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FIXED POINT THEOREMS IN FUZZY b-METRIC SPACES USING TWO DIFFERENT t-NORMS

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Abstract. The primary objective of this study is to derive some theorems in fuzzy b-metric spaces under some assumptions on t-norms satisfying rational contractions. Some consequence results of our main finding are also given. At last, to validate our main results, two examples with graphical representation are also presented.

Keywords: fuzzy b-metric space; fuzzy b-metric space; fixed point; t-norm.
2020 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

The most active and growing area of research in pure mathematics is the theory of fixed points. Many different types of nonlinear problems that arise in numerous scientific fields can be expressed as fixed point problems. The Banach [4] contraction principle is an important tool to deal problems of this kind. In general, fixed point theory has continued to be successful in posing and resolving a variety of problems and has made a significant contribution to many real-life problems. However, with some strong assumptions, many robust fixed point theorems have been established. The focus of research in recent years have been on understanding the

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principles of fixed point problems and easing the constraints on them by substituting weakened versions of these original, strong assumptions. That's why now days for scientists and mathematicians, it is a topic of significant interest (refer to see [25, 26, 29]).

Zadeh [40], in 1965, led down a lovely concept that stands for the justification of ambiguity, imprecision, and manipulation. Compared with classical set theory, this theory is far more intriguing and helpful. These methods are utilized across various scientific and technical domains, including navigation, image processing, fractals, and many more. Since then, various authors have significantly broadened the theory of fuzzy sets and its practical uses in order to utilize this idea in topology and analysis.

In 1975, Authors in [27] introduced fuzzy metric space. Fuzzy metric spaces are just one of numerous extensions of the metric and metric space. This modification broadens the probabilistic metric space to encompass fuzzy situations. George and Veeramani [9] introduced and modified the notion of a fuzzy metric space which has crucial implications for quantum particle physics, particularly in relation to the $E-$ infinity and string theories, see also [37]. This research establishes a strong basis for the extension of fixed-point theory in fuzzy metric space. Grabiec [11], in 1983, outlined the fuzzy metric's completeness property and extended the Banach contraction theorem in these spaces. Since then many generalizations [10, 12, 21, 22, 8, 7, 23, 28, 35, 18, 13, 14, 19, 20, 15, 16, 38, 17, 38, 39]. and extensions have been given by various authors

The notion of $b-$ metric was initiated from the works of Bourbaki [5] and Bakhtin [3]. Later, Czerwik [6] introduced and formally defined the notion of b-metric space. Examples and fixed point results about these spaces were discussed by different authors [2, 1, 31]. On the other hand, Sedghi and Shobe [33, 34] introduced the notion of fuzzy $b-$ metric space, which is in fact far wider than that of fuzzy metric spaces, by replacing the triangle inequality with weaker one i.e.

$$\mathbb{B}(g,z,t+u) \geq \mathfrak{W}(\mathbb{B}(g,e,\frac{t}{\lambda}),\mathbb{B}(e,z,\frac{u}{\lambda})) \text{ with } \lambda \geq 1.$$

In 2020, Oner and Sostak [30] laid out the properties and definition of strong fuzzy $b-$ metric spaces. Some fixed point results in complete fuzzy strong b-metric spaces was also proved by Kanwal et al.[24].

Next section, will provide an overview of some important concepts (such as t- norm) related to fuzzy metric spaces, b-metric spaces and fuzzy b-metric spaces. Additionally, several fundamental terms and results that will be relevant to the sequel are discussed.

2. Fundamental Concepts and Relevant Literature

Let's start by defining the terms "$t-$ norm or conjunction".

Definition 2.1. [32] Let $I = [0,1]$. A binary operation $\mathfrak{W} : I \times I \to I$ is said to be continuous conjunction or t-norm if the following conditions are satisfied:

(1) $\mathfrak{W}$ is continuous, commutative and associative,
(2) $\mathfrak{W}(g,1) = g$ for all $g \in [0,1]$, (boundary condition)
(3) $\mathfrak{W}(g,e) \leq \mathfrak{W}(h,k)$ for $g,e,h,k \in [0,1]$ such that $g \leq h$ and $e \leq k$. (Monotonicity)

There are three most commonly used t-norms in literature:

(1) $\mathfrak{W}_P(g,e) = ge$, is called product triangular norm
(2) $\mathfrak{W}_{\min}(g,e) = \min\{g,e\}$, is called minimum triangular norm
(3) $\mathfrak{W}_L(g,e) = \max\{g+e-1,0\}$, is called Lukasiewicz triangular norm

Definition 2.2. [27] Let $\mathfrak{W}$ be a continuous t-norm, $\mathbb{Y}$ is an arbitrary (nonempty) set and $\mathbb{B}$ is a fuzzy set on $\mathbb{Y}^2 \times (0,\infty)$. Then a 3-tuple $(\mathbb{Y},\mathbb{B},\mathfrak{W})$ is known as a fuzzy metric space if for all $t,s > 0$ and for all $g,e,z \in \mathbb{Y}$, following conditions hold:

B -1.) $\mathbb{B}(g,e,t) > 0$,
B -2.) $\mathbb{B}(g,e,t) = 1$ if and only if $g = e$,
B -3.) $\mathbb{B}(g,e,t) = \mathbb{B}(e,g,t)$,
B -4.) $\mathfrak{W}(\mathbb{B}(g,e,t),\mathbb{B}(e,z,s)) \leq \mathbb{B}(g,z,t+s)$,
B -5.) $\mathbb{B}(g,e,\cdot) : (0,\infty) \to [0,1]$ is continuous.

Example 2.3. Let $\mathbb{B} : \mathbb{Y} \times \mathbb{Y} \times \mathbb{R}^+ \cup \{0\} \longrightarrow [0,1]$, and define $\mathbb{B}$ for all $l \geq 0$, by

$$\mathbb{B}(g,e,l) = \frac{\min\{g,e\}+l}{\max\{g,e\}+l} \ \forall \, g,e \in \mathbb{Y}.$$

Then $\mathbb{B}$ is a fuzzy metric.

