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ON CERTAIN APPLICATIONS VIA COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED G_b-METRIC SPACES

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Abstract: In this paper, we prove certain coupled fixed point theorems for generalised mappings of the kind (ψ,ϕ)-contraction in complete G_b-metric spaces with partial order. Our findings are supported by a concrete illustration. We also provide a practical application of these findings to the resolution of integral equations, matrix equations, and homotopy theory.

Keywords: Coupled fixed point; g-mixed monotone property; partial ordering and G_b-completeness.

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1. Introduction

Today, mathematics and the practical sciences both heavily rely on the fixed point theory. The fixed point theory's most basic assumption is the Banach contraction principle [1]. As a result, it has been expanded by other mathematicians, who are also interested in fixed point theory in diverse metric spaces. Recently, it was discussed in ([2]-[9]) and references therein whether fixed points for contraction type mappings in partially ordered metric spaces existed.

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Applications to matrix equations, ordinary differential equations, and integral equations were also discussed.

In [4], Bhaskar and Lakshmikantham introduced the concepts of coupled fixed points and mixed monotone mappings, demonstrated various fixed point theorems for coupled fixed points for mixed monotone mappings, and talked about the existence and uniqueness of solutions for periodic boundary value issues. In [8], Lakshmikantham and Ciri\'c developed coupled coincidence and common fixed point theorems, which extend the theorems from [4]. They also proposed the idea of a mixed g-monotone mapping. Following that, certain coupled fixed point and coupled coincident point theorems in partially ordered metric spaces were provided, along with examples of how they applied to integral equations in ([10]-[14]).

As a generalisation of metric spaces, S. Czerwik presented the idea of b-metric spaces [15]. A generalised metric space, often known as a G-metric space, was defined by Mustafa and Sims [16]. On the other hand, Aghajani et al. [17] introduced the idea of G_b-metrics, which is known as a generalised b-metric space. There have also been important studies on G_b-metric spaces, as can be seen in ([18]-[29]).

In this paper,, For mixed g-monotone mappings obeying generalised (ψ,ϕ)-contractions in partially ordered G_b-metric spaces, we demonstrate coupled coincidence and common fixed point theorems. Our findings combine, generalise, and enhance a number of previously reported findings in the literature. Additionally, an application to homotopy, matrix equations, and integral equations is provided.

First, let's review the important concepts of G_b-metric spaces.

2. PRELIMINARIES

Definition 2.1: ([17]) Let $\mathscr{Q}$ be a non-empty set and $\kappa \geq 1$be a given real number. Suppose $G_b : \mathscr{Q}\times\mathscr{Q}\times\mathscr{Q} \to [0,\infty)$ be a function satisfying the following properties:

(g_0) $G_b(\rho,\tau,\sigma) = 0$ if $\rho = \tau = \sigma$;

(g_1) $0 < G_b(\rho,\rho,\tau)$ for any $\rho,\tau \in \mathscr{Q}$ with $\rho \neq \tau$;

(g_2) if $G_b(\rho,\rho,\tau) \leq G_b(\rho,\tau,\sigma)$ for all $\rho,\tau,\sigma \in \mathscr{Q}$ with $\tau \neq \sigma$;

(g_3) $G_b(\rho,\tau,\sigma) = G_b(P[\rho,\tau,\sigma])$, where P is a permutation of ρ,τ,σ (symmetry);

(g_4) $G_b(\rho,\tau,\sigma) \leq \kappa(G(\rho,\varsigma,\varsigma)+G(\varsigma,\tau,\sigma))$ for all $\rho,\tau,\sigma,\varsigma \in \mathscr{Q}$ (rectangle inequality)

then G_b is said to be a G_b-metric on $\mathscr{Q}$ and pair $(\mathscr{Q},G_b)$ is said to be a G_b-metric space or G_b is called a generalized b-metric .

Remark 2.2: It should be noted that the class of G_b-metric spaces is effectively larger than that of G-metric spaces. Indeed, each G-metric space is a G_b-metric space with $\kappa = 1$. The following example shows that a G_b-metric on $\mathscr{Q}$ need not be a G-metric on $\mathscr{Q}$.

Example 2.3: ([17]) Let $(\mathscr{Q},G)$ be a G-metric space. Consider $G_b(\rho,\tau,\sigma) = (G(\rho,\tau,\sigma))^s$, where $s > 1$ is a real number. Then, G_b is a G_b-metric with $\kappa = 2^{s-1}$, it is proved that $(\mathscr{Q},G_b)$ is not necessarily a G-metric space.

Definition 2.4: ([17]) A G_b- metric space $(\mathscr{Q},G_b)$ is said to be symmetric if
$G_b(\rho,\tau,\tau) = G_b(\tau,\rho,\rho)\ \forall\ \rho,\tau \in \mathscr{Q}$.

Definition 2.5: ([17]) Let $(\mathscr{Q},G_b)$ be a G_b-metric space, then for $\rho_0 \in \mathscr{Q}$, $\delta > 0$, the G_b-ball with center ρ_0 and radius δ is

$$B_{G_b}(\rho_0,\delta) = \{\tau \in \mathscr{Q} : G_b(\rho_0,\tau,\tau) < \delta\}$$

Definition 2.6: ([17]) Let $\mathscr{Q}$ be a G_b-metric space. A sequence $\{\rho_n\}$ in $\mathscr{Q}$ is called:

(a) G_b-Cauchy sequence if for every $\varepsilon > 0$, there is an integer $n_0 \in \mathbf{Z}^+$ such that for all $i,j,k \geq n_0$, $G_b\left(\rho_i,\rho_j,\rho_k\right) < \varepsilon$.

(b) G_b-convergent to a point $\rho \in \mathscr{Q}$ if for each $\varepsilon > 0$, there is an integer $n_0 \in \mathbf{Z}^+$ such that for all $i,j \geq n_0$, $G_b\left(\rho_i,\rho_j,\rho\right) < \varepsilon$.

A G_b-metric space on $\mathscr{Q}$ is said to be G_b-complete if every G_b-Cauchy sequence in $\mathscr{Q}$ is G_b-convergent in $\mathscr{Q}$.

Lemma 2.7: ([28]) If $(\mathscr{Q},G_b)$ be a G_b-metric space with $\kappa \geq 1$ and suppose that $\{\alpha_n\}$ is a G_b-convergent to α, then we have

$$\frac{1}{\kappa}G_b(\alpha,\beta,\beta) \leq \lim_{n\to\infty}\inf\ G_b(\alpha_n,\beta,\beta) \leq \lim_{n\to\infty}\sup G_b(\alpha_n,\beta,\beta) \leq \kappa G_b(\alpha,\beta,\beta).$$

In particular, if $\alpha = \beta$, then we have $\lim\limits_{n\to\infty} G_b(\alpha_n,\beta,\beta) = 0$.

For more properties of a G_b-metric we refer the reader to ([17], [28]).

Definition 2.8: ([4]) Let $\mathscr{Q}$ be a nonempty set and let $F : \mathscr{Q}^2 \to \mathscr{Q}$ be a mapping. An element (ρ, τ) is called a coupled fixed point of F if for $\rho, \tau \in \mathscr{Q}$

$$\begin{bmatrix} F(\rho,\tau) \\ F(\tau,\rho) \end{bmatrix} = \begin{bmatrix} \rho \\ \tau \end{bmatrix}$$

Definition 2.9: ([8]) Let $F : \mathscr{Q}^2 \to \mathscr{Q}$ and $f : \mathscr{Q} \to \mathscr{Q}$ be two mappings. An element (ρ, τ) is said to be a coupled coincident point of F and f if

$$\begin{bmatrix} F(\rho,\tau) \\ F(\tau,\rho) \end{bmatrix} = \begin{bmatrix} f\rho \\ f\tau \end{bmatrix}$$

Definition 2.10: ([8]) Let $F : \mathscr{Q}^2 \to \mathscr{Q}$ and $f : \mathscr{Q} \to \mathscr{Q}$ be two mappings. An element (ρ, τ) is said to be a coupled common point of F and f if

$$\begin{bmatrix} F(\rho,\tau) \\ F(\tau,\rho) \end{bmatrix} = \begin{bmatrix} f\rho \\ f\tau \end{bmatrix} = \begin{bmatrix} \rho \\ \tau \end{bmatrix}$$

Definition 2.11: ([8]) Let $\mathscr{Q}$ be a non-empty set. Then we say that the mappings $F : \mathscr{Q}^2 \to \mathscr{Q}$ and $f : \mathscr{Q} \to \mathscr{Q}$ are commutative if for all $\rho, \tau \in \mathscr{Q}$ such that $fF(\rho,\tau) = F(f\rho, f\tau)$ and $fF(\tau,\rho,) = F(f\tau, f\rho)$.

