Available online at http://scik.org
Adv. Fixed Point Theory, 2023, 13:23
https://doi.org/10.28919/afpt/8226
ISSN: 1927-6303

SOME GRAPHICAL FIXED POINT RESULTS IN A VARIABLE EXPONENT SEQUENCE SPACE

KENZA BENKIRANE*, ABDERRAHIM EL ADRAOUI, SAMIA BENNANI

Laboratory of Algebra, Analysis and Applications (LAMS), Faculty of Sciences Ben M'Sik, Hassan II University of Casablanca, Avenue Cdt Driss El Harti, BP 7955, Sidi Othmane, Casablanca, Morocco

Abstract. The main goal of this paper is to establish several fixed point results for certain mappings within variable exponent sequence spaces equipped with a graph. This will be achieved by integrating the principles of fixed point theory with those of graph theory.

Keywords: fixed point; variable exponent sequence spaces; graph; G-monotone; G-Kannan; G-Chatterjea.

2020 AMS Subject Classification: 47H09, 47H10, 47H40, 54H25.

1. Introduction and Preliminaries

In 1931, Orlicz [12] introduced the following vector space:

$$l_{p(.)} = \{(x_n)_n \subset \mathbb{R}^{\mathbb{N}} : \sum_{n=0}^{\infty} \frac{1}{p(n)} |\lambda x_n|^{p(n)} < \infty \text{ for some } \lambda > 0\};$$

where $p : \mathbb{N} \longrightarrow [1, \infty)$. Inspired by this example, Nakano [14, 15] defined, in 1950, the additive modular on universally continuous semi-ordered linear spaces. In 1959, Musielak and Orlicz [13] gave the formal definition of modular space, which is an abstraction of Orlicz space. In this generalization, we find, as examples, the variable exponent Lebesgue spaces $L_{p(.)}$. These

*Corresponding author

E-mail address: kenza.benkirane2-etu@etu.univh2c.ma

Received September 18, 2023

spaces of variable exponents provide the appropriate framework for mathematically formulating several problems across diverse specialties, for which the classic Lebesgue spaces were insufficient [6, 17, 18]. Many results on the fundamental properties of $L_{p(.)}$ were proved first by Kováčik and Rakosnık [10]. Fan and Zhao [7] later reproved these results. This study investigates fixed points within the context of variable exponent sequence spaces $l_{p(.)}$, special cases of the variable exponent spaces $L_{p(.)}$. These spaces were extensively studied (see for examples [14, 16, 2, 19, 26, 11]).

The study of fixed point theory is a highly dynamic and vibrant area of research. Among the various aspects of metric fixed point theory, the Banach contraction principle is the most renowned. After the publication of Banach's fixed point theorem [3], numerous mathematicians have dedicated their efforts to exploring potential extensions. Among these, the works of Kannan [21] and Chatterjea [5] emerge as sources of inspiration for a specific branch within the metric fixed point theory. The study of fixed points in variable exponent sequence spaces $l_{p(.)}$ for Banach mappings and Kannan mappings was first initiated in [8, 20, 29].

Alternatively, a fascinating approach to the theory of fixed points in specific general structures has been proposed by Jachymski [9] within the context of metric spaces endowed with a graph G.

In this paper, inspired by the ideas given in [1, 9, 20, 27, 28], we prove some fixed point theorems for G-Kannan and G-Chatterjea operators in the space $l_{p(.)}$ endowed with G. Finally, an example supports the main result.

Below, we outline the essential mathematical background materials required to establish the results presented in this paper.

Consider the variable exponent sequence spaces $l_{p(.)}$. Now, let us introduce the convex modular function ν, borrowed from [15, 16, 19, 20].

Proposition 1.1. *The mapping $\nu : l_{p(.)} \longrightarrow [0,\infty)$ such that*

$$\nu(x) = \nu((x_n)) = \sum_{n=0}^{\infty} \frac{1}{p(n)} |x_n|^{p(n)},$$

is said to be a convex modular function if it satisfies the following conditions:

(i): $\nu(x)=0$ *if and only if* $x=0$,

(ii): *if* $|\gamma|=1$ *then* $\nu(\gamma x)=\nu(x)$,

(iii): *for any* $t\in[0,1]$, $\nu(tx+(1-t)y)\le t\nu(x)+(1-t)\nu(y)$,

for any $x,y\in l_{p(.)}$.

Following that, we proceed to present the modular versions of several metric known properties.

Definition 1.2. [25]

(i): A sequence $(x_n)_n\subset l_{p(.)}$ is said to be ν-convergent to $x\in l_{p(.)}$ if $\nu(x_n-x)\longrightarrow 0$. Note that if the ν-limit exists, it is unique.

(ii): A sequence $(x_n)_n\subset l_{p(.)}$ is said to be ν-Cauchy if $\upsilon(x_p-x_q)\longrightarrow 0$ as $p,q\longrightarrow\infty$.

(iii): $C\subset l_{p(.)}$ is ν-closed if for any ν-convergent sequence $(x_n)_n\subset C$ to x one has $x\in C$.

For the sake of completeness, it is important to mention that convex modular functions satisfy the Fatou's property.

Fatou's property: for any $(z_n)_n\subseteq l_{p(.)}$ ν-convergent to z, we have

$$\nu(x-z)\le\liminf_{n\longrightarrow\infty}\nu(x-z_n),$$

for any $x\in l_{p(.)}$.

Next, let us introduce the following property, known as the "Δ_2-condition".

Definition 1.3. ν satisfies the Δ_2-condition if there exists $\alpha\ge 0$ such that

$$\nu(2x)\le\alpha\nu(x),$$

for any $x\in l_{p(.)}$.