Definition 2.4. [33] Let $\mathfrak{W}$ be a continuous conjunction, $b \geq 1$ is a real number, $\mathbb{Y}$ is an arbitrary (nonempty) set and $\mathbb{B}$ is a fuzzy set on $\mathbb{Y}^2 \times (0,\infty)$. Then a 3-tuple $(\mathbb{Y},\mathbb{B},\mathfrak{W})$ is known as a fuzzy $b-$ metric space if for all $t,s > 0$ and for all $g,e,z \in \mathbb{Y}$, following conditions hold:

FB-1.) $\mathbb{B}(g,e,t) > 0$,

FB-2.) $\mathbb{B}(g,e,t) = 1$ if and only if $g = e$,

FB-3.) $\mathbb{B}(g,e,t) = \mathbb{B}(e,g,t)$,

FB-4.) $\mathfrak{W}\left(\mathbb{B}\left(g,e,\frac{t}{b}\right),\mathbb{B}\left(e,z,\frac{s}{b}\right)\right) \leq \mathbb{B}(g,z,t+s)$,

FB-5.) $\mathbb{B}(g,e,\cdot) : (0,\infty) \to [0,1]$ is continuous.

Following are few examples of fuzzy $b-$metric spaces.

Example 2.5. [33] Suppose d is a $b-$ metric on $\mathbb{Y}$ and define $\mathbb{B}(g,e,t) = e^{-\frac{-d(g,e)}{t}}$. Define $t-$ norm as $g * e = ge \; \forall \; g,e \in [0,1]$. Then $\mathbb{B}$ is a fuzzy b-metric.

Example 2.6. [34] Suppose d is a $b-$ metric on $\mathbb{Y}$ and define $\mathbb{B}(g,e,t) = \frac{t}{t+d(g,e)}$. If we set $t-$ norm as $g * e = ge \; \forall \; g,e \in [0,1]$. Then $\mathbb{B}$ is a fuzzy b-metric.

Example 2.7. [34] Let $\mathbb{B}(g,e,t) = e^{\frac{-|g-e|^q}{t}}$, where $q > 1$ is a real number. Then $\mathbb{B}$ is a fuzzy $b-$ metric with $b = 2^{q-1}$.

If we set $q = 2$, in above example (Example 2.7),then it can be easily verify that $(\mathbb{Y},\mathbb{B},\mathfrak{W})$ is not a fuzzy metric space. It means that, in general, not every fuzzy b-metric on $\mathbb{Y}$ is a fuzzy metric on $\mathbb{Y}$.

Definition 2.8. [31] We say a function h defined from $\mathbb{R}$ to $\mathbb{R}$ be a b-non-decreasing function if for all $g,e \in \mathbb{R}$, $g > e$ implies $f(g) \geq f(e)$.

Lemma 2.9. [34] *Let $\mathbb{B}(g,e,\cdot)$ be a fuzzy b-metric space. Then $\mathbb{B}(g,e,t)$ is b non-decreasing with respect to t for all $g,e \in \mathbb{Y}$.*

Let us recollect the ideas of convergence, completeness and some important definitions and propositions in a fuzzy b-metric space.

Definition 2.10. [34] Let $(\mathbb{Y}, \mathbb{B}, \mathfrak{W})$ be a fuzzy b-metric space. Define an open sphere $B(g,r,t)$ with center at $g \in \mathbb{Y}$ and radius $r \in (0,1)$ as

$$B(g,r,t) = \{e \in \mathbb{Y} : \mathbb{B}(g,e,t) > 1-r\},\ \forall\, t > 0.$$

Definition 2.11. [34] Suppose $(\mathbb{Y}, \mathbb{B}, \mathfrak{W})$ is fuzzy b-metric space. Then we say that a sequence $\{g_i\} \in \mathbb{Y}$:

(1) converges to g if $\mathbb{B}(g_i, g, t) \to 1$ as $i \to \infty$ for each $t > 0$.

(2) is called a Cauchy sequence, if for all $t > 0$ and $\varepsilon \in (0,1)$, there exists $j_0 \in \mathbb{N}$ such that $1 - \varepsilon < \mathbb{B}\left(g_i, g_j, t\right)$ for all $i, j \geq j_0$.

Remark 2.12. Triplet $(\mathbb{Y}, \mathbb{B}, \mathfrak{W})$ is said to be complete fuzzy b-metric space, if every Cauchy sequence in $\mathbb{Y}$ is convergent.

Lemma 2.13. [34] *In a fuzzy $b-$metric space $(\mathbb{Y}, \mathbb{B}, \mathfrak{W})$, if a sequence $\{g_n\}$ in $\mathbb{Y}$ converges to g, then*

(1) g is always unique.

(2) it is a Cauchy sequence.

Lets recall the following proposition.

Proposition 2.14. [34] *Suppose we have a sequence $\{g_n\}$ converges to g in a fuzzy $b-$ metric space $(\mathbb{Y}, \mathbb{B}, \mathfrak{W})$. Then*

$$\mathbb{B}\left(g, e, \frac{t}{b}\right) \leq \limsup_{n\to\infty} \mathbb{B}(g_n, e, t) \leq \mathbb{B}(g, e, bt),$$
$$\mathbb{B}\left(g, e, \frac{t}{b}\right) \leq \liminf_{n\to\infty} \mathbb{B}(g_n, e, t) \leq \mathbb{B}(g, e, bt).$$

Lemma 2.15. [31] *Let $(\mathbb{Y}, \mathbb{B}, \mathfrak{W})$ be a fuzzy $b-$ metric space and $\{g_n\} \in \mathbb{Y}$ is a sequence. If there exists $0 < \lambda < \frac{1}{b}$ such that*

$$\mathbb{B}(g_n, g_{n+1}, t) \geq \mathbb{B}\left(g_{n-1}, g_n, \frac{t}{\lambda}\right), \quad n \in \mathbb{N}, t > 0,$$

and there exist $g_0, g_1 \in \mathbb{Y}$ and $v \in (0,1)$ such that

$$\lim_{n\to\infty} \mathfrak{W}_{i=n}^{\infty} \mathbb{B}\left(g_0, g_1, \frac{t}{v^i}\right) = 1, \quad t > 0.$$

Then $\{g_n\}$ is a Cauchy sequence.

Remark 2.16. In this paper, we'll additionally utilize a fuzzy b - metric space in the context of the Definition 2.4 with an extra constraint $\lim_{t\to\infty}\mathbb{B}(g,e,t)=1$.

The primary objective of this paper is to present two theorems, which guarantees the existence and uniqueness of fixed points under some assumptions on t - norms, within the context of a fuzzy b-metric spaces, satisfying rational contractions. In section 4, some consequence results of our main finding and an example is given to justify the stability and importance of our result.

3. Main Result

In this section, first we prove the following Lemma which is important in proving our main result. Secondly, we derive two results satisfying two different types rational contractions for two different type of t-norms, for single-valued continuous and discontinuous mappings.