Definition 2.12: ([4]) Let $(\mathscr{Q}, \leq)$ be a partially ordered set and a mapping $F : \mathscr{Q}^2 \to \mathscr{Q}$ is said to have the mixed monotone property if $F(\alpha,\beta)$ is monotone non-decreasing in α and is monotone non-increasing in β, that is for any $\alpha, \beta \in \mathscr{Q}$,

$$\alpha_1 \leq \alpha_2 \Rightarrow F(\alpha_1,\beta) \leq F(\alpha_2,\beta) \text{ for } \alpha_1,\alpha_2 \in \mathscr{Q},$$

$$\beta_1 \leq \beta_2 \Rightarrow F(\alpha,\beta_2) \leq F(\alpha,\beta_1) \text{ for } \beta_1,\beta_2 \in \mathscr{Q}.$$

Definition 2.13: ([8]) Let $(\mathscr{Q}, \leq)$ be a partially ordered set and $F : \mathscr{Q}^2 \to \mathscr{Q}$, $g : \mathscr{Q} \to \mathscr{Q}$ be mappings. The mapping F is said to have the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in the second argument, that is, for any $\alpha, \beta \in \mathscr{Q}$,

$$g\alpha_1 \leq g\alpha_2 \Rightarrow F(\alpha_1,\beta) \leq F(\alpha_2,\beta) \text{ for } \alpha_1,\alpha_2 \in \mathscr{Q},$$

$$g\beta_1 \leq g\beta_2 \Rightarrow F(\alpha,\beta_2) \leq F(\alpha,\beta_1) \text{ for } \beta_1,\beta_2 \in \mathscr{Q}.$$

Now we prove our main result.

3. Main Results

Let $(\mathscr{Q},\leq)$ be a partially ordered set and G_b be a G_b-metric on $\mathscr{Q}$ such that $(\mathscr{Q},G_b)$ is a complete G_b-metric space. Also the product space $\mathscr{Q}\times\mathscr{Q}$ endowed with the following partial order:

$$(u,v)\leq(x,y) \iff x\geq u,\ y\leq v \text{ for } (u,v),(x,y)\in\mathscr{Q}\times\mathscr{Q}.$$

Theorem 3.1: Let $(\mathscr{Q},\leq)$ be a partially ordered set and suppose there is a G_b-metric G_b on $\mathscr{Q}$ such that $(\mathscr{Q},G_b)$ is a complete G_b-metric space. Suppose $F:\mathscr{Q}^2\to\mathscr{Q}$, $g:\mathscr{Q}\to\mathscr{Q}$ are such that F is continuous and has the mixed g-monotone property. Assume also that there exist $\psi:[0,\infty)\to[0,\infty)$ is continuous, monotonically non-decreasing and $\phi:[0,\infty)\to[0,\infty)$ is lower semi-continuous with $\psi(t)=0=\phi(t)$ if and only if $t=0$, such that

$$\psi\left(2\kappa^4 G_b\left(F(a,b),F(x,y),F(x,y)\right)\right)\leq\psi\left(M(a,b,x,y)\right)-\phi\left(M(a,b,x,y)\right) \tag{1}$$

where

$$\begin{aligned} M(a,b,x,y) &= \frac{\lambda}{2}\left(G_b(ga,gx,gx)+G_b(gb,gy,gy)\right)\\ &\quad+\frac{\mu}{2}\left(\begin{array}{c}\frac{G_b(ga,F(a,b),F(a,b))G_b(gx,F(x,y),F(x,y))}{1+G_b(ga,gx,gx)}\\ +\frac{G_b(gb,F(b,a),F(b,a))G_b(gy,F(y,x),F(y,x))}{1+G_b(gb,gy,gy)}\end{array}\right)\end{aligned}$$

for all $x,y,a,b\in\mathscr{Q}$ for which $ga\leq gx$ and $gy\leq gb$ and λ,μ are nonnegative real numbers with $\lambda+\mu<1$. Suppose $F(\mathscr{Q}^2)\subseteq g(\mathscr{Q})$, g is continuous and commutes with F. If there exist $a_0,b_0\in\mathscr{Q}$ such that $ga_0\leq F(a_0,b_0)$ and $gb_0\geq F(b_0,a_0)$ then there exist $a,b\in\mathscr{Q}$ such that $F(a,b)=ga$ and $F(b,a)=gb$ that is, F and g have a coupled coincidence point.

Proof Let $a_0,b_0\in\mathscr{Q}$ such that $ga_0\leq F(a_0,b_0)$ and $gb_0\geq F(b_0,a_0)$.
Since $F(\mathscr{Q}^2)\subseteq g(\mathscr{Q})$, then we can choose $a_1,b_1\in\mathscr{Q}$ such that

$$ga_1=F(a_0,b_0) \text{ and } gb_1=F(b_0,a_0). \tag{2}$$

Taking into account $F(\mathscr{Q}^2)\subseteq g(\mathscr{Q})$, by continuing this process, we can construct sequences $\{a_n\}$, $\{b_n\}$ in $\mathscr{Q}$ such that

$$ga_{p+1}=F(a_p,b_p) \text{ and } gb_{p+1}=F(b_p,a_p). \tag{3}$$

We shall show that

$$ga_p \le ga_{p+1} \text{ and } gb_{p+1} \le gb_p \text{ for } p = 0,1,2,\cdots \tag{4}$$

For this purpose, we use the mathematical induction. Since, $ga_0 \le F(a_0,b_0)$ and $gb_0 \ge F(b_0,a_0)$ then by (2), we get $ga_0 \le ga_1$ and $gb_0 \ge gb_1$ that is, (4) holds for $p=0$.
We presume that (4) holds for some $p>0$. As F has the mixed g-monotone property and $ga_p \le ga_{p+1}$ and $gb_{p+1} \le gb_p$

$$ga_{p+1} = F(a_p,b_p) \le F(a_{p+1},b_p) \le F(a_{p+1},b_{p+1}) = ga_{p+2}$$

$$gb_{p+2} = F(b_{p+1},a_{p+1}) \le F(b_{p+1},a_p) \le F(b_p,a_p) = gb_{p+1}.$$

Thus, (4) holds for any $p \in \mathbb{N}$. Assume for some $p \in \mathbb{N}$, $ga_p = ga_{p+1}$ and $gb_p = gb_{p+1}$ then, by (4), (a_p,b_p) is a coupled coincidence point of F and g. From now on, assume for any $p \in \mathbb{N}$ that at least $ga_p \ne ga_{p+1}$ or $gb_p \ne gb_{p+1}$.
Due to (1)

$$\begin{aligned}\psi\left(2\kappa^4 G_b(ga_p,ga_{p+1},ga_{p+1})\right) &= \psi\left(2\kappa^4 G_b\left(F(a_{p-1},b_{p-1}),F(a_p,b_p),F(a_p,b_p)\right)\right)\\ &\le \psi\left(M(a_{p-1},b_{p-1},a_p,b_p)\right) - \phi\left(M(a_{p-1},b_{p-1},a_p,b_p)\right)\end{aligned}$$

where

$$\begin{aligned}M(a_{p-1},b_{p-1},a_p,b_p) &= \frac{\lambda}{2}\left(G_b(ga_{p-1},ga_p,ga_p)+G_b(gb_{p-1},gb_p,gb_p)\right)\\ &\quad+\frac{\mu}{2}\left(\begin{array}{c}\frac{G_b(ga_{p-1},F(a_{p-1},b_{p-1}),F(a_{p-1},b_{p-1}))G_b(ga_p,F(a_p,b_p),F(a_p,b_p))}{1+G_b(ga_{p-1},ga_p,ga_p)}\\ \\ +\frac{G_b(gb_{p-1},F(b_{p-1},a_{p-1}),F(b_{p-1},a_{p-1}))G_b(gb_p,F(b_p,a_p),F(b_p,a_p))}{1+G_b(gb_{p-1},gb_p,gb_p)}\end{array}\right)\\ &\le \frac{\lambda}{2}\left(G_b(ga_{p-1},ga_p,ga_p)+G_b(gb_{p-1},gb_p,gb_p)\right)\\ &\quad+\frac{\mu}{2}\left(\ G_b(ga_p,ga_{p+1},ga_{p+1})+G_b(gb_p,gb_{p+1},gb_{p+1})\ \right).\end{aligned}$$