The modular convex ν is said to satisfy the Δ_2-condition if and only if $p^+=\sup_{n\in\mathbb{N}}p(n)<\infty$ [15, 16, 19, 20, 22, 23].

The following definition of the Luxemburg norm is borrowed from [20, 19].

Definition 1.4. The Luxemburg norm on $l_{p(.)}$ is defined by:

$$||x||_\nu=\inf\left\{\lambda>0\,;\,\nu\left(\frac{1}{\lambda}x\right)\le 1\right\}.$$

$(l_{p(.)}, ||.||_\nu)$ is said to be a uniformly convex Banach space if and only if $1 < p^- < p^+ < \infty$ [19], where

$$p^- = \inf_{n\in\mathbb{N}} p(n) \text{ and } p^+ = \sup_{n\in\mathbb{N}} p(n).$$

Now, we recall the following results on graphs from [24]. Let us consider an arbitrary set V.

Definition 1.5.

- (a)**:** A digraph G is a pair $G = (V,E)$ where E is a subset of the Cartesian product $V \times V$.
- (b)**:** The elements of V are called vertices or nodes of G
- (c)**:** The elements of E are the edges also called oriented edges or arcs of G.
- (d)**:** An edge of the form (V,V) is a loop on V.
- (e)**:** Let $G = (V,E)$ be a digraph. G is transitive if, for any $x,y,z \in V$ such that $(x,y) \in E$ and $(y,z) \in E$, we have $(x,z) \in E$.
- (f)**:** Let $G = (V,E)$ be a digraph. G is reflexive if, $\Delta = \{(x,x); x \in V\} \subset E$. Otherwise, every vertex has a loop.

Consider the digraph $G = (V,E)$.

Definition 1.6. Let x,y be a vertex

- 1**:** x is said to be isolated if, for all vertex y such that $y \neq x$, we have neither $(x,y) \in E$ nor $(y,x) \in E$.
- 2**:** A path in G, from x to y, is a sequence of vertices $p = (a_i)_{0\leq i\leq n}$, $n \in \mathbb{N}^*$ such that $a_0 = x, a_n = y$ and $(a_i, a_{i+1}) \in E$, for all $i \in \{0,1,...,n-1\}$. The integer n is the length of the path p. If $x = y$ and $n > 1$, the path p is called a directed cycle. An acyclic digraph is a digraph which has no directed cycle.
- 3**:** We denote by $y \in [x]_G$ the fact that there exists a path in G connecting x to y.

Throughout this paper, we donate by $(l_{p(.)}, G)$ the variable exponent sequence space $l_{p(.)}$ and the digraph $G = (V,E)$ such that $V = l_{p(.)}$. We define the G-monotonicity of sequences in $l_{p(.)}$ as follows:

Definition 1.7. Let $(x_n)_n$ be a sequence in V.

1.: $(x_n)_n$ is said to be G-increasing if, for all $n \in \mathbb{N}$ $(x_n, x_{n+1}) \in E$.

2.: $(x_n)_n$ is said to be G-decreasing if, for all $n \in \mathbb{N}$ $(x_{n+1}, x_n) \in E$.

3.: $(x_n)_n$ is said to be G-monotone, if it is either G-increasing or G-decreasing.

Now we recall a useful type of continuity of mappings.

Definition 1.8. [24] A self-mapping f on $l_{p(.)}$ is called orbitally G-continuous if for all $x, y \in V$ and any seguence $(k_n)_{n\in\mathbb{N}}$ of positive integers

$$f^{k_n}x \longrightarrow y \text{ and } f^{k_n}x \in [f^{k_{n+1}}x]_G \Longrightarrow f(f^{k_n}x) \longrightarrow fy.$$

We conclude this section by the following Property (OSC):

Property: G satisfies the Property (OSC) if, for any G-monotone sequence $(x_n)_n$ that ν-converges to some $x \in V$ and where $x_{n+1} \in [x_n]_G$, it follows that $x \in [x_n]_G$ for any $n \in \mathbb{N}$.

2. MAIN RESULTS

In this section, we will prove some fixed point theorems for G-Kannan and G-Chatterjea operators in the space $l_{p(.)}$ endowed with G.

Denote for any $y \in l_{p(.)}$ the complete subgraph $G[O_f(y)]$ induced by the orbit $O_f(y) := \{f^n y : n \in \mathbb{N}\}$.

Let $(l_{p(.)}, G)$ be a variable exponent sequence space with digraph G. We introduce G-monotone G-Kannan mappings as follows:

Definition 2.1. The mapping $f : l_{p(.)} \longrightarrow l_{p(.)}$ is called a G-monotone G-Kannan mapping if f satisfies the following conditions:

•: f is G-monotone, that is, for every $a, b \in l_{p(.)}$

$$b \in [a]_G \Longrightarrow fb \in [fa]_G.$$

•: f is G-Kannan, that is, there exists $L \in [0, \frac{1}{2}[$ such that every $a, b \in l_{p(.)}$

$$b \in [a]_G \Longrightarrow \nu(fa - fb) \leq L(\nu(fa - a) + \nu(fb - b)).$$

Theorem 2.2. *Let C be a non empty ν-closed subset of $l_{p(.)}$ and $f : C \longrightarrow C$ be a mapping. Suppose that there exists $y \in C$ such that $\nu(fy - y) < \infty$ and $fy \in [y]_G$. If f is G-monotone*

G-Kannan mapping and G satisfies the property (OSC), then the sequence $(f^n y)$ ν-converges to some point $c \in C$. Moreover, c is the fixed point of f or $\nu(fc-c) = +\infty$.

