1}{2}e^{2} \geq 1\), \(\beta = \frac{1}{3}e^{2} \geq 1\) and \(\gamma = \frac{1}{6}e^{2} \geq 1\). It follows that \(d\) is a generalized 2-metric.

**Definition 1.5.** Let \(\{x_n\}_{n \in \mathbb{N}}\) be a sequence in a generalized 2-metric space \((X, d)\).

a) the sequence \(\{x_n\}_{n \in \mathbb{N}}\) is convergent to \(x \in X\) iff for all \(\xi \in X\),

\[
 \lim_{n \to \infty} d(x_n, x, \xi) = 0.
\]
b) the sequence \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \) iff for all \( \xi \in X \),

\[
\lim_{n,m \to \infty} d(x_n, x_m, \xi) = 0.
\]

2. Main Result

**Definition 2.1.** [8] A function \( \psi : [0, \infty) \to [0, \infty) \) is an altering distance function if it satisfies:

(i) \( \psi \) is continuous and non-decreasing.

(ii) \( \psi(t) = 0 \iff t = 0 \).

To this prove uniqueness and existence of a fixed point the definition was amended to include:

(iii) \( \psi \) is sublinear function.

Denote the class of all altering distances functions by \( \Psi \).

**Definition 2.2.** Let \((X, d)\) be a generalized \( 2 \)-metric space and \( T : X \to X \) is a contraction if there exists \( 0 \leq \lambda < 1 \) such that

\[
d(Tx, Ty, \xi) \leq \lambda d(x, y, \xi)
\]

for all \( x, y, \xi \in X \).

In [9], authors have proved a similar result in a \( b_2 \) metric space with the additional property that the set is partially ordered.

**Definition 2.3.** Let \((X, d)\) be a generalized \( 2 \)-metric space and a mapping \( T : X \to X \) is a \((\psi, \phi)\) generalized almost weakly contractive mapping if it satisfies the inequality

\[
\beta \psi(d(Tx, Ty, \xi)) \\
\leq \psi \left( \max \left\{ d(x, y, \xi), \frac{d(x, Tx, \xi)d(y, Ty, \xi)}{1 + d(x, y, \xi)}, \frac{d(x, Tx, \xi)d(y, Ty, \xi)}{1 + d(Tx, Ty, \xi)} \right\} \right) \\
- \phi \left( \max \left\{ d(x, y, \xi), \frac{d(x, Tx, \xi)d(y, Ty, \xi)}{1 + d(x, y, \xi)}, \frac{d(x, Tx, \xi)d(y, Ty, \xi)}{1 + d(Tx, Ty, \xi)} \right\} \right) \\
+ \mu \min \{d(x, Tx, \xi), d(x, Ty, \xi), d(y, Tx, \xi), d(y, Ty, \xi)\}
\]

(2)

where \( x, y, z \in X, \mu \geq 0 \) and \( \psi, \phi \in \Psi \).

**Theorem 2.4.** Let \((X, d)\) be a generalized complete \( 2 \)-metric space and \( T : X \to X \) be a \((\psi, \phi)\) generalized almost weakly contractive mapping. Then \( T \) has a unique fixed point.
Proof. Let \( x_0 \in X \) and define a sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( X \) by

\[
x_n = T x_{n-1},
\]

for all \( n \in \mathbb{N} \). If \( x_n = x_{n+1} \) for some \( n \in \mathbb{N} \) then we have a fixed point. We assume that \( x_n \neq x_{n+1} \) and we shall show that the sequence \( \{d(x_n, x_{n+1}, \xi)\}_{n \in \mathbb{N}} \) is a decreasing sequence of real numbers. By (2), we get

\[
\psi (d(x_n, x_{n+1}, \xi)) = \psi (d(T x_{n-1}, T x_n, \xi)) \leq \frac{1}{\beta} \left( \psi \left( \max \left\{ d(x_{n-1}, x_n, \xi), \frac{d(x_{n-1}, T x_{n-1}, \xi) d(x_n, T x_n, \xi)}{1 + d(T x_{n-1}, T x_n, \xi)} \right\} \right) - \phi \left( \max \left\{ d(x_{n-1}, x_n, \xi), \frac{d(x_{n-1}, T x_{n-1}, \xi) d(x_n, T x_n, \xi)}{1 + d(T x_{n-1}, T x_n, \xi)} \right\} \right) + \mu \min \left\{ d(x_{n-1}, T x_n, \xi), d(x_n, T x_n, \xi), d(x_{n-1}, T x_{n-1}, \xi), d(x_n, T x_{n-1}, \xi) \right\},
\]