Therefore,

$$\begin{aligned}
& \psi\left(2\kappa^4 G_b(ga_p, ga_{p+1}, ga_{p+1})\right) \\
\leq\ & \psi\left(\begin{array}{c} \frac{\lambda}{2}\left(G_b(ga_{p-1}, ga_p, ga_p) + G_b(gb_{p-1}, gb_p, gb_p)\right) \\ +\frac{\mu}{2}\left(\ G_b(ga_p, ga_{p+1}, ga_{p+1}) + G_b(gb_p, gb_{p+1}, gb_{p+1})\ \right) \end{array}\right) \\
& -\phi\left(\begin{array}{c} \frac{\lambda}{2}\left(G_b(ga_{p-1}, ga_p, ga_p) + G_b(gb_{p-1}, gb_p, gb_p)\right) \\ +\frac{\mu}{2}\left(\ G_b(ga_p, ga_{p+1}, ga_{p+1}) + G_b(gb_p, gb_{p+1}, gb_{p+1})\ \right) \end{array}\right) \\
\leq\ & \psi\left(\begin{array}{c} \frac{\lambda}{2}\left(G_b(ga_{p-1}, ga_p, ga_p) + G_b(gb_{p-1}, gb_p, gb_p)\right) \\ +\frac{\mu}{2}\left(\ G_b(ga_p, ga_{p+1}, ga_{p+1}) + G_b(gb_p, gb_{p+1}, gb_{p+1})\ \right) \end{array}\right).
\end{aligned}$$

By the property of ψ, we have that

$$\begin{aligned}
G_b(ga_p, ga_{p+1}, ga_{p+1}) \ \leq\ & \frac{\lambda}{4\kappa^4(1-\frac{\mu}{4\kappa^4})}\left(G_b(ga_{p-1}, ga_p, ga_p) + G_b(gb_{p-1}, gb_p, gb_p)\right) \\
& + \frac{\mu}{4\kappa^4(1-\frac{\mu}{4\kappa^4})} G_b(gb_p, gb_{p+1}, gb_{p+1}). \qquad (5)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
G_b(gb_p, gb_{p+1}, gb_{p+1}) \ \leq\ & \frac{\lambda}{4\kappa^4(1-\frac{\mu}{4\kappa^4})}\left(G_b(ga_{p-1}, ga_p, ga_p) + G_b(gb_{p-1}, gb_p, gb_p)\right) \\
& + \frac{\mu}{4\kappa^4(1-\frac{\mu}{4\kappa^4})} G_b(ga_p, ga_{p+1}, ga_{p+1}). \qquad (6)
\end{aligned}$$

Using (5), (6) and letting $\zeta = \frac{\lambda}{2\kappa^4-\mu} < 1$ to obtain

$$\begin{aligned}
& G_b(ga_p, ga_{p+1}, ga_{p+1}) + G_b(gb_p, gb_{p+1}, gb_{p+1}) \\
\leq\ & \zeta\left(G_b(ga_{p-1}, ga_p, ga_p) + G_b(gb_{p-1}, gb_p, gb_p)\right) \\
\leq\ & \zeta^2\left(G_b(ga_{p-2}, ga_{p-1}, ga_{p-1}) + G_b(gb_{p-2}, gb_{p-1}, gb_{p-1})\right) \\
& \vdots \\
\leq\ & \zeta^p\left(G_b(ga_0, ga_1, ga_1) + G_b(gb_0, gb_1, gb_1)\right) \to 0 \text{ as } p \to \infty.
\end{aligned}$$

Therefore

$$G_b(ga_p, ga_{p+1}, ga_{p+1}) \ \leq\ \zeta^p\left(G_b(ga_0, ga_1, ga_1) + G_b(gb_0, gb_1, gb_1)\right) \to 0 \text{ as } p \to \infty$$

and

$$G_b(gb_p, gb_{p+1}, gb_{p+1}) \leq \zeta^p \left(G_b(ga_0, ga_1, ga_1) + G_b(gb_0, gb_1, gb_1)\right) \to 0 \text{ as } p \to \infty.$$

By use of the rectangle inequality, for $q > p$, we get

$$\begin{aligned}
G_b\left(ga_p, ga_q, ga_q\right) &\leq \kappa\left(G_b\left(ga_p, ga_{p+1}, ga_{p+1}\right) + G_b\left(ga_{p+1}, ga_q, ga_q\right)\right), \\
&\leq \kappa G_b\left(ga_p, ga_{p+1}, ga_{p+1}\right) \\
&\qquad + \kappa^2\left(G_b\left(ga_{p+1}, ga_{p+2}, ga_{p+1}\right) + G_b\left(ga_{p+2}, ga_q, ga_q\right)\right), \\
&\leq \kappa G_b\left(ga_p, ga_{p+1}, ga_{p+1}\right) + \kappa^2 G_b\left(ga_{p+2}, ga_{p+2}, ga_{p+1}\right) \\
&\qquad + \cdots + \kappa^{q-1} G_b\left(ga_{q-1}, ga_q, ga_q\right), \\
&\leq \left(\kappa\zeta^p + \kappa^2\zeta^{p+1} + \cdots + \kappa^{q-1}\zeta^{q-1}\right)\begin{pmatrix} G_b(ga_0, ga_1, ga_1) \\ +G_b(gb_0, gb_1, gb_1) \end{pmatrix} \\
&\leq \left(\kappa\zeta^p + \kappa^2\zeta^{p+1} + \kappa^3\zeta^{p+2} + \cdots\right)\begin{pmatrix} G_b(ga_0, ga_1, ga_1) \\ +G_b(gb_0, gb_1, gb_1) \end{pmatrix} \\
&\leq \frac{\zeta^p}{\frac{1}{\kappa} - \zeta}\begin{pmatrix} G_b(ga_0, ga_1, ga_1) \\ +G_b(gb_0, gb_1, gb_1) \end{pmatrix} \to 0 \text{ as } p \to \infty.
\end{aligned}$$

By similar arguments, we obtain $G_b\left(gb_p, gb_q, gb_q\right) \to 0$ as $p, q \to \infty$. This shows that $\left\{ga_p\right\}, \left\{gb_p\right\}$ are Cauchy sequences in the G_b-metric space $(\mathscr{Q}, G_b)$. Since $(\mathscr{Q}, G_b)$ is complete, there exist $a, b \in \mathscr{Q}$ such that

$$\lim_{p\to\infty} ga_p = a \quad \lim_{p\to\infty} gb_p = b. \tag{7}$$

From (7) and the continuity of g, we have

$$\lim_{p\to\infty} g(ga_p) = ga \quad \lim_{p\to\infty} g(gb_p) = gb. \tag{8}$$

From (3) and the commutativity of F and g, we have

(9)

$$g(ga_{p+1}) = g(F(a_p, b_p)) = F(ga_p, gb_p) \text{ and } g(gb_{p+1}) = g(F(b_p, a_p)) = F(gb_p, ga_p).$$

Now we shall show that $ga = F(a,b)$ and $gb = F(b,a)$. By letting $p \to \infty$ in (10), by (7), (8) and the continuity of F, we obtain

$ga = \lim_{p\to\infty} g(ga_{p+1}) = \lim_{p\to\infty} g(F(a_p,b_p)) = \lim_{p\to\infty} F(ga_p, gb_p) = F(a,b)$

and $gb = \lim_{p\to\infty} g(gb_{p+1}) = \lim_{p\to\infty} g(F(b_p,a_p)) = \lim_{p\to\infty} F(gb_p, ga_p) = F(b,a)$.