Proof. Let $y \in C$. Suppose that $\nu(fy-y) < \infty$ and $fy \in [y]_G$. We will now establish the ν-convergence of the sequence $(f^n y)_n$. According to the ν-completeness of $l_{p(.)}$ it is enough to show that $(f^n y)_n$ is a ν-Cauchy sequence. Due to the G-monotonicity, we can infer that $f^2 y \in [fy]_G$. By induction on n, we get $f^{n+1}y \in [f^n y]_G$, for all $n \in \mathbb{N}$. Then $(f^n y)_n$ is a G-monotone sequence. As f is a G-Kannan, there exists a constant L where $L \in [0, \frac{1}{2}[$ such that

$$\nu(f^{n+1}y - f^n y) \leq L(\nu(f^{n+1}y - f^n y) + \nu(f^n y - f^{n-1}y)),$$

for any $n \in \mathbb{N}^*$. Thus

$$\nu(f^{n+1}y - f^n y) \leq \frac{L}{1-L}\nu(f^n y - f^{n-1}y),$$

for any $n \geq 1$. Set $k = \frac{L}{1-L}$, then

$$\nu(f^{n+1}y - f^n y) \leq k\nu(f^n y - f^{n-1}y),$$

for any $n \geq 1$. Therefore,

$$\nu(f^{n+1}y - f^n y) \leq k^n \nu(fy-y), \text{ for any } n \in \mathbb{N}.$$

Since $f^{n+1}y \in [f^n y]_G$; $f^{n+2}y \in [f^{n+1}y]_G$;...;$f^{n+h}y \in [f^{n+h-1}y]_G$, then $f^{n+h}y \in [f^n y]_G$. Then, since f is a G-Kannan mapping, we have

$$\begin{aligned}
\nu(f^{n+h}y - f^n y) &= \nu(f(f^{n+h-1}y) - f(f^{n-1}y)), \\
&\leq L(\nu(f^{n+h}y - f^{n+h-1}y) + \nu(f^n y - f^{n-1}y)), \\
&\leq L(k^{n+h-1} - k^{n-1})\nu(fy-y),
\end{aligned}$$

for all $n \in \mathbb{N}^*$ and $h \in \mathbb{N}$. Given that $k < 1$ and $\nu(fy-y) < \infty$, it follows that $(f^n y)_n$ is a ν-Cauchy sequence. Thus $(f^n y)_n$ ν-converges to some $c \in l_{p(.)}$. Since C is ν-closed subset of $l_{p(.)}$, we get $c \in C$.

Since G satisfies the property (OSC). And, since $(f^n y)_n$ ν-converges to some $c \in C$, then

$c \in [f^n y]_G$ for any n.

Let $n \in \mathbb{N}^*$. Since $c \in [f^n y]_G$ and f is G-Kannan mapping, then

$$\nu(f^n y - fc) \leq L(\nu(f^n y - f^{n-1} y) + \nu(fc - c)).$$

for any $n \geq 1$. Hence, for any $n \geq 1$

$$\nu(f^n y - fc) \leq L(k^{n-1} \nu(fy - y) + \nu(fc - c)). \tag{1}$$

According to Fatou's property, we obtain

$$\begin{aligned} \nu(fc - c) &\leq \liminf_{n \longrightarrow +\infty} \nu(f^n y - fc), \\ &\leq L\nu(fc - c). \end{aligned}$$

Assume that $\nu(fc - c) < \infty$. As $L < \frac{1}{2}$, we have $\nu(fc - c) = 0$.

Therefore, c is a fixed point of f. □

Example 2.3.

Consider ν as the modular convex defined on $l_{p(.)}$ by

$$\nu(x) = \nu((x_n)_n) = \sum_{n=0}^{\infty} \frac{1}{p(n)} |x_n|^{p(n)}.$$

And the non empty ν-closed subset C of $l_{p(.)}$ such that

$$C = \left\{(x_{k,n})_{(k,n) \in \mathbb{N}^* \times \mathbb{N}}\right\} \cup \{0\},$$

where $x_{k,n} = \frac{x_n}{k}$ such that $x_n = \left(\frac{1}{2}\right)^{\frac{n}{2}}$.

Let f be a self mapping defined on C by,

$$fx = \frac{x}{6}$$

Consider the graph G on $l_{p(.)}$ such that, for any $n \in \mathbb{N}$

$$E(G) = \{(0,0)\} \cup \{(0, (x_{k,n})) : k \in \mathbb{N}^*\} \cup \left\{\left((x_{k_1,n}), (x_{k_2,n})\right) : k_1, k_2 \in \mathbb{N}^* \text{ and } k_1 > k_2\right\}.$$

For $y=(x_{1,n})$, we have

$$\begin{aligned}
\nu(y-fy) &= \nu\left(x_{1,n}-\frac{x_{1,n}}{6}\right), \\
&= \nu\left(x_{1,n}\left(1-\frac{1}{6}\right)\right), \\
&= \sum_{n=0}^{+\infty}\frac{1}{p(n)}\left|\left(1-\frac{1}{6}\right)x_{1,n}\right|^{p(n)}, \\
&= \sum_{n=0}^{+\infty}\frac{1}{p(n)}\left|\frac{5}{6}\left(\frac{1}{2}\right)^{\frac{n}{2}}\right|^{p(n)}, \\
&\leq \sum_{n=0}^{+\infty}\frac{5}{6}\left(\frac{1}{2}\right)^{\frac{n}{2}}.
\end{aligned}$$

Since $\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^{\frac{n}{2}}<\infty$ and $\frac{5}{6}\in\mathbb{N}^*$ then, $\nu(y-fy)<\infty$.