Lemma 3.1. *If for some $\lambda\in(0,1)$ and $g,e\in\mathbb{Y}$,*

$$\frac{1}{\mathbb{B}(g,e,t)}\leq\frac{1}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)},\quad t>0, \tag{3.1}$$

then $g=e$.

Proof. Condition (3.1) gives that

$$\frac{1}{\mathbb{B}(g,e,t)}\leq\frac{1}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)},\quad t>0,$$

implies that

$$\mathbb{B}(g,e,t)\geq\mathbb{B}\left(g,e,\frac{t}{\lambda^n}\right),\quad n\in\mathbb{N},t>0.$$

taking limit $n\to\infty$, we get

$$\mathbb{B}(g,e,t)\geq\lim_{n\to\infty}\mathbb{B}\left(g,e,\frac{t}{\lambda^n}\right)=1,\quad t>0,$$

and by condition (B1) it follows that $g=e$. □

Theorem 3.2. *Let $f:\mathbb{Y}\to\mathbb{Y}$ be a map defined on complete fuzzy b-metric space $(\mathbb{Y},\mathbb{B},\mathfrak{W})$. Suppose, if there exists a $0<\lambda<\frac{1}{b}$ such that for all $g,e\in\mathbb{Y}$ and $t>0$,*

$$\mathbb{B}(fg,fe,t)\geq N(g,e,\frac{t}{\lambda}), \tag{3.2}$$

where

$$N(g,e,\frac{t}{\lambda}) = \min\left\{ \mathbb{B}\left(g,e,\tfrac{t}{\lambda}\right), \frac{\mathbb{B}\left(fg,g,\frac{t}{\lambda}\right)\mathbb{B}\left(fe,e,\frac{t}{\lambda}\right)}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)}, \frac{\mathbb{B}\left(fg,g,\frac{t}{\lambda}\right)[1+\mathbb{B}\left(fe,e,\frac{t}{\lambda}\right)]}{[1+\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)]} \right\} \tag{3.3}$$

and there exist $g_0 \in \mathbb{Y}$ *and* $v \in (0,1)$ *such that for all* $t > 0$

$$\lim_{n\to\infty} \mathfrak{W}_{i=n}^{\infty} \mathbb{B}\left(g_0, fg_0, \frac{t}{v^i}\right) = 1.$$

Then there exist a unique $g \in \mathbb{Y}$ *such that* $fg = g$.

Proof. Since $\mathbb{Y}$ is non-empty, therefore there exists $g_0 \in \mathbb{Y}$ and $g_{n+1} = fg_n, n \in \mathbb{N}$. By (3.2) for every $n \in \mathbb{N}$ and for all $t > 0$, with $g = g_n$ and $e = g_{n-1}$, we have

$$\mathbb{B}(g_{n+1}, g_n, t) = \mathbb{B}(fg_n, fg_{n-1}, t) \geq N(g_n, g_{n-1}, \frac{t}{\lambda}), \tag{3.4}$$

where

$$\begin{aligned} N(g_n, g_{n-1}, \frac{t}{\lambda}) &= \min\left\{ \begin{array}{c} \mathbb{B}\left(g_n, g_{n-1}, \frac{t}{\lambda}\right), \frac{\mathbb{B}\left(fg_n, g_n, \frac{t}{\lambda}\right)\mathbb{B}\left(fg_{n-1}, g_{n-1}, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g_n, g_{n-1}, \frac{t}{\lambda}\right)}, \\ \frac{\mathbb{B}\left(fg_n, g_n, \frac{t}{\lambda}\right)[1+\mathbb{B}\left(fg_{n-1}, g_{n-1}, \frac{t}{\lambda}\right)]}{[1+\mathbb{B}\left(g_n, g_{n-1}, \frac{t}{\lambda}\right)]} \end{array} \right\} \\ &= \min\left\{ \begin{array}{c} \mathbb{B}\left(g_n, g_{n-1}, \frac{t}{\lambda}\right), \frac{\mathbb{B}\left(g_{n+1}, g_n, \frac{t}{\lambda}\right)\mathbb{B}\left(g_n, g_{n-1}, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g_n, g_{n-1}, \frac{t}{\lambda}\right)}, \\ \frac{\mathbb{B}\left(g_{n+1}, g_n, \frac{t}{\lambda}\right)[1+\mathbb{B}\left(g_n, g_{n-1}, \frac{t}{\lambda}\right)]}{[1+\mathbb{B}\left(g_n, g_{n-1}, \frac{t}{\lambda}\right)]} \end{array} \right\} \\ &= \min\left\{\mathbb{B}\left(g_n, g_{n-1}, \frac{t}{\lambda}\right), \mathbb{B}\left(g_{n+1}, g_n, \frac{t}{\lambda}\right)\right\} \end{aligned}$$

If $\mathbb{B}\left(g_{n+1}, g_n, \frac{t}{\lambda}\right) < \mathbb{B}\left(g_n, g_{n-1}, \frac{t}{\lambda}\right)$, then $N(g_n, g_{n-1}, \frac{t}{\lambda}) = \mathbb{B}\left(g_{n+1}, g_n, \frac{t}{\lambda}\right)$. Therefore on using Eq (3.4) and by Lemma 3.1 it follows that $g_n = g_{n+1}$, $n \in \mathbb{N}$. This implies that $N(g_n, g_{n-1}, \frac{t}{\lambda}) = \mathbb{B}\left(g_n, g_{n-1}, \frac{t}{\lambda}\right)$, and so again Eq (3.4) gives

$$\mathbb{B}\left(g_{n+1}, g_n, t\right) \geq \mathbb{B}\left(g_n, g_{n-1}, \frac{t}{\lambda}\right), \quad n \in \mathbb{N}, t > 0$$

Thus we get $\{g_n\}$ is a Cauchy sequence (using Lemma 2.15.Therefore there exists $g \in \mathbb{Y}$ such that

$$\lim_{n\to\infty} g_n = g \text{ and } \lim_{n\to\infty} \mathbb{B}\left(g, g_n, t\right) = 1, \quad t > 0. \tag{3.5}$$