We have proved that F and g have a coupled coincidence point. This completes the proof of Theorem 3.1.

In the following theorem, we omit the continuity hypothesis of F. We need the following definition.

Definition 3.2: Let $(\mathscr{Q}, \leq)$ be a partially ordered G_b-metric set and G_b be a G_b-metric on $\mathscr{Q}$. We say that $(\mathscr{Q}, G_b, \leq)$ is regular if the following conditions hold:

(a) if non-decreasing sequence $a_p \to a$, then $a_p \leq a$ for all p,

(b) if non-increasing sequence $b_p \to b$, then $b \leq b_p$ for all p.

Theorem 3.3: Let $(\mathscr{Q}, \leq)$ be a partially ordered set and G_b be a G_b-metric on $\mathscr{Q}$ such that $(\mathscr{Q}, G_b, \leq)$ is regular. Suppose $F : \mathscr{Q}^2 \to \mathscr{Q}$, $g : \mathscr{Q} \to \mathscr{Q}$ are such that F has the mixed g-monotone property. Assume also that there exist $\psi : [0,\infty) \to [0,\infty)$ is continuous, monotonically non-decreasing and $\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$, such that

$$\psi\left(2\kappa^4 G_b\left(F(a,b), F(x,y), F(x,y)\right)\right) \leq \psi\left(M(a,b,x,y)\right) - \phi\left(M(a,b,x,y)\right)$$

where

$$\begin{aligned} M(a,b,x,y) &= \frac{\lambda}{2}\left(G_b(ga,gx,gx) + G_b(gb,gy,gy)\right) \\ &\quad + \frac{\mu}{2}\left(\begin{array}{c} \frac{G_b(ga,F(a,b),F(a,b))G_b(gx,F(x,y),F(x,y))}{1+G_b(ga,gx,gx)} \\ + \frac{G_b(gb,F(b,a),F(b,a))G_b(gy,F(y,x),F(y,x))}{1+G_b(gb,gy,gy)} \end{array}\right) \end{aligned}$$

for all $x,y,a,b \in \mathscr{Q}$ for which $ga \leq gx$ and $gy \leq gb$ and λ, μ are nonnegative real numbers with $\lambda + \mu < 1$. Suppose $F(\mathscr{Q}^2) \subseteq g(\mathscr{Q})$ and $(g(\mathscr{Q}), G_b)$ is a complete G_b-metric space. If there exist $a_0, b_0 \in \mathscr{Q}$ such that $ga_0 \leq F(a_0,b_0)$ and $gb_0 \geq F(b_0,a_0)$ then there exist $a,b \in \mathscr{Q}$ such that $F(a,b) = ga$ and $F(b,a) = gb$ that is, F and g have a coupled coincidence point.

Proof: Proceeding exactly as in Theorem 3.1, we have that $\{ga_n\}$, $\{gb_n\}$ in $\mathscr{Q}$ are Cauchy sequences in the complete G_b-metric space $(g(\mathscr{Q}),G_b)$. Then, there exist $a,b\in\mathscr{Q}$ such that

$$ga_p \to ga \text{ and } gb_p \to gb \tag{10}$$

since $\{ga_n\}$ is non-decreasing and $\{gb_n\}$ is non-increasing, then since $(\mathscr{Q},G_b,\leq)$ is regular we have $ga_p \leq ga$ and $gb_p \geq gb$ for all p. If $ga_p = ga$ and $gb_p = gb$ for some $p \geq 0$ then, $ga = ga_p \leq ga_{p+1} \leq ga = ga_p$ and $gb \leq gb_{p+1} \leq gb_p = gb$, which implies that

$$ga_p = ga_{p+1} = F(a_p,b_p) \text{ and } gb_p = gb_{p+1} = F(b_p,a_p)$$

that is (a_p,b_p) is coupled coincidence point of F and g. Then, we suppose that $(ga_p,gb_p) \neq (ga,gb)$ for all $p \geq 0$. By Lemma (2.7), we have

$$\begin{aligned}\frac{1}{\kappa}G_b(ga,F(a,b),F(a,b)) &\leq \liminf_{p\to\infty} G_b(ga_{p+1},F(a,b),F(a,b))\\ &\leq \liminf_{p\to\infty} G_b(F(a_p,b_p),F(a,b),F(a,b)).\end{aligned}$$

Now from (1) and applying ψ on both sides, we have that

$$\begin{aligned}\psi\left(2\kappa^3 G_b(ga,F(a,b),F(a,b))\right) &\leq \liminf_{p\to\infty}\psi\left(2\kappa^4 G_b(F(a_p,b_p),F(a,b),F(a,b))\right)\\ &\leq \liminf_{p\to\infty}\psi\left(M(a_p,b_p,a,b)\right) - \liminf_{p\to\infty}\phi\left(M(a_p,b_p,a,b)\right).\end{aligned} \tag{11}$$

Here using Eq. (10), we obtain that

$$\begin{aligned}\liminf_{p\to\infty} M(a_p,b_p,a,b) &= \liminf_{p\to\infty}\left(\begin{array}{c}\frac{\lambda}{2}\left(G_b(ga_p,ga,ga)+G_b(gb_p,gb,gb)\right)\\ +\frac{\mu}{2}\left(\begin{array}{c}\frac{G_b(ga_p,F(a_p,b_p),F(a_p,b_p))G_b(ga,F(a,b),F(a,b))}{1+G_b(ga_p,ga,ga)}\\ +\frac{G_b(gb_p,F(b_p,a_p),F(b_p,a_p))G_b(gb,F(b,a),F(b,a))}{1+G_b(gb_p,gb,gb)}\end{array}\right)\end{array}\right)\\ &\leq \limsup_{p\to\infty}\left(\begin{array}{c}\frac{\lambda}{2}\left(G_b(ga_p,ga,ga)+G_b(gb_p,gb,gb)\right)\\ +\frac{\mu}{2}\left(\begin{array}{c}\frac{G_b(ga_p,F(a_p,b_p),F(a_p,b_p))G_b(ga,F(a,b),F(a,b))}{1+G_b(ga_p,ga,ga)}\\ +\frac{G_b(gb_p,F(b_p,a_p),F(b_p,a_p))G_b(gb,F(b,a),F(b,a))}{1+G_b(gb_p,gb,gb)}\end{array}\right)\end{array}\right)\\ &= 0.\end{aligned}$$

From Eq. (11) conclude that

$$\psi\left(2\kappa^3 G_b(ga, F(a,b), F(a,b))\right) \leq \psi(0) - \phi(0) = 0$$

and hence we get that $G_b(ga, F(a,b), F(a,b)) = 0$ implies that $ga = F(a,b)$. Analogously, one find $F(b,a) = gb$. Thus, we proved that F and g have a coupled coincidence point. This completes the proof of Theorem 3.3.

Now, we shall prove the existence and uniqueness of coupled common fixed point. For a product $\mathscr{Q}$ of a partial ordered set $(\mathscr{Q}, \leq)$, we define a partial ordering in the following way: For all $(a,b),(x,y) \in \mathscr{Q}^2$

$$(a,b) \leq (x,y) \Leftrightarrow a \leq x, \;\; b \geq y$$

We say that (a,b) and (x,y) are comparable if

$$(a,b) \leq (x,y) \text{ or } (x,y) \leq (a,b).$$

Also, we say that (a,b) is equal to (x,y) if and only $a = x$ and $b = y$.

Theorem 3.4: In addition to hypotheses of Theorem 3.1, suppose that for all $(a,b),(x,y) \in \mathscr{Q}^2$, there exists $(u,v) \in \mathscr{Q}^2$ such that $(F(u,v), F(v,u))$ is comparable to $(F(a,b), F(b,a))$ and $(F(x,y), F(y,x))$. Then, F and g have a unique coupled common fixed point (a,b) such that $a = ga = F(a,b)$ and $b = gb = F(b,a)$.