Also,

$$\begin{aligned}
(fy,y) &= \left(\left(\frac{x_{1,n}}{6}\right),(x_{1,n})\right), \\
&= \left(\left(\frac{x_{1,n}}{6}\right),\left(\frac{x_{1,n}}{1}\right)\right).
\end{aligned}$$

Then, $(fy,y)\in E(G)$. Hence, $fy\in[y]_G$.

Therefore, there exists $y\in C$ such that $\nu(y-fy)<\infty$ and $fy\in[y]_G$.

One can see that, for $n\in\mathbb{N}$, we have

$$(f0,f0)=(0,0)\in E(G).$$

And,

$$\left(f\left(x_{k_1,n}\right),f\left(x_{k_2,n}\right)\right)=\left(\left(\frac{x_n}{6k_1}\right),\left(\frac{x_n}{6k_2}\right)\right)\in E(G),\ \text{ for any }\ k_1,k_2\in\mathbb{N}^*.$$

And,

$$(f(0)-f(x_{k,n}))=\left(0,\left(\frac{x_n}{6k}\right)\right)\in E(G),\ \text{ for any }\ k\in\mathbb{N}^*.$$

Then, for every $(a,b)\in E(G)$, $(fa,fb)\in E(G)$.

Therefore, for every $(a,b)\in C$

$$b\in[a]_G\Longrightarrow fb\in[fa]_G.$$

Then f is G-monotone.

Let $a,b \in C$ such that $b \in [a]_G$, then $(a,b) \in E(G)$.

If $(a,b) = \big((x_{k_1,n}),(x_{k_2,n})\big)$, then

$$\begin{aligned}
\nu(f(x_{k_1,n}) - f(x_{k_2,n})) &= \nu\left(\frac{x_{k_1,n}}{6} - \frac{x_{k_2,n}}{6}\right),\\
&= \nu\left(\frac{1}{6}\frac{x_n}{k_1} - \frac{1}{6}\frac{x_n}{k_2}\right),\\
&= \nu\left(\frac{1}{6}\frac{1}{k_1}\left(\frac{1}{2}\right)^{\frac{n}{2}} - \frac{1}{6}\frac{1}{k_2}\left(\frac{1}{2}\right)^{\frac{n}{2}}\right),\\
&= \nu\left(\frac{1}{6}\left(\frac{1}{2}\right)^{\frac{n}{2}}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\right),\\
&= \sum_{n=0}^{+\infty}\frac{1}{p(n)}\left|\frac{1}{6}\left(\frac{1}{2}\right)^{\frac{n}{2}}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\right|^{p(n)},\\
&= \sum_{n=0}^{+\infty}\frac{1}{p(n)}\left(\frac{1}{6}\left(\frac{1}{2}\right)^{\frac{n}{2}}\right)^{p(n)}\left|\frac{1}{k_1} - \frac{1}{k_2}\right|^{p(n)}.
\end{aligned}$$

Hence,

$$\nu(f(x_{k_1,n}) - f(x_{k_2,n})) = \sum_{n=0}^{+\infty}\frac{1}{p(n)}\left(\frac{1}{6}\left(\frac{1}{2}\right)^{\frac{n}{2}}\right)^{p(n)}\left|\frac{1}{k_1} - \frac{1}{k_2}\right|^{p(n)}.$$

On the other hand, we have

$$\begin{aligned}
\nu(x_{k_1,n} - f(x_{k_1,n})) &= \nu\left(\frac{x_n}{k_1} - \frac{1}{6}\frac{x_n}{k_1}\right),\\
&= \nu\left(\frac{1}{k_1}\left(\frac{1}{2}\right)^{\frac{n}{2}} - \frac{1}{6}\frac{1}{k_1}\left(\frac{1}{2}\right)^{\frac{n}{2}}\right),\\
&= \nu\left(\frac{1}{k_1}\frac{5}{6}\left(\frac{1}{2}\right)^{\frac{n}{2}}\right),\\
&= \sum_{n=0}^{+\infty}\frac{1}{p(n)}\left(\frac{5}{6}\frac{1}{k_1}\left(\frac{1}{2}\right)^{\frac{n}{2}}\right)^{p(n)}.
\end{aligned}$$

Then,

$$\nu(x_{k_1,n} - f(x_{k_1,n})) = \sum_{n=0}^{+\infty}\frac{1}{p(n)}\left(\frac{5}{6}\frac{1}{k_1}\left(\frac{1}{2}\right)^{\frac{n}{2}}\right)^{p(n)}.$$

Likewise,

$$\nu(x_{k_2,n} - f(x_{k_2,n})) = \sum_{n=0}^{+\infty} \frac{1}{p(n)} \left(\frac{5}{6} \frac{1}{k_2} \left(\frac{1}{2}\right)^{\frac{n}{2}} \right)^{p(n)}.$$

And since

$$\left(\frac{1}{k_1}\right)^{p(n)} + \left(\frac{1}{k_2}\right)^{p(n)} \geq \left| \frac{1}{k_1} - \frac{1}{k_2} \right|^{p(n)}.$$