since \( \frac{1}{\beta} < 1 \), we get

\[
\psi (d(x_n, x_{n+1}, \xi)) = \psi (d(T x_{n-1}, T x_n, \xi)) \leq \psi \left( \max \left\{ d(x_{n-1}, x_n, \xi), \frac{d(x_{n-1}, T x_{n-1}, \xi) d(x_n, T x_n, \xi)}{1 + d(T x_{n-1}, T x_n, \xi)} \right\} \right) - \phi \left( \max \left\{ d(x_{n-1}, x_n, \xi), \frac{d(x_{n-1}, T x_{n-1}, \xi) d(x_n, T x_n, \xi)}{1 + d(T x_{n-1}, T x_n, \xi)} \right\} \right) + \mu \min \left\{ d(x_{n-1}, T x_n, \xi), d(x_n, T x_n, \xi), d(x_{n-1}, T x_{n-1}, \xi), d(x_n, T x_{n-1}, \xi) \right\},
\]

Inequality (4), can be reduced since

\[
\min \left\{ d(x_{n-1}, T x_n, \xi), d(x_n, T x_n, \xi), d(x_{n-1}, T x_{n-1}, \xi), d(x_n, T x_{n-1}, \xi) \right\} = \min \left\{ d(x_{n-1}, x_{n+1}, \xi), d(x_n, x_{n+1}, \xi), d(x_{n-1}, x_n, \xi), d(x_n, x_n, \xi) \right\} = 0.
\]

Using (5), inequality (4) reduces to

\[
\psi (d(x_n, x_{n+1}, \xi)) \leq \psi \left( \max \left\{ d(x_{n-1}, x_n, \xi), d(x_n, x_{n+1}, \xi) \right\} \right) - \phi \left( \max \left\{ d(x_{n-1}, x_n, \xi), d(x_n, x_{n+1}, \xi) \right\} \right)
\]
Inequality (6) further reduces, if we assume that
\[ \max \{ d(x_{n-1}, x_n, \xi), d(x_n, x_{n+1}, \xi) \} = d(x_{n-1}, x_n, \xi) \]
for otherwise, we assume that
\[ \max \{ d(x_{n-1}, x_n, \xi), d(x_n, x_{n+1}, \xi) \} = d(x_n, x_{n+1}, \xi). \]
In the latter case, inequality (6), reduces to
\[ \psi(d(x_n, x_{n+1}, \xi)) \leq \psi(d(x_{n-1}, x_n, \xi)) - \varphi(d(x_{n-1}, x_n, \xi)) \]
(7)
It follows that \( 0 \leq -\varphi(d(x_n, x_{n+1}, \xi)) \) which leads to a contradiction. Thus
\[ \max \{ d(x_{n-1}, x_n, \xi), d(x_n, x_{n+1}, \xi) \} = d(x_{n-1}, x_n, \xi). \] Hence, we have
\[ \psi(d(x_n, x_{n+1}, \xi)) \leq \psi(d(x_{n-1}, x_n, \xi)) - \varphi(d(x_{n-1}, x_n, \xi)) \]
(8)
It follows that \( \{d(x_n, x_{n+1}, \xi)\}_{n \in \mathbb{N}} \) is a decreasing sequence.

We next shall show that \( \lim_{n \to \infty} d(x_n, x_{n+1}, \xi) = 0 \). Suppose that
\[ \lim_{n \to \infty} d(x_n, x_{n+1}, \xi) = r \] where \( r > 0 \) then taking limit as \( n \to \infty \) in inequality (7) we get
\[ \psi(r) \leq \psi(r) - \varphi(r) \]
(9)
which is a contradiction unless we have that \( r = 0 \) thus
\[ \lim_{n \to \infty} d(x_n, x_{n+1}, \xi) = 0. \]