Next, We will show that $fg = g$. i.e. g is a fixed point of f. Let $\sigma_1 \in (\lambda b, 1)$ and $\sigma_2 = 1 - \sigma_1$. By Eq. (3.2) we have

$$\begin{aligned}\mathbb{B}(fg,g,t) \geq& \mathfrak{W}\left(\mathbb{B}\left(fg, fg_n, \frac{t\sigma_1}{b}\right), \mathbb{B}\left(g_{n+1}, g, \frac{t\sigma_2}{b}\right)\right)\\ \geq& \mathfrak{W}\left(N\left(g, g_n, \frac{t\sigma_1}{b}\right), \mathbb{B}\left(g_{n+1}, g, \frac{t\sigma_2}{b}\right)\right)\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}N\left(g, g_n, \frac{t\sigma_1}{b}\right) &= \min\left\{\begin{array}{c}\mathbb{B}\left(g, g_n, \frac{t\sigma_1}{b}\right), \frac{\mathbb{B}\left(fg,g,\frac{t\sigma_1}{b}\right)\mathbb{B}\left(fg_n,g_n,\frac{t\sigma_1}{b}\right)}{\mathbb{B}\left(g,g_n,\frac{t\sigma_1}{b}\right)},\\ \frac{\mathbb{B}\left(fg,g,\frac{t\sigma_1}{b}\right)[1+\mathbb{B}\left(fg_n,g_n,\frac{t\sigma_1}{b}\right)]}{[1+\mathbb{B}\left(g,g_n,\frac{t\sigma_1}{b}\right)]}\end{array}\right\}\\ &= \min\left\{\begin{array}{c}\mathbb{B}\left(g, g_n, \frac{t\sigma_1}{b}\right), \frac{\mathbb{B}\left(fg,g,\frac{t\sigma_1}{b}\right)\mathbb{B}\left(g_{n+1},g_n,\frac{t\sigma_1}{b}\right)}{\mathbb{B}\left(g,g_n,\frac{t\sigma_1}{b}\right)},\\ \frac{\mathbb{B}\left(fg,g,\frac{t\sigma_1}{b}\right)[1+\mathbb{B}\left(g_{n+1},g_n,\frac{t\sigma_1}{b}\right)]}{[1+\mathbb{B}\left(g,g_n,\frac{t\sigma_1}{b}\right)]}\end{array}\right\}\end{aligned}$$

Taking $n \to \infty$ and using Eq.(3.5), we get

$$\lim_{n\to\infty} N\left(g, g_n, \frac{t\sigma_1}{b}\right) = \min\left\{1, \mathbb{B}\left(fg, g, \frac{t\sigma_1}{b}\right)\right\} \tag{3.7}$$

Thus in Eq.(3.6), on taking $n \to \infty$ and using Eq. (3.7), we have

$$\mathbb{B}(fg,g,t) \geq \mathfrak{W}\left(\mathbb{B}\left(g, fg, \frac{t\sigma_1}{b\lambda}\right), 1\right) = \mathbb{B}\left(g, fg, \frac{t}{v}\right), \quad t > 0, \tag{3.8}$$

where $v = \frac{b\lambda}{\sigma_1} \in (0,1)$. So for all $t > 0$,

$$\mathbb{B}(fg,g,t) \geq \mathbb{B}\left(fg, g, \frac{t}{v}\right),$$

and hence it follows that $fg = g$ (by Lemma 3.1).

For uniqueness, suppose that $g \neq e$ are two fixed points for f, that is, $fg = g$ and $fe = e$. From Eq. (3.2), we have

$$\mathbb{B}(fg, fe, t) \geq N(g, e, \frac{t}{\lambda}), \tag{3.9}$$

where

$$\begin{aligned}&N(g,e,\frac{t}{\lambda})\\ &= \min\left\{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right), \frac{\mathbb{B}\left(fg,g,\frac{t}{\lambda}\right)\mathbb{B}\left(fe,e,\frac{t}{\lambda}\right)}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)}, \frac{\mathbb{B}\left(fg,g,\frac{t}{\lambda}\right)[1+\mathbb{B}\left(fe,e,\frac{t}{\lambda}\right)]}{[1+\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)]}\right\}\end{aligned}$$

$$= \min\left\{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right), \frac{\mathbb{B}\left(g,g,\frac{t}{\lambda}\right)\mathbb{B}\left(e,e,\frac{t}{\lambda}\right)}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)}, \frac{\mathbb{B}\left(g,g,\frac{t}{\lambda}\right)[1+\mathbb{B}\left(e,e,\frac{t}{\lambda}\right)]}{[1+\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)]}\right\}$$

$$= \min\left\{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right), \frac{1}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)}, \frac{2}{[1+\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)]}\right\} \tag{3.10}$$

Case - 1 If $\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)=1$, then Eq. (3.10) implies that $N(g,e,\frac{t}{\lambda})=1$. Consequently we get $\mathbb{B}(fg,fe,t)=1$. This is possible only if $g=e$. Hence the proof.

Case - 2 If $0<\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)\neq 1$, and $\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)<\min\left\{\frac{1}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)}, \frac{2}{[1+\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)]}\right\}$, then $N(g,e,\frac{t}{\lambda})=\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)$. Hence from Eq. (3.9), we get

$$\mathbb{B}(fg,fe,t)\geq \mathbb{B}(g,e,\frac{t}{\lambda})=\mathbb{B}(g,e,\frac{t}{\lambda})$$

This is possible ony if $g=e$.

Case - 3 If $0<\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)\neq 1$, and $\frac{1}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)}<\min\left\{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right), \frac{2}{[1+\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)]}\right\}$ or $\frac{2}{[1+\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)]}<\min\left\{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right), \frac{1}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)}\right\}$ then

$$N(g,e,\frac{t}{\lambda})=\frac{1}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)} \quad \text{or} \quad N(g,e,\frac{t}{\lambda})=\frac{2}{[1+\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)]}$$

Thus from inequalities (3.9), we get either

$$\mathbb{B}(fg,fe,t)\geq \frac{1}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)} \quad \text{or} \quad \mathbb{B}(fg,fe,t)\geq \frac{2}{[1+\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)]}$$

implies that

$$\mathbb{B}(fg,fe,t)\geq \frac{1}{\mathbb{B}\left(fg,fe,\frac{t}{\lambda}\right)} \quad \text{or} \quad \mathbb{B}(fg,fe,t)\geq \frac{2}{[1+\mathbb{B}\left(fg,fe,\frac{t}{\lambda}\right)]}$$

Consequently on using the condition (3.1) (in both cases), we get

$$\mathbb{B}(fg,fe,t)\geq 1$$

This implies that $g=e$. This completes the proof.

□

In our next Theorem, we refine the contraction and will make use of $\mathfrak{W}_{\min}$ conjunction to get unique fixed point for self maps in fuzzy b-metric space.

