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FIXED-POINTS OF INTERPOLATIVE KANNAN TYPE CONTRACTIONS IN BIPOLAR METRIC SPACES

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Abstract: In this paper, we introduce (λ, α)-interpolative and (λ, α, β)-interpolative Kannan type contractions and establish some fixed-point theorems in bipolar metric spaces. Additionally, these theorems expand and apply a number of intriguing findings from metric fixed-point theory to the bipolar metric setting.

Keywords: fixed-point; iterative methods; interpolative; contraction; bipolar metric spaces.

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1. INTRODUCTION

Metric fixed-point theory is gaining prominence in mathematics as a result of its extensive applications in the areas of applied mathematics and the sciences. The use of fixed-point theory to the study of non-linear processes has many advantages. There are numerous generalizations of the idea of a metric space in literature. Mutlu and Gurdal [3] presented one of the most recent generalizations, the bipolar metric space, with the idea that distances

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frequently occur between elements of two distinct sets rather than between points of a single set in real-world applications. Bipolar metrics were created to define these different distances as a result. Examples of fundamental distances include those between lines and points in a Euclidean space, those between points and sets in metric spaces, those between a class of students and a group of activities, lifetime mean distances between individuals and locations, and many more. On fixed-point in bipolar metric spaces, numerous researchers have written numerous publications that can be found, to mention a few, in [2, 3, 4, 6, 7, 8, 9, 16] and the references therein. The existence and advancement of fixed-point theorems are a result of this novel idea of generalization and improvement of metric spaces. In light of this, bipolar metric fixed-point theory is a current study subject that is receiving a lot of interest and appears to have a bright future.

In 1968, Kannan introduced an interesting type of contraction mapping which is not continuous and it poses a fixed-point [11]. Kannan's theorem asserts that if $(\mathfrak{A}, \Delta)$ be a complete metric space and let $\Gamma: \mathfrak{A} \to \mathfrak{A}$ be a mapping such that there exists $k < \frac{1}{2}$ satisfying

$$\Delta(\Gamma p, \Gamma q) \leq k[\Delta(p, \Gamma p) + \Delta(q, \Gamma q)] \tag{1.1}$$

for all $p, q \in \mathfrak{A}$. Then, Γ has a unique fixed-point $r \in \mathfrak{A}$.

Kannan's theorem has been generalized in different ways by many authors (see [10, 13-15]); one of the latest generalizations was given by Karapinar in [5]. Karapinar introduced a Kannan type contraction mapping called interpolative Kannan type contraction and proved a fixed-point result on it.

Definition 1.1 (see [5]) Let $(\mathfrak{A}, \Delta)$ be a metric space. A self-mapping $\Gamma: (\mathfrak{A}, \Delta) \to (\mathfrak{A}, \Delta)$ is said to be an interpolative Kannan type contraction if there exist a constant $\lambda \in [0,1), \alpha \in (0,1)$ such that

$$\Delta(\Gamma q, \Gamma p) \leq \lambda\big(\Delta(p, \Gamma p)\big)^{\alpha}\big(\Delta(\Gamma q, q)\big)^{1-\alpha} \tag{1.2}$$

Theorem 1.2 (see [5]) Let $(\mathfrak{A}, \Delta)$ be a complete metric space and $\Gamma: (\mathfrak{A}, \Delta) \to (\mathfrak{A}, \Delta)$ be an interpolative Kannan type contraction mapping. Then, Γ has a unique fixed-point.

In this article, we introduce the so-called (λ, α) -interpolative and (λ, α, β) -interpolative Kannan contractions and establish the reality of fixed-point s for contravariant mappings on bipolar metric spaces. We demonstrate how some well-known classical conclusions can be readily recovered under the choice of convenient constants.

2. PRELIMINARIES

Throughout this essay, the terms $\mathbb{N}$ and $\mathbb{R}$ refer to the sets of all positive integers and the sets of all real numbers, respectively. To specifically denote the set of all positive real numbers, we write $\mathbb{R}^+ = [0, +\infty)$. We go through some basic ideas and definitions in mathematics to make this paper self-sufficient.