We have,

$$\begin{aligned}
& \nu(x_{k_1,n} - f(x_{k_1,n})) + \nu(x_{k_2,n} - f(x_{k_2,n})) \\
= \ & \sum_{n=0}^{+\infty} \frac{1}{p(n)} \left(\left(\frac{5}{6} \frac{1}{k_1} \left(\frac{1}{2}\right)^{\frac{n}{2}} \right)^{p(n)} + \left(\frac{5}{6} \frac{1}{k_2} \left(\frac{1}{2}\right)^{\frac{n}{2}} \right)^{p(n)} \right), \\
= \ & \sum_{n=0}^{+\infty} \frac{1}{p(n)} \left(\frac{5}{6} \left(\frac{1}{2}\right)^{\frac{n}{2}} \right)^{p(n)} \left(\left(\frac{1}{k_1}\right)^{p(n)} + \left(\frac{1}{k_2}\right)^{p(n)} \right), \\
\geq \ & \sum_{n=0}^{+\infty} 5^{p(n)} \frac{1}{p(n)} \left(\frac{1}{6} \left(\frac{1}{2}\right)^{\frac{n}{2}} \right)^{p(n)} \left| \frac{1}{k_1} - \frac{1}{k_2} \right|^{p(n)}, \\
\geq \ & \sum_{n=0}^{+\infty} 5 \frac{1}{p(n)} \left(\frac{1}{6} \left(\frac{1}{2}\right)^{\frac{n}{2}} \right)^{p(n)} \left| \frac{1}{k_1} - \frac{1}{k_2} \right|^{p(n)}, \\
\geq \ & 5\nu(f(x_{k_1,n}) - f(x_{k_2,n})).
\end{aligned}$$

Then, for $(a,b) = \big((x_{k_1,n}),(x_{k_2,n})\big)$ we have

$$\nu(fa - fb) \leq \frac{1}{5}(\upsilon(a - fa) + \upsilon(b - fb)).$$

If $(a,b) = (0,0)$, we have

$$\nu(fa - fb) = \nu(f0 - f0) = 0.$$

Since

$$0 \leq \frac{1}{5}(\nu(a - fa) + \nu(b - fb)),$$

then, for $(a,b) = (0,0)$, we have

$$\nu(fa - fb) \leq \frac{1}{5}(\nu(a - fa) + \nu(b - fb)).$$

If $(a,b)=(0,(x_{k,n}))$, we have

$$\begin{aligned}
\nu(fa-fb) &= \nu(f(0)-f(x_{k,n})),\\
&= \sum_{n=0}^{+\infty}\frac{1}{p(n)}\left|\frac{x_{k,n}}{6}\right|^{p(n)},\\
&= \sum_{n=0}^{+\infty}\frac{1}{p(n)}\left|\frac{x_n}{6k}\right|^{p(n)},\\
&= \sum_{n=0}^{+\infty}\frac{1}{p(n)}\left|\frac{1}{5}\frac{5x_n}{6k}\right|^{p(n)},\\
&\leq \frac{1}{5}\sum_{n=0}^{+\infty}\frac{1}{p(n)}\left|\frac{5}{6}\frac{x_n}{k}\right|^{p(n)}.\\
&\leq \frac{1}{5}(\upsilon(0-f(0))+\upsilon(x_{k,n}-f(x_{k,n}))).
\end{aligned}$$

Then, for $(a,b)=(0,(x_{k,n}))$, we have

$$\nu(fa-fb)\leq\frac{1}{5}(\nu(a-fa)+\nu(b-fb)).$$

In all cases, $\nu(fa-fb)\leq\frac{1}{5}(\nu(a-fa)+\nu(b-fb))$, proving that f is a G-Kannan mapping with constant $L=\frac{1}{5}\in[0,\frac{1}{2}[$.

$(x_{k,n})_k$ is G-monotone sequence, for $\left(\left(\frac{x_n}{k+1}\right),\left(\frac{x_n}{k}\right)\right)\in E(G)$ we have $\left(\left(\frac{x_n}{6(k+1)}\right),\left(\frac{x_n}{6k}\right)\right)\in E(G)$.

This sequence is ν-converging to $0\in C$. Indeed,

$$\begin{aligned}
\nu(x_{k,n}-0) &= \sum_{n=0}^{+\infty}\frac{1}{p(n)}\left|\frac{1}{k}\left(\frac{1}{2}\right)^{\frac{n}{2}}\right|^{p(n)},\\
&\leq \sum_{n=0}^{+\infty}\frac{1}{k}\left(\frac{1}{2}\right)^{\frac{n}{2}}.
\end{aligned}$$

Since $\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^{\frac{n}{2}}<\infty$, then

$$\frac{1}{k}\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^{\frac{n}{2}}\longrightarrow 0$$

when $k\longrightarrow+\infty$.

And, $\big(0,(x_{k,n})\big)\in E(G)$ for any $k\in\mathbb{N}^*$. Then, G has the property (OSC).

From Theorem 2.2 f has a fixed point in C.

The following corollary is the modular version of the Kannan's theorem in $l_{p(.)}$.

Corollary 2.4. *Let C be a non empty ν-closed subset of $l_{p(.)}$ and $f : C \longrightarrow C$ be a G-monotone G-Kannan mapping where $\nu(fx-x) < \infty$ and $fx \in [x]_G$ for any $x \in C$. If G satisfies the property OSC then, for any $y \in C$, the sequence $(f^n y)$ υ-converges to the unique fixed point c of f. Moreover, we have $c \in [f^n y]_G$ and the following hold:*

$$\nu(f^n y - c) \leq Lk^{n-1}\nu(fy-y),$$

for any $y \in C$ and $n \in \mathbb{N}^*$, where $k = \frac{L}{1-L}$.