Theorem 3.3. *Let $f:\mathbb{Y}\to\mathbb{Y}$ be a map defined on complete fuzzy b-metric space $(\mathbb{Y},\mathbb{B},\mathfrak{W}_{\min})$. Suppose, if there exists a $0<\lambda<\frac{1}{b}$ such that for all $g,e\in\mathbb{Y}$ and $t>0$,*

$$\mathbb{B}(fg,fe,t)\geq \min\left\{\frac{\mathbb{B}\left(fg,e,\frac{2t}{\lambda}\right)\mathbb{B}\left(fe,g,\frac{t}{\lambda}\right)}{\mathbb{B}\left(g,fg,\frac{t}{\lambda}\right)},\frac{\mathbb{B}\left(fg,g,\frac{t}{\lambda}\right)\mathbb{B}\left(fe,e,\frac{t}{\lambda}\right)}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)}\right\} \tag{3.11}$$

for all $g,e\in\mathbb{Y},t>0$. Then there exist a unique $g\in\mathbb{Y}$ such that $fg=g$.

Proof. Let $g_0\in\mathbb{Y}$ and $g_{n+1}=fg_n, n\in\mathbb{N}$. By (3.11) with $g=g_n$ and $e=g_{n-1}$, for every $n\in\mathbb{N}$ and every $t>0$, we have

$$\begin{aligned}
&\mathbb{B}(fg_n,fg_{n-1},t)=\mathbb{B}(g_{n+1},g_n,t)\\
&\geq \min\left\{\frac{\mathbb{B}\left(fg_n,g_{n-1},\frac{2t}{\lambda}\right)\mathbb{B}\left(fg_{n-1},g_n,\frac{t}{\lambda}\right)}{\mathbb{B}\left(g_n,fg_n,\frac{t}{\lambda}\right)},\frac{\mathbb{B}\left(fg_n,g_n,\frac{t}{\lambda}\right)\mathbb{B}\left(fg_{n-1},g_{n-1},\frac{t}{\lambda}\right)}{\mathbb{B}\left(g_n,g_{n-1},\frac{t}{\lambda}\right)}\right\}\\
&\geq \min\left\{\frac{\mathbb{B}\left(g_{n+1},g_{n-1},\frac{2t}{\lambda}\right)\mathbb{B}\left(g_n,g_n,\frac{t}{\lambda}\right)}{\mathbb{B}\left(g_n,g_{n+1},\frac{t}{\lambda}\right)},\frac{\mathbb{B}\left(g_{n+1},g_n,\frac{t}{\lambda}\right)\mathbb{B}\left(g_n,g_{n-1},\frac{t}{\lambda}\right)}{\mathbb{B}\left(g_n,g_{n-1},\frac{t}{\lambda}\right)}\right\} \qquad (3.12)
\end{aligned}$$

On using condition (B 4) (of Definition 2.4) and assumption that $\mathfrak{W}=\mathfrak{W}_{\min}$, we have

$$\begin{aligned}
&\mathbb{B}(g_{n+1},g_n,t)\\
&\geq \min\left\{\frac{\min\left\{\mathbb{B}\left(g_{n+1},g_n,\frac{t}{b\lambda}\right),\mathbb{B}\left(g_n,g_{n-1},\frac{t}{b\lambda}\right)\right\}}{\mathbb{B}\left(g_n,g_{n+1},\frac{t}{\lambda}\right)},\mathbb{B}\left(g_{n+1},g_n,\frac{t}{\lambda}\right)\right\}\\
&> \min\left\{\frac{\mathbb{B}\left(g_{n+1},g_n,\frac{t}{b\lambda}\right),\mathbb{B}\left(g_n,g_{n-1},\frac{t}{b\lambda}\right)}{\mathbb{B}\left(g_n,g_{n+1},\frac{t}{b\lambda}\right)},\mathbb{B}\left(g_{n+1},g_n,\frac{t}{b\lambda}\right)\right\}\\
&\geq \min\left\{\mathbb{B}\left(g_n,g_{n-1},\frac{t}{b\lambda}\right),\mathbb{B}\left(g_{n+1},g_n,\frac{t}{b\lambda}\right)\right\} \qquad (3.13)
\end{aligned}$$

If $\mathbb{B}\left(g_{n+1},g_n,\frac{t}{b\lambda}\right)<\mathbb{B}\left(g_n,g_{n-1},\frac{t}{b\lambda}\right)$, then on using Eq (3.13) and by Lemma 3.1 it follows that $g_n=g_{n+1}$, $n\in\mathbb{N}$. This proves the result.

Thus assume that $\mathbb{B}\left(g_{n+1},g_n,\frac{t}{b\lambda}\right)>\mathbb{B}\left(g_n,g_{n-1},\frac{t}{b\lambda}\right)$ and so from Eq (3.13), we get

$$\mathbb{B}(g_{n+1},g_n,t)\geq\mathbb{B}\left(g_n,g_{n-1},\frac{t}{b\lambda}\right),\quad n\in\mathbb{N},t>0$$

Thus we get $\{g_n\}$ is a Cauchy sequence (using Lemma 2.15.Therefore there exists $g\in\mathbb{Y}$ such that

$$\lim_{n\to\infty}g_n=g \text{ and } \lim_{n\to\infty}\mathbb{B}(g,g_n,t)=1,\quad t>0. \tag{3.14}$$

Next we prove that g is a fixed point for f.

Let $\sigma_1 \in (\lambda b^2, 1)$ and $\sigma_2 = 1-\sigma_1$. On using property (B 4) for $\mathfrak{W} = \mathfrak{W}_{\min}$, we have

$$\mathbb{B}(fg,g,t) \geq \min\left\{\mathbb{B}\left(fg, fg_n, \frac{t\sigma_1}{b}\right), \mathbb{B}\left(g_{n+1}, g, \frac{t\sigma_2}{b}\right)\right\} \tag{3.15}$$