***Definition* 2.1** (see [3]) Let $\mathfrak{A}$ and $\mathfrak{B}$ be non-empty sets. A function $\Delta: \mathfrak{A} \times \mathfrak{B} \to \mathbb{R}^+$ is a bipolar metric on the pair $(\mathfrak{A}, \mathfrak{B})$, if it satisfies the following conditions:

(b1) $\Delta(p, q) = 0 \Leftrightarrow p = q$, whenever $(p, q) \in (\mathfrak{A}, \mathfrak{B})$.

(b2) $\Delta(p, q) = \Delta(q, p)$, whenever $p, q \in \mathfrak{A} \cap \mathfrak{B}$.

(b3) $\Delta(p_1, q_2) \leq \Delta(p_1, q_1) + \Delta(p_2, q_1) + \Delta(p_2, q_2), \forall\, p_1, p_2 \in \mathfrak{A}$ and $\forall\, q_1, q_2 \in \mathfrak{B}$.

The triple $(\mathfrak{A}, \mathfrak{B}, \Delta)$ is called a bipolar metric space. In specifically, a space is said to be disjointed if $\mathfrak{A} \cap \mathfrak{B} = \emptyset$, and joint otherwise. The left pole and the right pole of $(\mathfrak{A}, \mathfrak{B}, \Delta)$ are the sets $\mathfrak{A}$ and $\mathfrak{B}$, respectively.

Example 2.2 (see [3]) Consider the case when $(\mathfrak{A}, \Delta)$ is a metric space. Consequently, $(\mathfrak{A}, \mathfrak{A}, \Delta)$ is a bipolar metric space. But if $(\mathfrak{A}, \mathfrak{B}, \Delta)$ is a bipolar metric space with $\mathfrak{A} = \mathfrak{B}$, then $(\mathfrak{A}, \Delta)$ is a metric space.

Definition 2.3 (see [3]) Let $(\mathfrak{A}, \mathfrak{B}, \Delta)$ be a bipolar metric space. Then,

1) $\mathfrak{A}$ = set of left points; $\mathfrak{B}$ = set of right points; $\mathfrak{A} \cap \mathfrak{B}$ = set of central points. In particular, if $\mathfrak{A} \cap \mathfrak{B} = \emptyset$, the space is called disjoint, and otherwise it is called joint. Unless otherwise stated, we shall work with joint spaces.
2) A sequence (p_n) on the set $\mathfrak{A}$ is called a left sequence, and a sequence (q_n) on $\mathfrak{B}$ is called a right sequence. In a bipolar metric space, a left or a right sequence is called simply a sequence.

3) A sequence (p_n) is said to be convergent to a point p if and only if (p_n) is a left sequence, $\lim_{n\to\infty} \Delta(p_n p) = 0$ and $p \in \mathfrak{B}$, or (p_n) is a right sequence, $\lim_{n\to\infty} \Delta(p, p_n) = 0$ and $p \in \mathfrak{A}$.
4) A bisequence (p, q_n) on $(\mathfrak{A}, \mathfrak{B}, \Delta)$ is a sequence on the set $\mathfrak{A} \times \mathfrak{B}$. Furthermore, if the sequences (p_n) and (q_n) are convergent, then the bisequence (p_n, q_n) is said to be convergent. In addition, if (p) and (q_n) converge to a common point $r \in \mathfrak{A} \cap \mathfrak{B}$, then (p_n, q_n) is called biconvergent.
5) A bisequence (p_n, q_n) is a Cauchy bisequence if $\lim_{n\to\infty} \Delta(p_n, q_n) = 0$.
6) A bipolar metric space is called complete if every Cauchy bisequence is convergent, hence biconvergent.

Example 2.4 (see [3]) Assume that $\mathfrak{B}$ is the class of all nonempty compact subsets of $\mathbb{R}$ and that $\mathfrak{A}$ is the class of all singleton subsets of $\mathbb{R}$. We define $\Delta: \mathfrak{A} \times \mathfrak{B} \to \mathbb{R}^+$as $\Delta(p, A) = |p - \inf(A)| + |p - \sup(A)|$. The triple $(\mathfrak{A}, \mathfrak{B}, \Delta)$ is a complete bipolar metric space.