We next shall prove that \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \). From the modified rectangular inequality we obtain,
\[ d(x_n, x_{m}, \xi) \leq \alpha d(x_n, x_m, x_{n+1}) + \beta d(x_m, \xi, x_{n+1}) + \gamma d(\xi, x_n, x_{n+1}) \]
\[ \leq \alpha d(x_n, x_{n+1}, x_m) + \beta \alpha d(x_m, x_{m+1}, \xi) + \beta^2 d(x_{n+1}, x_{m+1}, \xi) \]
\[ + \beta \gamma d(x_m, x_{m+1}, x_{n+1}) + \gamma d(x_n, x_{n+1}, \xi) \]
(10)
Using properties of the altering distance functions we get,

\[
\psi (d(x_n, x_m, \xi)) \\
\leq \alpha \psi (d(x_n, x_{n+1}, x_m)) + \beta \alpha \psi (d(x_m, x_{m+1}, \xi)) + \beta^2 \psi (d(x_{n+1}, x_{m+1}, \xi)) \\
+ \beta \gamma \psi (d(x_m, x_{m+1}, x_{n+1})) + \gamma \psi (d(x_n, x_{n+1}, \xi))
\]

(11)

Using inequality (2) in (11) we get

\[
\psi (d(x_n, x_m, \xi)) \\
\leq \alpha \psi (d(x_n, x_{n+1}, x_m)) + \beta \alpha \psi (d(x_m, x_{m+1}, \xi)) \\
+ \beta \psi \left( \max \left\{ d(x_n, x_m, \xi), \frac{d(x_n, TX_n, \xi) d(x_m, TX_m, \xi)}{1 + d(TX_n, TX_m, \xi)} \right\} \right) \\
- \beta \phi \left( \max \left\{ d(x_n, x_m, \xi), \frac{d(x_n, TX_n, \xi) d(x_m, TX_m, \xi)}{1 + d(TX_n, TX_m, \xi)} \right\} \right) \\
+ \beta \mu \min \left\{ d(x_n, TX_n, \xi), d(x_n, TX_n, \xi) + d(x_m, TX_n, \xi), d(x_m, TX_m, \xi) \right\}
\]

(12)

Taking \( m, n \to \infty \) we get,

\[
\lim_{m,n\to\infty} \max \left\{ d(x_n, x_m, \xi), \frac{d(x_n, TX_n, \xi) d(x_m, TX_m, \xi)}{1 + d(TX_n, TX_m, \xi)} \right\} \\
= \lim_{m,n\to\infty} \max \left\{ d(x_n, x_m, \xi), \frac{d(x_n, x_{n+1}, \xi) d(x_{n+1}, x_m, \xi)}{1 + d(x_{n+1}, x_m, \xi)} \right\} \\
= \lim_{m,n\to\infty} d(x_n, x_m, \xi)
\]

(13)

and

\[
\lim_{m,n\to\infty} \min \left\{ d(x_n, TX_n, \xi), d(x_n, TX_n, \xi) d(x_m, TX_m, \xi), d(x_m, TX_m, \xi) \right\} \\
= \lim_{m,n\to\infty} \min \left\{ d(x_n, x_{n+1}, \xi), d(x_n, x_{n+1}, \xi), d(x_{n+1}, x_{n+1}, \xi), d(x_{n+1}, x_{n+1}, \xi) \right\} \\
= 0
\]

(14)

Taking \( m, n \to \infty \) in (12), using (13) and (14) we get

\[
\psi \left( \lim_{m,n\to\infty} d(x_n, x_m, \xi) \right) \leq \beta \psi \left( \lim_{m,n\to\infty} d(x_n, x_m, \xi) \right) - \beta \phi \left( \lim_{m,n\to\infty} d(x_n, x_m, \xi) \right)
\]

(15)
Inequality (15) is only true if \( \lim_{m,n \to \infty} d(x_n, x_m, \xi) = 0 \). Thus we conclude that \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \). Since \((X, d)\) is complete there exists \( x' \in X \) such that 
\[
\lim_{n \to \infty} d(x_n, x', \xi) = 0.
\]

We now show that \( Tx' = x' \). Replacing \( x_n = x_{n+1}, x_m = Tx' \) in inequality (15), we get

\[
(16) \quad \psi \left( \lim_{n \to \infty} d(x_{n+1}, Tx', \xi) \right) \leq \beta \psi \left( \lim_{n \to \infty} d(x_{n+1}, T^2x', \xi) \right) - \beta \phi \left( \lim_{n \to \infty} d(x_{n+1}, T^2x', \xi) \right)
\]