From Eq. (3.11) for $n \in \mathbb{N}, t>0$, we have

$$\begin{aligned}
&\mathbb{B}\left(fg, fg_n, \frac{t\sigma_1}{b}\right)\\
&\geq \min\left\{\frac{\mathbb{B}\left(fg, g_n, \frac{2t\sigma_1}{b\lambda}\right)\mathbb{B}\left(fg_n, g, \frac{t\sigma_1}{b\lambda}\right)}{\mathbb{B}\left(g, fg, \frac{t\sigma_1}{b\lambda}\right)}, \frac{\mathbb{B}\left(fg, g, \frac{t\sigma_1}{b\lambda}\right)\mathbb{B}\left(fg_n, g_n, \frac{t\sigma_1}{b\lambda}\right)}{\mathbb{B}\left(g, g_n, \frac{t\sigma_1}{b\lambda}\right)}\right\}\\
&\geq \min\left\{\frac{\mathbb{B}\left(fg, g_n, \frac{2t\sigma_1}{b\lambda}\right)\mathbb{B}\left(g_{n+1}, g, \frac{t\sigma_1}{b\lambda}\right)}{\mathbb{B}\left(g, fg, \frac{t\sigma_1}{b\lambda}\right)}, \frac{\mathbb{B}\left(fg, g, \frac{t\sigma_1}{b\lambda}\right)\mathbb{B}\left(g_{n+1}, g_n, \frac{t\sigma_1}{b\lambda}\right)}{\mathbb{B}\left(g, g_n, \frac{t\sigma_1}{b\lambda}\right)}\right\}\\
&\geq \min\left\{\begin{array}{c}\frac{\min\left\{\mathbb{B}\left(fg, g, \frac{t\sigma_1}{b^2\lambda}\right), \mathbb{B}\left(g, g_n, \frac{t\sigma_1}{b^2\lambda}\right)\right\}\mathbb{B}\left(g_{n+1}, g, \frac{t\sigma_1}{b\lambda}\right)}{\mathbb{B}\left(g, fg, \frac{t\sigma_1}{b\lambda}\right)},\\ \frac{\mathbb{B}\left(fg, g, \frac{t\sigma_1}{b\lambda}\right)\mathbb{B}\left(g_{n+1}, g_n, \frac{t\sigma_1}{b\lambda}\right)}{\mathbb{B}\left(g, g_n, \frac{t\sigma_1}{b\lambda}\right)}\end{array}\right\}
\end{aligned}$$

Taking $n \to \infty$ and using Eq.(3.14), we get

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{B}\left(fg, fg_n, \frac{t\sigma_1}{b}\right) &\geq \min\left\{\frac{\min\left\{\mathbb{B}\left(fg, g, \frac{t\sigma_1}{b^2\lambda}\right), 1\right\}}{\mathbb{B}\left(g, fg, \frac{t\sigma_1}{b\lambda}\right)}, \mathbb{B}\left(fg, g, \frac{t\sigma_1}{b\lambda}\right)\right\}\\
&> \min\left\{1, \mathbb{B}\left(fg, g, \frac{t\sigma_1}{b\lambda}\right)\right\}.
\end{aligned} \tag{3.16}$$

On taking $n \to \infty$ in Eq.(3.15) and using Eq. (3.16), we have

$$\begin{aligned}
\mathbb{B}(fg,g,t) &\geq \min\left\{1, \mathbb{B}\left(fg, g, \frac{t\sigma_1}{b\lambda}\right), 1\right\}\\
&\geq \mathbb{B}\left(fg, g, \frac{t}{v}\right), \quad t>0,
\end{aligned}$$

where $v = \frac{b\lambda}{\sigma_1} \in (0,1)$.Therefore Lemma 3.1 implies that $fg = g$.

For uniqueness, suppose that g and e are fixed points for f, that is, $fg = g$ and $fe = e$.

By Eq. (3.11), we get

$$\begin{aligned}
\mathbb{B}(g,e,t) &= \mathbb{B}(fg,fe,t)\\
&\geq \min\left\{\frac{\mathbb{B}\left(fg, e, \frac{2t}{\lambda}\right)\mathbb{B}\left(fe, g, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g, fg, \frac{t}{\lambda}\right)}, \frac{\mathbb{B}\left(fg, g, \frac{t}{\lambda}\right)\mathbb{B}\left(fe, e, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g, e, \frac{t}{\lambda}\right)}\right\}
\end{aligned}$$

$$\geq \min\left\{ \frac{\mathbb{B}\left(fg,e,\frac{2t}{\lambda}\right)\mathbb{B}\left(e,g,\frac{t}{\lambda}\right)}{\mathbb{B}\left(g,g,\frac{t}{\lambda}\right)}, \frac{\mathbb{B}\left(g,g,\frac{t}{\lambda}\right)\mathbb{B}\left(e,e,\frac{t}{\lambda}\right)}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)} \right\}$$

$$\geq \min\left\{ \min\left\{ \mathbb{B}\left(fg,g,\frac{t}{b\lambda}\right), \mathbb{B}\left(fg,e,\frac{t}{b\lambda}\right) \right\}, \frac{1}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)} \right\}$$

$$\geq \min\left\{ 1, \mathbb{B}\left(fg,e,\frac{t}{b\lambda}\right), \frac{1}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)} \right\} \tag{3.17}$$

This implies that either

$$\mathbb{B}(g,e,t) = 1 \text{ or } \mathbb{B}(g,e,t) = \mathbb{B}\left(fg,e,\frac{t}{b\lambda}\right) \text{ or } \mathbb{B}(g,e,t) = \frac{1}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)}$$

In all three cases, if we use conditions of Definition 2.4 and Lemma 3.1, we get that $g = e$ i.e fixed point is unique. This completes the proof of the Theorem. □

4. Corollaries and Examples

Here first we present some consequences of our main finding. Some of them are new in nature and few are generalized version of previous derived results. Later we furnish two examples in support of our main findings. Consider for all $g,e \in \mathbb{Y}, t > 0$,

$$N(g,e,\frac{t}{\lambda}) = \min\left\{ \mathbb{B}\left(g,e,\frac{t}{\lambda}\right), \frac{\mathbb{B}\left(fg,g,\frac{t}{\lambda}\right)\mathbb{B}\left(fe,e,\frac{t}{\lambda}\right)}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)}, \frac{\mathbb{B}\left(fg,g,\frac{t}{\lambda}\right)[1+\mathbb{B}\left(fe,e,\frac{t}{\lambda}\right)]}{[1+\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)]} \right\}$$

In general, either

$$N(g,e,\frac{t}{\lambda}) = \mathbb{B}\left(g,e,\frac{t}{\lambda}\right)$$

or

$$N(g,e,\frac{t}{\lambda}) = \frac{\mathbb{B}\left(fg,g,\frac{t}{\lambda}\right)\mathbb{B}\left(fe,e,\frac{t}{\lambda}\right)}{\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)}$$

or

$$N(g,e,\frac{t}{\lambda}) = \frac{\mathbb{B}\left(fg,g,\frac{t}{\lambda}\right)[1+\mathbb{B}\left(fe,e,\frac{t}{\lambda}\right)]}{[1+\mathbb{B}\left(g,e,\frac{t}{\lambda}\right)]}$$

On making use of above three equalities in Theorem 3.2, we get the following three results.