Definition 2.5 (see [3]) Let $(\mathfrak{A}_1, \mathfrak{B}_1)$ and $(\mathfrak{A}_2, \mathfrak{B}_2)$ be two pair of sets. A map $\Gamma: \mathfrak{A}_1 \cup \mathfrak{B}_1 \to \mathfrak{A}_2 \cup \mathfrak{B}_2$ is called

1) covariant if $\Gamma(\mathfrak{A}_1) \subseteq \mathfrak{A}_2$ and $\Gamma(\mathfrak{B}_1) \subseteq \mathfrak{B}_2$, and it is denoted as $\Gamma: (\mathfrak{A}_1, \mathfrak{B}_1) \rightrightarrows (\mathfrak{A}_2, \mathfrak{B}_2)$.
2) contravariant if $\Gamma(\mathfrak{A}_1) \subseteq \mathfrak{B}_2$ and $\Gamma(\mathfrak{B}_1) \subseteq \mathfrak{A}_2$, and it is denoted as $\Gamma: (\mathfrak{A}_1, \mathfrak{B}_1) \rightleftarrows (\mathfrak{A}_2, \mathfrak{B}_2)$.

Definition 2.6 (see [3]) A covariant or a contravariant map Γ from the bipolar metric space $(\mathfrak{A}_1, \mathfrak{B}_1, \Delta_1)$ to the bipolar metric space $(\mathfrak{A}_2, \mathfrak{B}_2, \Delta_2)$ is continuous, if and only if $p_n \to q$ on $(\mathfrak{A}_1, \mathfrak{B}_1, \Delta_1)$ implies $\Gamma(p_n) \to \Gamma(q)$ on $(\mathfrak{A}_2, \mathfrak{B}_2, \Delta_2)$.

3. MAIN RESULTS

We start with the following definitions.

Definition 3.1 Let $(\mathfrak{A}, \mathfrak{B}, \Delta)$ be a bipolar metric space and $\Gamma: (\mathfrak{A}, \mathfrak{B}, \Delta) \rightleftarrows (\mathfrak{A}, \mathfrak{B}, \Delta)$ a contravariant self-map. We shall call Γ a (λ, α)-interpolative Kannan contraction, if there exist $\lambda \in [0,1), \alpha \in (0,1)$ such that

$$\Delta(\Gamma q, \Gamma p) \leq \lambda\big(\Delta(p, \Gamma p)\big)^{\alpha}\big(\Delta(\Gamma q, q)\big)^{1-\alpha} \quad (3.1)$$

for all $(p, q) \in \mathfrak{A} \times \mathfrak{B}$, with $p \neq q$.

Definition 3.2 Let $(\mathfrak{A}, \mathfrak{B}, \Delta)$ be a bipolar metric space and $\Gamma: (\mathfrak{A}, \mathfrak{B}, \Delta) \rightleftarrows (\mathfrak{A}, \mathfrak{B}, \Delta)$ a contravariant self-map. We shall call Γ a (λ, α, β)-interpolative Kannan contraction, if there exist $\lambda \in [0,1), \alpha, \beta \in (0,1)$, $\alpha + \beta < 1$ such that

$$\Delta(\Gamma q, \Gamma p) \leq \lambda\big(\Delta(p, \Gamma p)\big)^{\alpha}\big(\Delta(\Gamma q, q)\big)^{\beta} \quad (3.2)$$

for all $(p, q) \in \mathfrak{A} \times \mathfrak{B}$, with $p \neq q$.

Theorem 3.3 Let $(\mathfrak{A}, \mathfrak{B}, \Delta)$ be a complete bipolar metric space and $\Gamma: (\mathfrak{A}, \mathfrak{B}, \Delta) \rightleftarrows (\mathfrak{A}, \mathfrak{B}, \Delta)$ be a contravariant continuous (λ, α)-interpolative Kannan contraction with $\lambda \in [0,1), \alpha \in (0,1)$. Then, $\Gamma: \mathfrak{A} \cup \mathfrak{B} \to \mathfrak{A} \cup \mathfrak{B}$ has a unique fixed-point.