Proof: The set of coupled coincidence points of F and g is not empty due to Theorem 3.1. Assume, now, (a,b) and (x,y) are two coupled coincidence points of F and g, that is,

$$F(a,b) = ga \quad F(x,y) = gx$$
$$F(b,a) = gb \quad F(y,x) = gy.$$

We shall show that (ga, gb) and (gx, gy) are equal. By assumption, there exists $(u,v) \in \mathscr{Q}^2$ such that $(F(u,v), F(v,u))$ is comparable to $(F(a,b), F(b,a))$ and $(F(x,y), F(y,x))$.
Define sequences $\{gu_p\}$ and $\{gv_p\}$ such that $u_0 = u$ and $v_0 = v$ and for any $p \geq 1$

$$gu_p = F(u_{p-1}, v_{p-1}) \text{ and } gv_p = F(v_{p-1}, u_{p-1}) \; \forall \; p.$$

Further, set $a_0 = a$, $b_0 = b$ and $x_0 = x$, $y_0 = y$ and on the same way define the sequences $\{ga_p\}$, $\{gb_p\}$ and $\{gx_p\}$ and $\{gy_p\}$. Then, it is easy that

$$F(a,b) = ga_p \quad F(x,y) = gx_p$$
$$F(b,a) = gb_p \quad F(y,x) = gy_p \ \forall\ p \geq 1.$$

Since $(F(a,b), F(b,a)) = (ga_1, gb_1) = (ga, gb)$ is comparable to
$(F(u,v), F(v,u)) = (gu_1, gv_1)$ then it is easy to show $(ga, gb) \geq (gu_1, gv_1)$.
Recursively, we get that $(gu_p, gv_p) \leq (ga, gb)\ \forall\ p$.
From (1), we have

$$\begin{aligned} \psi\left(2\kappa^4 G_b(gu_{p+1}, ga, ga)\right) &= \psi\left(2\kappa^4 G_b\left(F(u_p, v_p), F(a,b), F(a,b)\right)\right) \\ &\leq \psi\left(M(u_p, v_p, a, b)\right) - \phi\left(M(u_p, v_p, a, b)\right) \end{aligned}$$

where

$$\begin{aligned} M(u_p, v_p, a, b) &= \frac{\lambda}{2}\left(G_b(gu_p, ga, ga) + G_b(gv_p, gb, gb)\right) \\ &\quad + \frac{\mu}{2}\left(\begin{array}{c} \frac{G_b(gu_p, F(u_p,v_p), F(u_p,v_p))G_b(ga, F(a,b), F(a,b))}{1+G_b(gu_p, ga, ga)} \\ \\ + \frac{G_b(gv_p, F(v_p,u_p), F(v_p,u_p))G_b(gb, F(b,a), F(b,a))}{1+G_b(gv_p, gb, gb)} \end{array}\right) \\ &= \frac{\lambda}{2}\left(G_b(gu_p, ga, ga) + G_b(gv_p, gb, gb)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \psi\left(2\kappa^4 G_b(gu_{p+1}, ga, ga)\right) &\leq \psi\left(\frac{\lambda}{2}\left(G_b(gu_p, ga, ga) + G_b(gv_p, gb, gb)\right)\right) \\ &\quad -\phi\left(\frac{\lambda}{2}\left(G_b(gu_p, ga, ga) + G_b(gv_p, gb, gb)\right)\right) \\ &\leq \psi\left(\frac{\lambda}{2}\left(G_b(gu_p, ga, ga) + G_b(gv_p, gb, gb)\right)\right). \end{aligned}$$

By the property of ψ, we have that

$$G_b(gu_{p+1}, ga, ga) \leq \frac{\lambda}{4\kappa^4}\left(G_b(gu_p, ga, ga) + G_b(gv_p, gb, gb)\right).$$

Similarly, we can prove that

$$G_b(gv_{p+1},gb,gb)\leq\frac{\lambda}{4\kappa^4}\left(G_b(gu_p,ga,ga)+G_b(gv_p,gb,gb)\right)$$

and hence

$$\begin{aligned}G_b(gu_{p+1},ga,ga)+G_b(gv_{p+1},gb,gb) &\leq \frac{\lambda}{2\kappa^4}\left(G_b(gu_p,ga,ga)+G_b(gv_p,gb,gb)\right)\\ &\leq (\frac{\lambda}{2\kappa^4})^2\left(G_b(gu_{p-1},ga,ga)+G_b(gv_{p-1},gb,gb)\right)\\ &\vdots\\ &\leq (\frac{\lambda}{2\kappa^4})^p\left(G_b(gu_0,ga,ga)+G_b(gv_0,gb,gb)\right)\\ &\quad\to 0 \text{ as } p\to\infty.\end{aligned}$$

This yields that

$$\lim_{p\to\infty}G_b(gu_{p+1},ga,ga)=0 \text{ and } \lim_{p\to\infty}G_b(gv_{p+1},gb,gb)=0. \tag{12}$$

Analogously, we may show that

$$\lim_{p\to\infty}G_b(gu_{p+1},gx,gx)=0 \text{ and } \lim_{p\to\infty}G_b(gv_{p+1},gy,gy)=0. \tag{13}$$

Combining (12) and (13) yields that (ga,gb) and (gx,gy) are equal. Since $ga=F(a,b)$ and $gb=F(b,a)$, by commutativity of F and g we have

$$\begin{aligned}ga'&=g(ga)=gF(a,b)=F(ga,gb)=F(a',b')\\ gb'&=g(gb)=gF(b,a)=F(gb,ga)=F(b',a')\end{aligned}$$

where, $a'=ga$ and $b'=gb$. Thus, (a',b') is a coupled coincidence point of F and g. Consequently, (ga,gb) and (ga',gb') are equal.
We deduce $a'=ga=ga'$ and $b'=gb=gb'$. Therefore, (a',b') is a coupled common fixed of F and g. Its uniqueness follows easily from (1).

Corollary 3.5: Let $F:\mathscr{Q}^2\to\mathscr{Q}$ has mixed monotone property on $\mathscr{Q}$ satisfying :

$$\psi\left(2\kappa^4G_b\left(F(a,b),F(x,y),F(x,y)\right)\right)\leq\psi\left(M(a,b,x,y)\right)-\phi\left(M(a,b,x,y)\right)$$

where

$$M(a,b,x,y) = \frac{\lambda}{2}\left(G_b(a,x,x)+G_b(b,y,y)\right) + \frac{\mu}{2}\left(\begin{array}{c} \frac{G_b(a,F(a,b),F(a,b))G_b(x,F(x,y),F(x,y))}{1+G_b(a,x,x)} \\ +\frac{G_b(b,F(b,a),F(b,a))G_b(y,F(y,x),F(y,x))}{1+G_b(b,y,y)} \end{array}\right)$$

for all $(x,y) \leq (a,b)$ where λ, μ are nonnegative real numbers with $\lambda + \mu < 1$. Also suppose either

(a) F is continuous, or

(b) $\mathscr{Q}$ has the following properties:

(i) if non-decreasing sequence $a_p \to a$, then $a_p \leq a$ for all p,

(ii) if non-increasing sequence $b_p \to b$, then $b \leq b_p$ for all p.

If there exist $a_0, b_0 \in \mathscr{Q}$ such that $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$ then F has a coupled fixed point $(a,b) \in \mathscr{Q} \times \mathscr{Q}$.

Corollary 3.6: Let $F : \mathscr{Q}^2 \to \mathscr{Q}$ has mixed monotone property on $\mathscr{Q}$ satisfying :

$$\psi\left(2\kappa^4 G_b\left(F(a,b),F(x,y),F(x,y)\right)\right) \leq \psi\left(\frac{G_b(a,x,x)+G_b(b,y,y)}{2}\right)$$

for all $(x,y) \leq (a,b)$. Also suppose either

(a) F is continuous, or

(b) $\mathscr{Q}$ has the following properties:

(i) if non-decreasing sequence $a_p \to a$, then $a_p \leq a$ for all p,

(ii) if non-increasing sequence $b_p \to b$, then $b \leq b_p$ for all p.

If there exist $a_0, b_0 \in \mathscr{Q}$ such that $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$ then F has a coupled fixed point $(a,b) \in \mathscr{Q} \times \mathscr{Q}$.