Corollary 4.1. *Let $f : \mathbb{Y} \to \mathbb{Y}$ be a map defined on complete fuzzy b-metric space $(\mathbb{Y}, \mathbb{B}, \mathfrak{W})$. Suppose, if there exists a $0 < \lambda < \frac{1}{b}$ such that for all $g,e \in \mathbb{Y}$ and $t > 0$,*

$$\mathbb{B}(fg,fe,t) \geq \mathbb{B}(g,e,\frac{t}{\lambda}),$$

and there exist $g_0 \in \mathbb{Y}$ *and* $v \in (0,1)$ *such that for all* $t > 0$.

$$\lim_{n\to\infty} \mathfrak{W}_{i=n}^{\infty} \mathbb{B}\left(g_0, fg_0, \frac{t}{v^i}\right) = 1.$$

Then f *has a unique fixed point in* $\mathbb{Y}$.

Corollary 4.2. *Let* $f : \mathbb{Y} \to \mathbb{Y}$ *be a map defined on complete fuzzy b-metric space* $(\mathbb{Y}, \mathbb{B}, \mathfrak{W})$. *Suppose, if there exists a* $0 < \lambda < \frac{1}{b}$ *such that for all* $g, e \in \mathbb{Y}$ *and* $t > 0$,

$$\mathbb{B}(fg, fe, t) \geq \frac{\mathbb{B}\left(fg, g, \frac{t}{\lambda}\right) \mathbb{B}\left(fe, e, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g, e, \frac{t}{\lambda}\right)},$$

and there exist $g_0 \in \mathbb{Y}$ *and* $v \in (0,1)$ *such that for all* $t > 0$

$$\lim_{n\to\infty} \mathfrak{W}_{i=n}^{\infty} \mathbb{B}\left(g_0, fg_0, \frac{t}{v^i}\right) = 1.$$

Then f *has a unique fixed point in* $\mathbb{Y}$.

Corollary 4.3. *Let* $f : \mathbb{Y} \to \mathbb{Y}$ *be a map defined on complete fuzzy b-metric space* $(\mathbb{Y}, \mathbb{B}, \mathfrak{W})$. *Suppose, if there exists a* $0 < \lambda < \frac{1}{b}$ *such that for all* $g, e \in \mathbb{Y}$ *and* $t > 0$,

$$\mathbb{B}(fg, fe, t) \geq \frac{\mathbb{B}\left(fg, g, \frac{t}{\lambda}\right) \left[1 + \mathbb{B}\left(fe, e, \frac{t}{\lambda}\right)\right]}{\left[1 + \mathbb{B}\left(g, e, \frac{t}{\lambda}\right)\right]},$$

and there exist $g_0 \in \mathbb{Y}$ *and* $v \in (0,1)$ *such that for all* $t > 0$

$$\lim_{n\to\infty} \mathfrak{W}_{i=n}^{\infty} \mathbb{B}\left(g_0, fg_0, \frac{t}{v^i}\right) = 1.$$

Then f *has a unique fixed point in* $\mathbb{Y}$.

Corollary 4.4. *Let* $f : \mathbb{Y} \to \mathbb{Y}$ *be a map defined on complete fuzzy b-metric space* $(\mathbb{Y}, \mathbb{B}, \mathfrak{W})$. *Suppose, if there exists a* $0 < \lambda < \frac{1}{b}$ *such that for all* $g, e \in \mathbb{Y}$ *and* $t > 0$,

$$\mathbb{B}(fg, fe, t) \geq k_1 \frac{\mathbb{B}\left(fg, e, \frac{2t}{\lambda}\right) \mathbb{B}\left(fe, g, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g, fg, \frac{t}{\lambda}\right)} + k_2 \frac{\mathbb{B}\left(fg, g, \frac{t}{\lambda}\right) \mathbb{B}\left(fe, e, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g, e, \frac{t}{\lambda}\right)}$$

where $k_1, k_2 > 0$ *with* $k_1 + k_2 > 1$, *and there exist* $g_0 \in \mathbb{Y}$ *and* $v \in (0,1)$ *such that for all* $t > 0$

$$\lim_{n\to\infty} \mathfrak{W}_{i=n}^{\infty} \mathbb{B}\left(g_0, fg_0, \frac{t}{v^i}\right) = 1.$$

Then f *has a unique fixed point in* $\mathbb{Y}$.

Proof. For all $k_1, k_2 > 0$, we have

$$\begin{aligned}\mathbb{B}(fg, fe, t) &\geq k_1 \frac{\mathbb{B}\left(fg, e, \frac{2t}{\lambda}\right)\mathbb{B}\left(fe, g, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g, fg, \frac{t}{\lambda}\right)} + k_2 \frac{\mathbb{B}\left(fg, g, \frac{t}{\lambda}\right)\mathbb{B}\left(fe, e, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g, e, \frac{t}{\lambda}\right)} \\ &\geq (k_1 + k_2) \min\left\{\frac{\mathbb{B}\left(fg, e, \frac{2t}{\lambda}\right)\mathbb{B}\left(fe, g, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g, fg, \frac{t}{\lambda}\right)}, \frac{\mathbb{B}\left(fg, g, \frac{t}{\lambda}\right)\mathbb{B}\left(fe, e, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g, e, \frac{t}{\lambda}\right)}\right\}.\end{aligned}$$

Since $k_1 + k_2 > 1$, then

$$\mathbb{B}(fg, fe, t) \geq \min\left\{\frac{\mathbb{B}\left(fg, e, \frac{2t}{\lambda}\right)\mathbb{B}\left(fe, g, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g, fg, \frac{t}{\lambda}\right)}, \frac{\mathbb{B}\left(fg, g, \frac{t}{\lambda}\right)\mathbb{B}\left(fe, e, \frac{t}{\lambda}\right)}{\mathbb{B}\left(g, e, \frac{t}{\lambda}\right)}\right\}.$$