Proof: Let $p_0 \in \mathfrak{A}$ and $q_0 \in \mathfrak{B}$; for each nonnegative integer n, we employ one of the iterative approaches described below to define sequences $\{p_n\}$ and $\{q_n\}$:

$$q_n = \Gamma p_n, \;\; p_{n+1} = \Gamma q_n \quad (3.3)$$

Then, we have

$$\begin{aligned}\Delta(p_n, q_n) &= \Delta(\Gamma q_{n-1}, \Gamma p_n)\\ &\leq \lambda\big(\Delta(p_n, \Gamma p_n)\big)^{\alpha}\big(\Delta(\Gamma q_{n-1}, q_{n-1})\big)^{1-\alpha}\\ &= \lambda\big(\Delta(p_n, q_n)\big)^{\alpha}\big(\Delta(p_n, q_{n-1})\big)^{1-\alpha}\end{aligned}$$

i.e. $$\big(\Delta(p_n, q_n)\big)^{1-\alpha} \leq \lambda\big(\Delta(p_n, q_{n-1})\big)^{1-\alpha} \quad (3.4)$$

Hence

$$\Delta(p_n, q_n) \leq \lambda^{\frac{1}{1-\alpha}}\Delta(p_n, q_{n-1}) \leq \lambda\Delta(p_n, q_{n-1}) \quad (3.5)$$

for all integer $n \geq 1$.

We also acquire

$$\begin{aligned}\Delta(p_n, q_{n-1}) &= \Delta(\Gamma q_{n-1}, \Gamma p_{n-1})\\ &\leq \lambda\big(\Delta(p_{n-1}, \Gamma p_{n-1})\big)^{\alpha}\big(\Delta(\Gamma q_{n-1}, q_{n-1})\big)^{1-\alpha}\\ &= \lambda\big(\Delta(p_{n-1}, q_{n-1})\big)^{\alpha}\big(\Delta(p_n, q_{n-1})\big)^{1-\alpha}\end{aligned}$$

Hence

$$\left(\Delta(p_n, q_{n-1})\right)^{\alpha} \le \lambda\left(\Delta(p_{n-1}, q_{n-1})\right)^{\alpha} \tag{3.6}$$

i.e.

$$\Delta(p_n, q_{n-1}) \le \lambda^{\frac{1}{\alpha}}\Delta(p_{n-1}, q_{n-1}) \le \lambda\Delta(p_{n-1}, q_{n-1})$$

for all integer $n \ge 1$. Moreover, it is easy to see that

$$\Delta(p_n, q_n) \le \lambda^{2n}\Delta(p_0, q_0),$$
$$\Delta(p_n, q_{n-1}) \le \lambda^{2n-1}\Delta(p_0, q_0). \tag{3.7}$$

Hence, for all positive integers m and n, we have

(1) If $m > n$, we have

$$\begin{aligned}\Delta(p_n, q_m) &\le \Delta(p_n, q_n) + \Delta(p_{n+1}, q_n) + \Delta(p_{n+1}, q_m)\\ &\le \lambda^{2n}\Delta(p_0, q_0) + \lambda^{2n+1}\Delta(p_0, q_0) + \Delta(p_{n+1}, q_m)\\ &\le (\lambda^{2n} + \lambda^{2n+1})\Delta(p_0, q_0)\\ &\quad +\Delta(p_{n+1}, q_{n+1}) + \Delta(p_{n+2}, q_{n+1}) + \Delta(p_{n+2}, q_m)\\ &\le (\lambda^{2n} + \lambda^{2n+1})\Delta(p_0, q_0) + (\lambda^{2n+2} + \lambda^{2n+3})\Delta(p_0, q_0)\\ &\quad +\Delta(p_{n+2}, q_m)\\ &\le \left(\lambda^{2n} + \lambda^{2n+1} + \lambda^{2n+2} + \lambda^{2n+3} + \cdots + \lambda^{2(m-n)}\right)\Delta(p_0, q_0)\\ &= \lambda^{2n}\left(\frac{1-\lambda^{2(m-n)+1}}{1-\lambda}\right)\Delta(p_0, q_0) \to 0 \text{ as } m, n \to \infty, \text{ since } \lambda < 1.\end{aligned}$$

(2) If $m < n$, we have

$$\begin{aligned}\Delta(p_n, q_m) &\le \Delta(p_{m+1}, q_m) + \Delta(p_{m+1}, q_{m+1}) + \Delta(p_n, q_{m+1})\\ &\le \lambda^{2m+1}\Delta(p_0, q_0) + \lambda^{2m+2}\Delta(p_0, q_0) + \Delta(p_n, q_{m+1})\\ &\le (\lambda^{2m+1} + \lambda^{2m+2})\Delta(p_0, q_0) + \Delta(p_n, q_{m+1})\\ &\le \left(\lambda^{2m+1} + \lambda^{2m+2} + \cdots + \lambda^{2(m-n-1)}\right)\Delta(p_0, q_0)\\ &= \lambda^{2m+1}\left(\frac{1-\lambda^{2(m-n)+1}}{1-\lambda}\right)\Delta(p_0, q_0) \to 0 \text{ as } m, n \to \infty, \text{ since } \lambda < 1.\end{aligned}$$