It follows that

\[
(17) \quad \psi \left( d(x', Tx', \xi) \right) \leq \beta \psi \left( d(x', T^2x', \xi) \right) - \beta \phi \left( d(x', T^2x', \xi) \right)
\]

which leads to a contradiction, unless we have \( d(x', Tx', \xi) = 0 \) i.e., \( Tx' = x' \). To prove uniqueness of \( x' \), we assume that \( x'' \) is a fixed point of \( T \) such that \( x' \neq x'' \). From inequality (2),

\[
\beta \psi (d(x', x'', \xi)) \leq \psi \left( \max \left\{ d(x', x'', \xi), d(x', T^2x', \xi), d(x', x', \xi) \right\} \right) - \phi \left( d(x', x'', \xi) \right)
\]

It follows that

\[
(18) \quad \beta \psi (d(x', x'', \xi)) \leq \psi \left( d(x', x'', \xi) \right) - \phi \left( d(x', x'', \xi) \right)
\]

is a contradiction unless \( d(x', x'', \xi) = 0 \) which implies that \( x' = x'' \). \( \square \)

**Corollary 2.5.** Let \((X, d)\) be a generalized complete 2-metric space and a mapping \( T : X \to X \) be a self mapping. If there exists \( \psi, \phi \in \Psi \) such that

\[
(19) \quad \beta \psi (d(Tx, Ty, \xi)) \leq \psi \left( \max \left\{ d(x, y, \xi), d(x, Tx, \xi) \right\} \right) - \phi \left( d(x, Ty, \xi) \right)
\]

where \( x, y, z \in X \) and \( \psi, \phi \in \Psi \). Then \( T \) has a unique fixed point.
Proof. Follows from theorem 2.4 by taking $\mu = 0$. \hfill \square

**Corollary 2.6.** Let $(X, d)$ be a generalized complete 2-metric space and a mapping $T : X \rightarrow X$ be a self mapping. If there exists $\varphi \in \Psi$ such that

$$
\beta(d(Tx, Ty, \xi)) \\
\leq \left( \max \left\{ d(x, y, \xi), \frac{d(x, Tx, \xi)d(y, Ty, \xi)}{1+d(x, y, \xi)}, \frac{d(x, Tx, \xi)d(y, Ty, \xi)}{1+d(x, y, \xi)} \right\} \right) \\
- \varphi \left( \max \left\{ d(x, y, \xi), \frac{d(x, Tx, \xi)d(y, Ty, \xi)}{1+d(x, y, \xi)}, \frac{d(x, Tx, \xi)d(y, Ty, \xi)}{1+d(x, y, \xi)} \right\} \right)
$$

(21)

where $x, y, z \in X$ and $\varphi \in \Psi$. Then $T$ has a unique fixed point.

Proof. Follows from theorem 2.4 by taking $\psi$- the identity mapping \hfill \square

**Example 2.7.** Let $X = [0, 1]$ and define

$$
d(x, y, z) = \begin{cases} 
0, & \text{if at least two of the three points are the same} \\
e^{|x-y|+|y-z|+|z-x|}, & \text{otherwise}
\end{cases}
$$

(22)

It can be shown that $d$ is a generalized 2-metric.

d is a generalized 2-metric.

Let $T : X \rightarrow X$ be defined by

$$
T(x) = \sin x
$$

then for $x \neq y \neq z \in X$,

$$
|Tx - Ty| + |Ty - z| + |z - Tx|
= |\sin x - \sin y| + |\sin y - \sin(\sin^{-1} z)| + |\sin x - \sin(\sin^{-1} z)|
\leq |x - y| + |y - \sin^{-1} z| + |x - \sin^{-1} z|
\leq |x - y| + |y - z| + |x - z|
$$

Since the exponential function is increasing, it follows that

$$
e^{|Tx - Ty| + |Ty - z| + |z - Tx|} \leq e^{|x - y| + |y - z| + |x - z|}
$$

(23)
Let \( \psi(t) = t \) then it follows that for some \( \beta \geq 1 \)

\[
\beta \psi(d(Tx, Ty, z)) \\
\leq \psi(d(x, y, z)) \\
\leq \psi \left( \max \left\{ d(x, y, z), \frac{d(x, Tx, z)d(y, Ty, z)}{1+d(x, y, z)}, \frac{d(x, Tx, z)d(y, Ty, z)}{1+d(Tx, Ty, z)} \right\} \right)
\]

It follows from theorem (2.4), that \( T \) has a unique fixed point in \( X \).

3. Conclusion

In this paper, we proved the existence and uniqueness of a fixed point for a \((\psi, \phi)\)-weakly contractive mapping in a generalized 2-metric space by further imposing a sublinearity property on the class of all altering distance functions.

Conflict of Interests

The authors declare that there is no conflict of interests.

References