Proof. According to Theorem 2.2, we deduce the existence of the fixed point c of f. To prove the uniqueness of this fixed point. Let $c^{'}$ be another fixed point of f. According to the inequality (1):

$$\nu(f^n y - fc) \leq L(k^{n-1}\nu(fy-y) + \nu(fc-c)),$$

according to Fatou's property, we obtain

$$\begin{aligned} \nu(c-c^{'}) &= \nu(fc-c^{'}), \\ &\leq \liminf_{n \longrightarrow +\infty} \nu(f^n y - fc^{'}), \\ &\leq L\nu(fc^{'}-c^{'}). \end{aligned}$$

Since $c^{'}$ is a fixed point of f, then $\nu(fc^{'}-c^{'}) = 0$.
which implies,

$$\nu(c-c^{'}) = 0.$$

Therefore, $c = c^{'}$. This proves the uniqueness of the fixed point of f.
Again, by using the inequality (1) above combined with the Fatou's property, we get

$$\begin{aligned} \nu(f^n y - c) &\leq L(k^{n-1}\nu(fy-y) + \nu(fc-c)), \\ &\leq Lk^{n-1}\nu(fy-y), \end{aligned}$$

where $k = \frac{L}{1-L}$. Then,

$$\nu(f^n y - c) \leq Lk^{n-1}\nu(fy-y),$$

for any $y \in C$ and $n \in \mathbb{N}^*$. □

Now, let introduced the G-monotone G-Chatterjea mappings as follows:

Definition 2.5. The mapping $f : l_{p(.)} \longrightarrow l_{p(.)}$ is called a G-monotone G-Chatterjea mapping if f satisfies the following conditions:

(a)**:** f is G-monotone, that is, for any $a,b \in l_{p(.)}$

$$b \in [a]_G \Longrightarrow fb \in [fa]_G.$$

(b)**:** f is G-Chatterjea, that is, there exists $0 < L < \frac{1}{\alpha} \leq \frac{1}{2}$ such that every $a,b \in l_{p(.)}$

$$b \in [a]_G \Longrightarrow \nu(fa - fb) \leq L(\nu(fa - b) + \nu(a - fb)),$$

where α is the constant of Δ_2-condition.

Theorem 2.6. *Let C be a non empty ν-closed subset of $l_{p(.)}$ and $f : C \longrightarrow C$ be a mapping. Suppose there exists $y \in C$ such that $\nu(fy - y) < \infty$ and $fy \in [y]_G$. If f is G-monotone G-Chatterjea mapping and ν satisfying the Δ_2-condition, then the sequence $(f^n y)_n$ ν-converges to some $c \in C$.*
Morever, if f is orbitally G-continuous and G satisfies the property (OSC) with $\nu(fc - c) < \infty$ then c is a fixed point of f.

Proof. Let $y \in C$. Suppose that $\nu(fy - y) < \infty$ and $fy \in [y]_G$. We will now establish the ν-convergence of the sequence $(f^n y)_n$. According to the ν-completeness of $l_{p(.)}$ it is enough to show that $(f^n y)_n$ is a ν-Cauchy sequence. Since f is a G-monotone then $f^2 y \in [fy]_G$. By induction on n, we get $f^{n+1} y \in [f^n y]_G$, for all $n \in \mathbb{N}$. Then $(f^n y)_n$ is a G-monotone sequence. Since f is a G-Chatterjea mapping there exists $0 < L < \frac{1}{\alpha} \leq \frac{1}{2}$ such that

$$\begin{aligned} \nu(f^{n+1}y - f^n y) &\leq L(\nu(f^{n+1}y - f^{n-1}y) + \nu(f^n y - f^n y)), \\ &\leq L\nu(f^{n+1}y - f^{n-1}y), \end{aligned}$$

for any $n \geq 1$. We have,

$$\begin{aligned} \nu(f^{n+1}y - f^{n-1}y) &= \nu\left(\frac{1}{2}2(f^{n+1}y - f^n y) + \frac{1}{2}2(f^n y - f^{n-1}y)\right), \\ &\leq \frac{1}{2}\nu(2(f^{n+1}y - f^n y)) + \frac{1}{2}\nu(2(f^n y - f^{n-1}y)), \\ &\leq \frac{1}{2}\alpha\nu(f^{n+1}y - f^n y) + \frac{1}{2}\alpha\nu(f^n y - f^{n-1}y), \end{aligned}$$

for any $n \geq 1$. Then,

$$\nu(f^{n+1}y - f^n y) \leq \frac{\alpha L}{2}\nu(f^{n+1}y - f^n y) + \frac{\alpha L}{2}\nu(f^n y - f^{n-1}y),$$

for any $n \geq 1$. Hence,

$$\nu(f^{n+1}y - f^n y) \leq \frac{\frac{\alpha L}{2}}{1 - \frac{\alpha L}{2}}\nu(f^n y - f^{n-1}y),$$

for any $n \geq 1$. Set $k = \frac{\frac{\alpha L}{2}}{1-\frac{\alpha L}{2}}$, then

$$\nu(f^{n+1}y - f^n y) \leq k\, \nu(f^n y - f^{n-1}y),$$

for any $n \in \mathbb{N}^*$. Therefore,

$$\nu(f^{n+1}y - f^n y) \leq k^n\, \nu(fy - y),$$

for any $n \in \mathbb{N}^*$. Since $f^{n+1}y \in [f^n y]_G$; $f^{n+2}y \in [f^{n+1}y]_G$;...;$f^{n+h}y \in [f^{n+h-1}y]_G$ then $f^{n+h}y \in [f^n y]_G$. Then, since f is G-Chatterjea mapping, we have