This indicates that $\Delta(p_n, q_m)$ can be made arbitrarily small by large m and n, and hence (p_n, q_m) is a Cauchy bisequence in $(\mathfrak{A}, \mathfrak{B})$. The bisequence (p_n, q_m) biconverges to some $p^* \in \mathfrak{A} \cap \mathfrak{B}$ such that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = p^*$ due to the completeness of $(\mathfrak{A}, \mathfrak{B}, \Delta)$. Also $\lim_{n\to\infty} \Gamma p_n = \lim_{n\to\infty} q_n = \sigma^* \in \mathfrak{D} \cap \mathfrak{E}$ implies that Γp_n has a unique limit p^*, and $p_n \to p^*$. Now, $\Gamma p_n \to \Gamma p^*$ is implied by the continuity of Γ. Consequently, $\Gamma p^* = p^*$.

We shall now prove the uniqueness of the fixed-point. If $q^* \in \mathfrak{A} \cap \mathfrak{B}$ is another fixed-point of Γ, that is, $\Gamma q^* = q^*$, then we get

$$\Delta(q^*, p^*) = \Delta(\Gamma q^*, \Gamma p^*) \leq \lambda\big(\Delta(p^*, \Gamma p^*)\big)^{\alpha}\big(\Delta(\Gamma q^*, q^*)\big)^{1-\alpha}$$

$$\leq \lambda\big(\Delta(p^*, p^*)\big)^{\alpha}\big(\Delta(q^*, q^*)\big)^{1-\alpha}) \tag{3.8}$$

Therefore, $\Delta(q^*, p^*) \leq 0$, and hence, $p^* = q^*$.

Theorem 3.4 Let $(\mathfrak{A}, \mathfrak{B}, \Delta)$ be a complete bipolar metric space and $\Gamma: (\mathfrak{A}, \mathfrak{B}, \Delta) \rightleftarrows (\mathfrak{A}, \mathfrak{B}, \Delta)$ be a contravariant continuous (λ, α, β)-interpolative Kannan contraction with $\lambda \in [0,1), \alpha, \beta \in (0,1)$. Then, $\Gamma: \mathfrak{A} \cup \mathfrak{B} \to \mathfrak{A} \cup \mathfrak{B}$ has a unique fixed-point.

Proof: Following the steps of proof of Theorem 3.3, we construct the sequences $\{p_n\}$ and $\{q_n\}$ by iterating

$$q_n = \Gamma p_n, \;\; p_{n+1} = \Gamma q_n,$$

where $p_0 \in \mathfrak{A}$ and $q_0 \in \mathfrak{B}$ are arbitrary starting points. Then, we have

$$\Delta(p_n, q_n) = \Delta(\Gamma q_{n-1}, \Gamma p_n)$$

$$\leq \lambda\big(\Delta(p_n, \Gamma p_n)\big)^{\alpha}\big(\Delta(\Gamma q_{n-1}, q_{n-1})\big)^{\beta}$$

$$= \lambda\big(\Delta(p_n, q_n)\big)^{\alpha}\big(\Delta(p_n, q_{n-1})\big)^{\beta} \tag{3.9}$$

Since $\beta < 1 - \alpha$, we have

$$\big(\Delta(p_n, q_n)\big)^{1-\alpha} \leq \lambda\big(\Delta(p_n, q_{n-1})\big)^{\beta} \leq \lambda\big(\Delta(p_n, q_{n-1})\big)^{1-\alpha}$$

i.e.

$$\Delta(p_n, q_n) \leq \lambda^{\frac{1}{1-\alpha}}\Delta(p_n, q_{n-1}) \leq \lambda\Delta(p_n, q_{n-1}) \tag{3.10}$$

for all integer $n \geq 1$.