Example 3.7: Let $\mathscr{Q} = [0,1]$ be endowed with the usual ordering and define $G_b : \mathscr{Q}^3 \to \mathbb{R}^+$ by $G_b(\alpha,\beta,\gamma) = (|\alpha-\beta|+|\alpha-\gamma|+|\beta-\gamma|)^2$ for all $\alpha,\beta,\gamma \in \mathscr{Q}$. Then $(\mathscr{Q}, G_b)$ is a complete G_b-metric space with $\kappa = 2$, according to Example 2.3. Let $F : \mathscr{Q}^2 \to \mathscr{Q}$ and $g : \mathscr{Q} \to \mathscr{Q}$ be given by $F(\alpha,\beta) = \frac{\alpha+\beta}{16\sqrt{2}}$,and $g(\alpha) = \frac{\alpha}{2}$ for all $\alpha,\beta \in \mathscr{Q}$,
also $\psi, \phi : [0,\infty) \to [0,\infty)$ as $\psi(t) = t$ and $\phi(t) = \frac{t}{2}$ for all $t \in [0,\infty)$. We will check that the contraction (1) is satisfied for all $\alpha,\beta,\rho,\tau \in \mathscr{Q}$ satisfying $g\alpha \leq g\rho$ and $g\tau \leq g\beta$.

In this case, we have

$$\begin{aligned}
\psi\left(2\kappa^4 G_b\left(F(\alpha,\beta),F(\rho,\tau),F(\rho,\tau)\right)\right) &= 2\kappa^4\left(2|F(\alpha,\beta)-F(\rho,\tau)|\right)^2 \\
&= 2\kappa^4\left(2|\frac{\alpha+\beta}{16\sqrt{2}}-\frac{\rho+\tau}{16\sqrt{2}}|\right)^2 \\
&\leq \frac{2\kappa^4}{8\kappa^4}\left(2|\frac{\alpha}{2}-\frac{\rho}{2}|+|\frac{\beta}{2}-\frac{\tau}{2}|\right)^2 \\
&\leq \frac{4\kappa^4}{8\kappa^4}\left(\left(2|\frac{\alpha}{2}-\frac{\rho}{2}|\right)^2+\left(2|\frac{\beta}{2}-\frac{\tau}{2}|\right)\right)^2 \\
&\leq \frac{1}{2}\left(G_b(g\alpha,g\rho,g\rho)+G_b(g\beta,g\tau,g\tau)\right) \\
&= \psi\left(M(\alpha,\beta,\rho,\tau)\right)-\phi\left(M(\alpha,\beta,\rho,\tau)\right).
\end{aligned}$$

Thus, inequality (1) satisfies with constant $\lambda=\frac{1}{2}$ and $\mu=0$. Other conditions in Theorem 3.1 are satisfied. It follows that $(0,0)$ is coupled fixed point of F and g in $\mathscr{Q}\times\mathscr{Q}$.

3.1. Application to Integral Equations.

In this section, we study the existence of a unique solution to a nonlinear integral equation, as an application to Corollary 3.6.

Consider the following integral equation:

$$\chi(t)=\int_a^b\left(\Gamma_1(t,s)+\Gamma_2(t,s)\right)\left(\Lambda(s,\chi(s))+\Upsilon(s,\chi(s))\right)ds+z(t),\ t\in I=[a,b] \tag{14}$$

We will analyze Eq. (14) under the following assumptions:

- (a) $\Gamma_1,\Gamma_2\in C(I\times I,\mathbb{R})$ and $\Gamma_1(t,s)\leq 0$, $\Gamma_2(t,s)\geq 0$;
- (b) $z\in C(I,\mathbb{R})$;
- (c) $\Lambda,\Upsilon\in C(I\times\mathbb{R},\mathbb{R})$
- (d) There exist constants $\theta,\tau>0$ such that for all $\chi,\zeta\in\mathbb{R}$ and $\chi\geq\zeta$

$$0\leq\Lambda(t,\chi)-\Lambda(t,\zeta)\leq\frac{\theta}{\sqrt{2}\kappa}|\chi-\zeta|$$

  and

$$-\frac{\tau}{\sqrt{2}\kappa}|\chi-\zeta|\leq\Upsilon(t,\chi)-\Upsilon(t,\zeta)\leq 0;$$

- (e) $2\kappa^2\max\{\theta,\tau\}||\Gamma_1-\Gamma_2||_\infty^2\leq 1$,

  where $||\Gamma_1-\Gamma_2||_\infty=\sup\{(\Gamma_1(t,s)-\Gamma_2(t,s)):t,s\in I\}$;

(f) There exist $(x,y) \in C(I,\mathbb{R}) \times C(I,\mathbb{R})$ a coupled lower and upper solution of the integral equation (14) if $x(t) \leq y(t)$ and

$$x(t) \leq \int_a^b \Gamma_1(t,s)\left(\Lambda(s,x(s)) + \Upsilon(s,y(s))\right) ds + \int_a^b \Gamma_2(t,s)\left(\Lambda(s,y(s)) + \Upsilon(s,x(s))\right) ds + z(t)$$

and

$$y(t) \geq \int_a^b \Gamma_1(t,s)\left(\Lambda(s,y(s)) + \Upsilon(s,x(s))\right) ds + \int_a^b \Gamma_2(t,s)\left(\Lambda(s,x(s)) + \Upsilon(s,y(s))\right) ds + z(t)$$

for all $t \in I$.

Theorem 3.1.1: Under assumptions $(a)-(f)$, Eq. (14) has a unique solution in $C(I,\mathbb{R})$.

Proof: Let $\mathscr{Q} := C(I,\mathbb{R})$. $\mathscr{Q}$ is a partially ordered set if we define the following order relation in $\mathscr{Q}$:

$$\chi, \zeta \in C(I,\mathbb{R}), \chi \leq \zeta \iff \chi(t) \leq \zeta(t)\ \forall\, t \in I.$$

And $(\mathscr{Q}, G_b)$ is a complete G_b-metric space with $\kappa = 2$ which G_b-metric
$G_b(\chi,\zeta,\tau) = \sup_{t\in I}\left(|\chi(t)-\zeta(t)| + |\chi(t)-\tau(t)| + |\zeta(t)-\tau(t)|\right)^2\ \forall\, \chi,\zeta,\tau \in \mathscr{Q}$.
Now define on $\mathscr{Q} \times \mathscr{Q}$ the following partial order: for $(\chi,\zeta),(\rho,\tau) \in \mathscr{Q}^2$,

$$(\chi,\zeta) \leq (\rho,\tau) \iff \chi(t) \leq \rho(t) \text{ and } \zeta(t) \geq \tau(t),\ \forall\, t \in I.$$

Obviously, for any $(\chi,\zeta) \in \mathscr{Q}^2$, the functions $\max\{\chi,\zeta\}$ is upper bound and $\min\{\chi,\zeta\}$ is lower bound of χ and ζ. Therefore, for every $(\chi,\zeta),(\rho,\tau) \in \mathscr{Q}^2$, there exists the element $(\max\{\chi,\rho\},\min\{\zeta,\tau\})$ which is comparable to (χ,ζ) and (ρ,τ). Define now the mappings $\psi : [0,\infty) \to [0,\infty)$ as $\psi(t) = t$ and $F : \mathscr{Q}^2 \to \mathscr{Q}$ by

$$F(\chi,\zeta)(t) = \int_a^b \Gamma_1(t,s)\left(\Lambda(s,\chi(s)) + \Upsilon(s,\zeta(s))\right) ds + \int_a^b \Gamma_2(t,s)\left(\Lambda(s,\zeta(s)) + \Upsilon(s,\chi(s))\right) ds + z(t).$$