$$\begin{aligned}
\nu(f^{n+h}y - f^n y) &= \nu(f(f^{n+h-1}y - f(f^{n-1}y),\\
&\leq L(\nu(f^{n+h}y - f^{n-1}y) + \nu(f^{n+h-1}y - f^n y)),
\end{aligned}$$

for any $n \geq 1$ and $h \in \mathbb{N}$. Since,

$$\begin{aligned}
\nu(f^{n+h}y - f^{n-1}y) &= \nu(f^{n+h}y - f^n y + f^n y - f^{n-1}y),\\
&= \nu\left(\frac{1}{2}2(f^{n+h}y - f^n y) + \frac{1}{2}2(f^n y - f^{n-1}y)\right),\\
&\leq \frac{1}{2}\nu(2(f^{n+h}y - f^n)) + \frac{1}{2}\nu(2(f^n y - f^{n-1}y)),\\
&\leq \frac{\alpha}{2}\nu(f^{n+h}y - f^n y) + \frac{\alpha}{2}\nu(f^n y - f^{n-1}y,)\\
&\leq \frac{\alpha}{2}\nu(f^{n+h}y - f^n y) + \frac{\alpha}{2}k^n\nu(fy - y),
\end{aligned}$$

for any $n \geq 1$ and $h \in \mathbb{N}$. And,

$$\begin{aligned}
\nu(f^{n+h-1}y - f^n y) &= \nu(f^{n+h-1}y - f^{n+h}y + f^{n+h}y - f^n y),\\
&= \nu\left(\frac{1}{2}2(f^{n+h-1}y - f^{n+h}y) + \frac{1}{2}2(f^{n+h}y - f^n y)\right),\\
&\leq \frac{1}{2}\nu(2(f^{n+h-1}y - f^{n+h}y)) + \frac{1}{2}\nu(2(f^{n+h}y - f^n y)),
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\alpha}{2}\nu(f^{n+h-1}y - f^{n+h}y) + \frac{\alpha}{2}\nu(f^{n+h}y - f^{n}y), \\
&\leq \frac{\alpha}{2}k^{n+h-1}\nu(fy - y) + \frac{\alpha}{2}\nu(f^{n+h}y - f^{n}y),
\end{aligned}$$

for any $n \geq 1$ and $h \in \mathbb{N}$. Then,

$$\begin{aligned}
\nu(f^{n+h}y - f^{n}y) &\leq L(\nu(f^{n+h}y - f^{n-1}y) + \nu(f^{n+h-1}y - f^{n}y)), \\
&\leq \frac{L\alpha}{2}\nu(f^{n+h}y - f^{n}y) + \frac{L\alpha}{2}k^{n}\nu(fy - y) + \frac{L\alpha}{2}k^{n+h-1}\nu(fy - y) \\
&+ \frac{L\alpha}{2}\nu(f^{n+h}y - f^{n}y),
\end{aligned}$$

for any $n \in \mathbb{N}^*$ and $h \in \mathbb{N}$. Hence,

$$\nu(f^{n+h}y - f^{n}y) - \frac{L\alpha}{2}\nu(f^{n+h}y - f^{n}y) - \frac{L\alpha}{2}\nu(f^{n+h}y - f^{n}y) \leq \frac{L\alpha}{2}(k^{n} + k^{n+h-1})\nu(fy - y),$$

for any $n \in \mathbb{N}^*$ and $h \in \mathbb{N}$. Thus,

$$(1 - L\alpha)\nu(f^{n+h}y - f^{n}y) \leq \frac{L\alpha}{2}(k^{n} + k^{n+h-1})\nu(fy - y),$$

for any $n \in \mathbb{N}^*$ and $h \in \mathbb{N}$. Therefore,

$$\nu(f^{n+h}y - f^{n}y) \leq \frac{\frac{L\alpha}{2}}{(1 - L\alpha)}(k^{n} + k^{n+h-1})\nu(fy - y),$$

for any $n \geq 1$ and $h \in \mathbb{N}$. Given that $k < 1$ and $\nu(fy - y) < \infty$, it follows that $(f^{n}y)_{n}$ is a ν-Cauchy sequence. Then, $(f^{n}y)_{n}$ ν-converges to some $c \in l_{p(.)}$. Since C is a ν-closed subset of $l_{p(.)}$, we get $c \in C$.

Consider that G fulfills the property (OSC) and $\nu(fc - c) < \infty$.

As $f^{n}y \in [f^{n+1}y]_{G}$ and $(f^{n}y)_{n}$ ν-converges to $c \in C$, we have $c \in [f^{n}y]_{G}$ for any $n \in \mathbb{N}$. Since f is G-Chatterjea, then

$$\nu(f^{n}y - fc) \leq L(\nu(f^{n}y - c) + \nu(f^{n-1}y - fc)).$$

By using the Fatou's Property, we have

$$\begin{aligned}
\nu(c - fc) &\leq \liminf_{n \longrightarrow +\infty} \nu(f^{n}y - fx), \\
&\leq L \liminf_{n \longrightarrow +\infty} \nu(f^{n}y - c) + L \liminf_{n \longrightarrow +\infty} \nu(f^{n-1}y - fc).
\end{aligned}$$

Consider that f is orbitally G-continuous. Since $(f^{n-2}y)$ ν-converges to c and $f^{n-2}y \in [f^{n-1}y]_G$, then $f(f^{n-2}y)$ ν-converges to fc.