We also acquire

$$\Delta(p_n, q_{n-1}) = \Delta(\Gamma q_{n-1}, \Gamma p_{n-1})$$

$$\leq \lambda\big(\Delta(p_{n-1}, \Gamma p_{n-1})\big)^{\alpha}\big(\Delta(\Gamma q_{n-1}, q_{n-1})\big)^{\beta}$$

$$= \lambda\big(\Delta(p_{n-1}, q_{n-1})\big)^{\alpha}\big(\Delta(p_{n}, q_{n-1})\big)^{\beta}$$

Since $\alpha < 1 - \beta$, we have

$$\big(\Delta(p_n, q_n)\big)^{1-\beta} \leq \lambda\big(\Delta(p_n, q_{n-1})\big)^{\alpha} \leq \lambda\big(\Delta(p_n, q_{n-1})\big)^{1-\beta}$$

i.e.

$$\Delta(p_n, q_n) \leq \lambda^{\frac{1}{1-\beta}}\Delta(p_n, q_{n-1}) \leq \lambda\Delta(p_n, q_{n-1}) \tag{3.11}$$

for all integer $n \geq 1$.

As already elaborated in the proof of Theorem 3.3, the classical procedure leads to the existence of a unique fixed-point $q^* \in \mathfrak{A} \cap \mathfrak{B}$.

AUTHORS' CONTRIBUTIONS

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Mutlu, K. Ozkan, U. Gurdal, Coupled fixed-point theorems on bipolar metric spaces, Eur. J. Pure Appl. Math. 10 (2017), 655-667.

[2] A. Mutlu, K. Ozkan, U. Gurdal, Locally and weakly contractive principle in bipolar metric spaces, TWMS J. Appl. Eng. Math. 10 (2020), 379-388.

[3] A. Mutlu, U. Gurdal, Bipolar metric spaces and some fixed-point theorems, J. Nonlinear Sci. Appl. 9 (2016), 5362-5373.

[4] B.S. Rao, G.N.V. Kishore, G.K. Kumar, Geraghty type contraction and common coupled fixed-point theorems in bipolar metric spaces with applications to homotopy, Int. J. Math. Trends Technol. 63 (2018), 25-34.

[5] E. Karapinar, Revisiting the Kannan Type Contractions via Interpolation, Adv. Theory Nonlinear Anal. Appl. 2 (2018), 85-87. https://doi.org/10.31197/atnaa.431135.

[6] G.N.V. Kishore, D.R. Prasad, B.S. Rao, et al. Some applications via common coupled fixed-point theorems in bipolar metric spaces, J. Critic. Rev. 7 (2019), 601-607.

[7] G.N.V. Kishore, K.P.R. Rao, A. Sombabu, et al. Related results to hybrid pair of mappings and applications in bipolar metric spaces, J. Math. 2019 (2019), 8485412. https://doi.org/10.1155/2019/8485412.

[8] G.N.V. Kishore, K.P.R. Rao, H. IsIk, et al. Covarian mappings and coupled fiexd point results in bipolar metric spaces, Int. J. Nonlinear Anal. Appl. 12 (2021), 1-15. https://doi.org/10.22075/ijnaa.2021.4650.

[9] G.N.V. Kishore, R.P. Agarwal, B. Srinuvasa Rao, et al. Caristi type cyclic contraction and common fixed point theorems in bipolar metric spaces with applications, Fixed Point Theory Appl. 2018 (2018), 21. https://doi.org/10.1186/s13663-018-0646-z.

[10] G.E. Hardy, T.D. Rogers, A generalization of a fixed point theorem of Reich, Can. Math. Bull. 16 (1973), 201-206. https://doi.org/10.4153/cmb-1973-036-0.

[11] R. Kannan, Some results on fixed-points, Bull. Calcutta Math. Soc. 60 (1968), 71-76.

[12] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, Fund. Math. 3 (1922), 133-181.

[13] S. Reich, Kannan's fixed-point theorem, Boll. Un. Mat. Ital. 4 (1971), 1-11.

[14] S. Reich, Fixed-point s of contractive functions, Boll. Un. Mat. Ital. 5(1972), 26-42.

[15] S. Reich, Some remarks concerning contraction mappings. Can. Math. Bull. 14 (1971), 121-124.

[16] U. Gurdal, A. Mutlu, K. Ozkan, Fixed-point results for αψ-contractive mappings in bipolar metric spaces, J. Inequal. Spec. Funct. 11 (2020), 64-75.

[17] Y.U. Gaba, E. Karapınar, A new approach to the interpolative contractions, Axioms. 8 (2019), 110. https://doi.org/10.3390/axioms8040110.