On applying Theorem 3.3, we get result. □

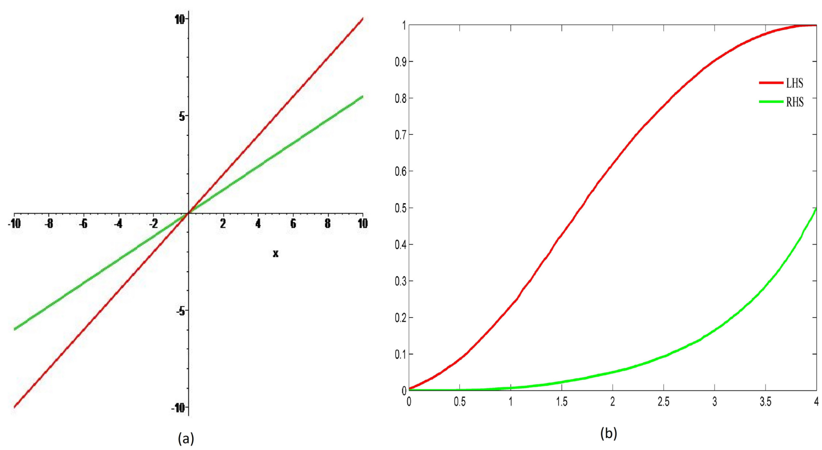

FIGURE 1. Graph of the function $f(g)$ and the inequality (3.2).

Example 4.5. Let $\mathbb{Y} = [0,4]$. Define

$$\mathbb{B}(g, e, t) = \exp\left(-\frac{(g-e)^2}{t}\right) \quad \text{and} \quad f(g) = \frac{3}{5}g$$

for all $g, e \in \mathbb{Y}$, and $t > 0$. Clearly the triplet $(\mathbb{Y}, \mathbb{B}, \mathfrak{W})$ is a fuzzy b-metric space with $b = 2^{2-1} = 2$.

Also, $f(0) = 0$ is the only one fixed point of f in $\mathbb{Y} = [0,4]$ (see Figure 1).

Now from eq (3.2), for all $g, e \in [0,1]$ and for $\lambda \in (\frac{1}{5}, \frac{1}{2})$, we have

$$\mathbb{B}(fg, fe, t) = \exp\left(-\frac{(3/5)^2(g-e)^2}{t}\right)$$

$$= \exp\left(-\frac{9(g-e)^2}{25t}\right)$$
$$\geq N\left(g, e, \frac{t}{\lambda}\right).$$

In order to get the better understanding of the example, Graphical representation of function $f(g)$ and the inequality (3.2) are given in Fig. 1.

Example 4.6. Let $\mathbb{Y} = (0,1], \mathbb{B}(g,e,t) = e^{-\frac{(g-e)^2}{t}}$ and define $\mathfrak{W} = \mathfrak{W}_p(a,b) = ab$. Then $(\mathbb{Y}, \mathbb{B}, \mathfrak{W})$ is a complete b-fuzzy metric space with $b = 2$.

Let us define a function

$$h(g) = \begin{cases} \frac{3}{2}, & g \in (0,1) \\ 1, & g = 1 \end{cases}$$

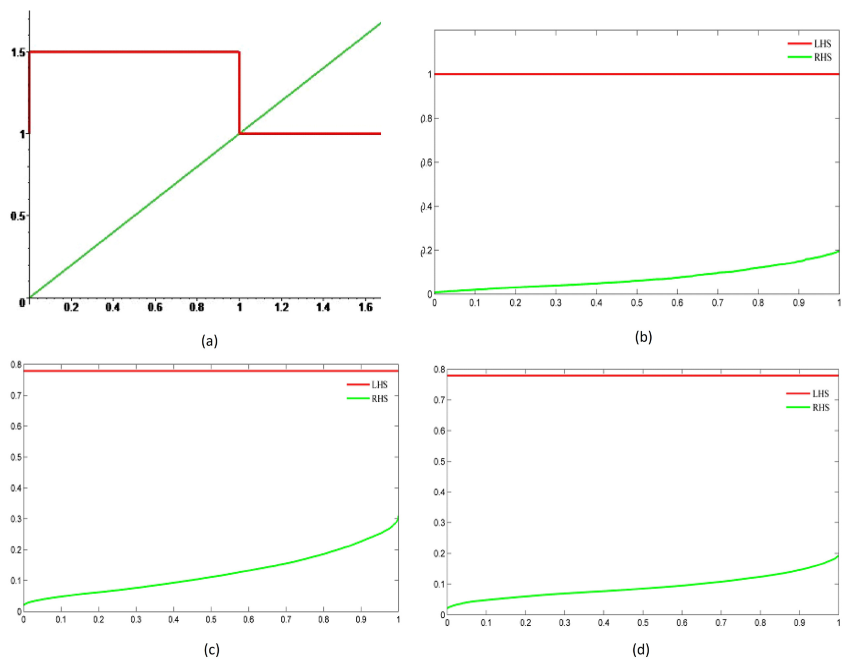

FIGURE 2. Graph of the function $h(g)$ and the inequality (3.11) for different cases.

Case- E_1: Let us start with trivial case, that is, if $g = e = 1$. Then $\mathbb{B}(hg, he, t) = 1 = \min\{1,1\}, t > 0$. Thus Condition (3.11) of Theorem 3.3 satisfied.

Case- E_2: If $g, e \in (0,1)$, then for any $0 < \lambda < \frac{1}{4} = \frac{1}{b^2}$, and by referring Fig 2 -(b), we can see that condition (3.11) of Theorem 3.3 holds good.

Case- E_3: If $g \in (0,1)$ and $e = 1$, then for any $0 < \lambda < \frac{1}{4} = \frac{1}{b^2}$, Condition (3.11) of Theorem 3.3 holds good (refer to see Fig 2 -(c)).

Case- E_4: If $e \in (0,1)$ and $g = 1$, then for any $0 < \lambda < \frac{1}{4} = \frac{1}{b^2}$, Condition (3.11) of Theorem 3.3 holds good. (refer to see Fig 2 -(d))

Thus all the conditions of Theorem 3.3 are satisfied. Also 1 is the unique fixed point of the map $h(g)$.

5. Conclusion

In this article, two theorems (Theorem 3.2 and Theorem 3.3) are proved under two different t- norms which guarantee the existence and uniqueness of fixed points in fuzzy b-metric spaces satisfying rational expression. We have also presented some consequence results. Some of them are extension of existing results of literature, such as Grabiec [11] and Gupta and Mani [17], and few of them are new in nature (such as Corollary 4.3 and Corollary 4.4). In support of our main findings, two examples (Example 2.3 and Example 2.5) are demonstrated with graphical representation of inequalities.

Conflict of Interests

The authors declare that there is no conflict of interests.

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