Then obviously, F has the mixed monotone property. In what follows, we estimate
$G_b(F(\chi,\zeta),F(\rho,\tau),F(\rho,\tau))$ for $\chi \geq \rho$ and $\zeta \leq \tau$. Indeed, as F has the mixed monotone

property, $F(\chi,\zeta) \geq F(\rho,\tau)$ and we have

$$\begin{aligned}
&\psi\left(2\kappa^4 G_b\left(F(\chi,\zeta),F(\rho,\tau),F(\rho,\tau)\right)\right) = 2\kappa^4 G_b\left(F(\chi,\zeta),F(\rho,\tau),F(\rho,\tau)\right)\\
= \;& 8\kappa^4 \sup_{t\in I} |F(\chi,\zeta)(t) - F(\rho,\tau)(t)|^2\\
= \;& 8\kappa^4 \sup_{t\in I} (F(\chi,\zeta)(t) - F(\rho,\tau)(t))^2\\
= \;& 8\kappa^4 \sup_{t\in I} \begin{pmatrix} \int_a^b \Gamma_1(t,s)\left(\Lambda(s,\chi(s)) + \Upsilon(s,\zeta(s))\right) ds \\ + \int_a^b \Gamma_2(t,s)\left(\Lambda(s,\zeta(s)) + \Upsilon(s,\chi(s))\right) ds \\ - \int_a^b \Gamma_1(t,s)\left(\Lambda(s,\rho(s)) + \Upsilon(s,\tau(s))\right) ds \\ - \int_a^b \Gamma_2(t,s)\left(\Lambda(s,\tau(s)) + \Upsilon(s,\rho(s))\right) ds \end{pmatrix}^2\\
= \;& 8\kappa^4 \sup_{t\in I} \begin{pmatrix} \int_a^b \Gamma_1(t,s)\left[(\Lambda(s,\chi(s)) - \Lambda(s,\rho(s))) - (\Upsilon(s,\tau(s)) - \Upsilon(s,\zeta(s)))\right] ds \\ + \int_a^b \Gamma_2(t,s)\left[(\Lambda(s,\zeta(s)) - \Lambda(s,\tau(s))) - (\Upsilon(s,\rho(s)) - \Upsilon(s,\chi(s)))\right] ds \end{pmatrix}^2\\
\leq \;& 8\kappa^4 \sup_{t\in I} \begin{pmatrix} \int_a^b \Gamma_1(t,s)\left[\frac{\theta}{\sqrt{2}\kappa}|\chi(s)-\rho(s)| + \frac{\tau}{\sqrt{2}\kappa}|\tau(s)-\zeta(s)|\right] ds \\ + \int_a^b (-\Gamma_2(t,s))\left[\frac{\theta}{\sqrt{2}\kappa}|\zeta(s)-\tau(s)| + \frac{\tau}{\sqrt{2}\kappa}|\chi(s)-\rho(s)|\right] ds \end{pmatrix}^2\\
\leq \;& 4\kappa^2 \max\{\theta,\tau\} \sup_{t\in I} \begin{pmatrix} \int_a^b (\Gamma_1(t,s) - \Gamma_2(t,s))|\chi(s)-\rho(s)| ds \\ + \int_a^b (\Gamma_1(t,s) - \Gamma_2(t,s))|\zeta(s)-\tau(s)| ds \end{pmatrix}^2
\end{aligned} \tag{15}$$

Defining $(X) = \int_a^b (\Gamma_1(t,s) - \Gamma_2(t,s))|\chi(s)-\rho(s)|ds$

and $(Y) = \int_a^b (\Gamma_1(t,s) - \Gamma_2(t,s))|\zeta(s)-\tau(s)|ds$ and using the Cauchy-Schwartz inequality in (X) we obtain

$$\begin{aligned}
(X) \;&\leq\; \left(\int_a^b (\Gamma_1(t,s) - \Gamma_2(t,s))^2 ds\right)^{\frac{1}{2}} \left(\int_a^b |\chi(s)-\rho(s)|^2 ds\right)^{\frac{1}{2}}\\
&\leq\; ||\Gamma_1(t,s) - \Gamma_2(t,s)||_\infty (|\chi(s)-\rho(s)|)
\end{aligned} \tag{16}$$

Similarly, we can obtain the following estimate for (Y) :

$$(Y) \leq ||\Gamma_1(t,s) - \Gamma_2(t,s)||_\infty(|\zeta(s) - \tau(s)|). \tag{17}$$

By (15)-(17) and assumption (e) , we get

$$\begin{aligned}
& \psi\left(2\kappa^4 G_b\left(F(\chi,\zeta),F(\rho,\tau),F(\rho,\tau)\right)\right) \\
\leq \;& 4\kappa^2 \max\{\theta,\tau\}||\Gamma_1(t,s) - \Gamma_2(t,s)||^2_\infty \left(\; |\chi-\rho| + |\zeta-\tau| \; \right)^2 \\
\leq \;& \frac{1}{2}\left[(2|\chi-\rho|)^2 + (2|\zeta-\tau|)^2\right] \\
\leq \;& \frac{1}{2}\left(G_b(\chi,\rho,\rho) + G_b(\zeta,\tau,\tau)\right) \\
\leq \;& \psi\left(\frac{G_b(\chi,\rho,\rho) + G_b(\zeta,\tau,\tau)}{2}\right).
\end{aligned}$$

This proves that the operator F satisfies the contractive condition appearing in Corollary 3.6. Finally, let (x,y) be a coupled lower and upper solution of the integral equation (14) then, by assumption (f), we have $x \leq F(x,y) \leq F(y,x) \leq y$. Corollary 3.6 gives us that F has a unique coupled fixed point $(\chi,\zeta) \in \mathscr{Q} \times \mathscr{Q}$. Since $x \leq y$, Corollary 3.6 says us that $\chi = \zeta$ and this implies $\chi = F(\chi,\chi)$ and χ is the unique solution of Eq. (14).

3.2. Applications to Matrix Equations.

In this section, we study the existence and uniqueness of solutions $(\mathscr{A},\mathscr{B})$ to the system of matrix equations:

$$\begin{cases} \mathscr{A} = Q + C_1^\star \mathscr{A} C_1 - D_1^\star \mathscr{B} D_1 \\ \\ \mathscr{B} = Q + C_1^\star \mathscr{B} C_1 - D_1^\star \mathscr{A} D_1 \end{cases} \tag{18}$$

where $C_1, D_1 \in \mathscr{M}(n)$: the set of all $n \times n$ matrices, $Q \in \mathscr{P}(n)$: the set of all $n \times n$ positive definite matrices, and $\mathscr{H}(n)$ is the set of all $n \times n$ Hermitian matrices. We endow $\mathscr{H}(n)$ with the partial order $\leq$ given by $M,N \in \mathscr{H}(n)$, $M \leq N \Leftrightarrow N - M \in \mathscr{P}(n)$. For a fixed $P \in \mathscr{P}(n)$, we consider $||H||_{1,P} = tr\left(P^{\frac{1}{2}} H P^{\frac{1}{2}}\right)$ for all $H \in \mathscr{H}(n)$ where tr is the trace operator. The space $\mathscr{H}(n)$ equipped with the G_b-metric induced by $||.||_{1,P}$ is a complete G_b- metric space for any positive definite matrix P (see. [30]).

The following lemma will be useful for our application.

Lemma 3.2.1: Let $M \geq 0$ and $N \geq 0$ be $n \times n$ matrices. Then, we have $0 \leq tr(MN) = tr(NM) \leq ||M||tr(N)$, where $||.||$ is the spectral norm.

Theorem 3.2.2: Suppose that there exists $P \in \mathscr{P}(n)$ such that $16\kappa^4 \max\{||P^{\frac{-1}{2}}C_1^{\star}PC_1P^{\frac{-1}{2}}||, ||P^{\frac{-1}{2}}D_1^{\star}PD_1P^{\frac{-1}{2}}||\} < 1$. Suppose also that $0 \leq C_1^{\star}QC_1$ and $Q \leq D_1^{\star}QD_1$. Then, the system (18) has one and only one solution $(X,Y) \in \mathscr{H}(n) \times \mathscr{H}(n)$.

Proof Consider the mappings $F : \mathscr{H}(n) \times \mathscr{H}(n) \to \mathscr{H}(n)$ defined by $F(X,Y) = Q + C_1^{\star}XC_1 - D_1^{\star}YD_1$ for all $X,Y \in \mathscr{H}(n)$.