Which implies,

$$\nu(f^{n-1}y - fc) \longrightarrow 0,$$

when $n \longrightarrow +\infty$. Hence,

$$\nu(c - fc) \leq 0.$$

Since $\nu : l_{p(.)} \longrightarrow [0, +\infty)$, then

$$\begin{aligned} \nu(c - fc) = 0 \quad &\Longrightarrow \quad c - fc = 0, \\ &\Longrightarrow \quad c = fc. \end{aligned}$$

We conclude that c is a fixed point of f. $\square$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] K. Chaira, A. Eladraoui, M. Kabil, et al. Kannan fixed point theorem on generalized metric space with a graph, Appl. Math. Sci. 13 (2019), 263–274. https://doi.org/10.12988/ams.2019.9226.

[2] C. Bardaro, J. Musielak, G. Vinti, Nonlinear integral operators and applications, Walter de Gruyter, 2003. https://doi.org/10.1515/9783110199277.

[3] S. Banach, Sur les opérations dans les ensembles abstraits et leurs applications, Fund. Math. 3 (1922), 133–181.

[4] K. Benkirane, A. Eladraoui, S. Bennani, Kannan fixed point theorem in the variable exponent sequence spaces lp(.) with a graph, in: International Conference on Research in Applied Mathematics and Computer Science (ICRAMCS 2022), March 24-26, 2022, Casablanca, Morocco.

[5] S.K. Chatterjea, Fixed point theorems, C.R. Acad. Bulgare Sci. 25 (1972), 727–730.

[6] L. Diening, P. Harjulehto, P. Hästö, et al. Lebesgue and Sobolev spaces with variable exponents, Springer Berlin Heidelberg, Berlin, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-18363-8.

[7] X. Fan, D. Zhao, On the spaces $Lp(x)(\Omega)$ and $Wm, p(x)(\Omega)$, J. Math. Anal. Appl. 263 (2001), 424–446. https://doi.org/10.1006/jmaa.2000.7617.

[8] S.J.H. Ghoncheh, Some fixed point theorems for Kannan mapping in the modular spaces, Cienc. eNatura: Santa Maria. 37 (2015), 462–466.

[9] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc. 136 (2008), 1359–1373.

[10] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. 41 (1991), 592–618. http://dml.cz/dmlcz/102493.

[11] M.A. Khamsi, O.D. Méndez, S. Reich, Modular geometric properties in variable exponent spaces, Mathematics. 10 (2022), 2509. https://doi.org/10.3390/math10142509.

[12] W. Orlicz, Über konjugierte exponentenfolgen, Stud. Math. 3 (1931), 200–211.

[13] J. Musielak, W. Orlicz, On Modular spaces, Stud. Math. 18 (1959), 49–65. http://eudml.org/doc/216946.

[14] H. Nakano, Modulared sequence spaces, Proc. Japan Acad. Ser. A Math. Sci. 27 (1951), 508–512. https://doi.org/10.3792/pja/1195571225.

[15] H. Nakano, Modulared semi-ordered linear spaces, Maruzen, Tokyo, 1950.

[16] V. Klee, Summability in $l(p_1, p_2, ...)$ spaces, Stud. Math. 25 (1965), 277–280.

[17] K.R. Rajagopal, M. Růzǐčka, On the modeling of electrorheological materials, Mech. Res. Commun. 23 (1996), 401–407. https://doi.org/10.1016/0093-6413(96)00038-9.

[18] M. Růžička, Electrorheological fluids: modeling and mathematical theory, Springer, Berlin, Heidelberg, 2000. https://doi.org/10.1007/BFb0104029.

[19] K. Sundaresan, Uniform convexity of Banach spaces $l(p_i)$, Stud. Math. 39 (1971), 227–231.

[20] A. Abdou, M. Khamsi, Fixed points of Kannan maps in the variable exponent sequence spaces $l_{p(.)}$, Mathematics. 8 (2020), 76. https://doi.org/10.3390/math8010076.

[21] R. Kannan, Some results on fixed points–II, Amer. Math. Mon. 76 (1969), 405–408. https://doi.org/10.1080/00029890.1969.12000228.

[22] M.A. Khamsi, W.M. Kozlowski, Fixed point theory in modular function spaces, Springer, Cham, 2015. https://doi.org/10.1007/978-3-319-14051-3.

[23] J. Musielak, Orlicz spaces and modular spaces, Lecture Notes in Mathematics, Vol. 1034, Springer, Berlin, Heidelberg, 1983. https://doi.org/10.1007/BFb0072210.

[24] K. Chaira, A. Eladraoui, M. Kabil, et al. Kannan fixed point theorem on generalized metric space with a graph, Appl. Math. Sci. 13 (2019), 263–274. https://doi.org/10.12988/ams.2019.9226.

[25] M.A. Khamsi, W.M. Kozlowski, Fixed point theory in modular function spaces, Springer, Cham, 2015. https://doi.org/10.1007/978-3-319-14051-3.

[26] D. Waterman, T. Ito, F. Barber, et al. Reflexivity and summability: the Nakano $l(p_i)$ spaces, Stud. Math. 33 (1969), 141–146. http://eudml.org/doc/217377.

[27] K. Fallahi, A. Aghanians, Fixed points for Chatterjea contractions on a metric Space with a graph, Int. J. Nonlinear Anal. Appl. 7 (2016), 49–58. https://doi.org/10.22075/ijnaa.2016.449.

[28] B. Samet. M. Turinici, Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications, Commun. Math. Anal. 13 (2012), 82–97.

[29] A.A.N. Abdou, M.A. Khamsi, Fixed point theorems in modular vector spaces, J. Nonlinear Sci. Appl. 10 (2017), 4046–4057. https://doi.org/10.22436/jnsa.010.08.01.