For all $X_j, Y_j \in \mathscr{H}(n)$, $j = 1,2$ with $X_1 \leq Y_1$ and $Y_2 \leq X_2$ and $\psi : [0,\infty) \to [0,\infty)$ as $\psi(t) = ||t||_{1,P}$. By using Lemma 3.2.1, we have

$$
\begin{aligned}
& \psi\left(2\kappa^4 G_b\left(F(Y_1,Y_2), F(X_1,X_2), F(X_1,X_2)\right)\right) \\
= \;& 8\kappa^4 \left(||F(Y_1,Y_2) - F(X_1,X_2)|||_{1,P}\right)^2 \\
= \;& 8\kappa^4 \left(||C_1^{\star}(Y_1 - X_1)C_1 - D_1^{\star}(Y_2 - X_2)D_1||_{1,P}\right)^2 \\
= \;& 8\kappa^4 \left(tr\left[P^{\frac{1}{2}}\left(C_1^{\star}(Y_1 - X_1)C_1 - D_1^{\star}(Y_2 - X_2)D_1\right)P^{\frac{1}{2}}\right]\right)^2 \\
= \;& 8\kappa^4 \left(tr[C_1^{\star}PC_1(Y_1 - X_1)] + tr[D_1^{\star}PD_1(Y_2 - X_2)]\right)^2 \\
= \;& 8\kappa^4 \left(tr\left[C_1^{\star}PC_1P^{\frac{-1}{2}}P^{\frac{1}{2}}(Y_1 - X_1)P^{\frac{-1}{2}}P^{\frac{1}{2}}\right] + tr\left[D_1^{\star}PD_1P^{\frac{-1}{2}}P^{\frac{1}{2}}(Y_2 - X_2)P^{\frac{-1}{2}}P^{\frac{1}{2}}\right]\right)^2 \\
\leq \;& 8\kappa^4 \left(||P^{\frac{-1}{2}}C_1PC_1^{\star}P^{\frac{-1}{2}}||tr(P^{\frac{1}{2}}(Y_1 - X_1)P^{\frac{1}{2}}) + ||P^{\frac{-1}{2}}D_1PD_1^{\star}P^{\frac{-1}{2}}||tr(P^{\frac{1}{2}}(Y_2 - X_2)P^{\frac{1}{2}})\right)^2 \\
\leq \;& 8\kappa^4 \left(||P^{\frac{-1}{2}}C_1PC_1^{\star}P^{\frac{-1}{2}}||||Y_1 - X_1||_{1,P} + ||P^{\frac{-1}{2}}D_1PD_1^{\star}P^{\frac{-1}{2}}||||Y_2 - X_2||_{1,P}\right)^2 \\
\leq \;& \frac{1}{2}(2||Y_1 - X_1||_{1,P})^2 + (2||Y_2 - X_2||_{1,P})^2 \\
\leq \;& \psi\left(\frac{G_b(Y_1,X_1,X_1) + G_b(Y_2,X_2,X_2)}{2}\right).
\end{aligned}
$$

Thus, we proved that the contractive condition given in Corollary 3.6 is satisfied. Moreover, from $0 \leq C_1^{\star}QC_1$ and $Q \leq D_1^{\star}QD_1$ we have letting $Q \leq F(Q,0)$ and $0 \geq F(0,Q)$. Corollary 3.6. F has a coupled fixed point. Then there exist $X,Y \in \mathscr{H}(n)$ such that $F(X,Y) = X$ and $F(Y,X) = Y$.

3.3. Applications to Homotopy.

In this section, we study the existence of an unique solution to Homotopy theory.

Theorem 3.3.1: Let $(\mathscr{Q}, G_b)$ be complete G_b-metric space, U and $\overline{U}$ be an open and closed subset of $\mathscr{Q}$ such that $U \subseteq \overline{U}$. Suppose $H_b : \overline{U}^2 \times [0,1] \to \mathscr{Q}$ be an operator with following conditions are satisfying,

- (τ_0) $x \neq H_b(x,y,\chi)$, $y \neq H_b(y,x,\chi)$, for each $x,y \in \partial U$ and $\chi \in [0,1]$ (Here ∂U is boundary of U in $\mathscr{Q}$)
- (τ_1) $\psi\left(2\kappa^4 G_b\left(H_b(x,y,\chi), H_b(a,b,\chi), H_b(a,b,\chi)\right)\right) \leq \psi\left(\frac{G_b(x,a,a)+G_b(y,b,b)}{2}\right)$ for all $x,y,a,b \in \overline{U}$ and $\chi \in [0,1]$, where $\psi : [0,\infty) \to [0,\infty)$ is continuous, non-decreasing and $\psi(t) = 0 \iff t = 0$,
- (τ_2) $\exists\, M \geq 0 \ni G_b(H_b(x,y,\chi), H_b(x,y,\zeta), H_b(x,y,\zeta)) \leq M|\kappa - \zeta|$ for every $x,y \in \overline{U}$and $\chi, \zeta \in [0,1]$.

Then $H_b(.,0)$ has a coupled fixed point $\iff$ $H_b(.,1)$ has a coupled fixed point.

Proof Let the set $\mathscr{B} = \left\{\ \chi \in [0,1] : H_b(x,y,\chi) = x, H_b(y,x,\chi) = y \text{ for some } x,y \in U\ \right\}$. Since $H_b(.,0)$ has a coupled fixed point in U^2, we have that $(0,0,) \in \mathscr{B}^2$. So that $\mathscr{B}$ is non-empty set. Now we show that $\mathscr{B}$ is both closed and open in $[0,1]$ and hence by the connectedness $\mathscr{B} = [0,1]$. As a result, $H_b(.,1)$ has a coupled fixed point in U^2. First we show that $\mathscr{B}$ closed in $[0,1]$. To see this, Let $\{\chi_p\}_{p=1}^{\infty} \subseteq \mathscr{B}$ with $\chi_p \to \chi \in [0,1]$ as $p \to \infty$. We must show that $\chi \in \mathscr{B}$. Since $\chi_p \in \mathscr{B}$ for $p = 0,1,2,3,\cdots$, there exists sequences $\{x_p\}, \{y_p\} \subseteq U$ with $x_p = H_b(x_p,y_p,\chi_p)$, $y_p = H_b(y_p,x_p,\chi_p)$. Consider

$$\begin{aligned}
& G_b(x_p,x_{p+1},x_{p+1}) \\
= \;& G_b\left(H_b(x_p,y_p,\chi_p), H_b(x_{p+1},y_{p+1},\chi_{p+1}), H_b(x_{p+1},y_{p+1},\chi_{p+1})\right) \\
\leq \;& \kappa \begin{pmatrix} G_b\left(H_b(x_p,y_p,\chi_p), H_b(x_{p+1},y_{p+1},\chi_p), H_b(x_{p+1},y_{p+1},\chi_p)\right) \\ G_b\left(H_b(x_{p+1},y_{p+1},\chi_p), H_b(x_{p+1},y_{p+1},\chi_{p+1}), H_b(x_{p+1},y_{p+1},\chi_{p+1})\right) \end{pmatrix} \\
\leq \;& \kappa G_b\left(H_b(x_p,y_p,\chi_p), H_b(x_{p+1},y_{p+1},\chi_p), H_b(x_{p+1},y_{p+1},\chi_p)\right) + \kappa M|\chi_p - \chi_{p+1}|.
\end{aligned}$$

Letting $p \to \infty$, we get

$$\lim_{p\to\infty} G_b(x_p,x_{p+1},x_{p+1}) \;\leq\; \lim_{p\to\infty} \kappa G_b\left(H_b(x_p,y_p,\chi_p), H_b(x_{p+1},y_{p+1},\chi_p), H_b(x_{p+1},y_{p+1},\chi_p)\right).$$

Since ψ is continuous and non-decreasing, we obtain

$$\begin{aligned}\lim_{p\to\infty}\psi\left(2\kappa^3G_b(x_p,x_{p+1},x_{p+1})\right) &\leq \lim_{p\to\infty}\psi\left(2\kappa^4G_b\left(\begin{array}{c}H_b(x_p,y_p,\chi_p),H_b(x_{p+1},y_{p+1},\chi_p),\\ H_b(x_{p+1},y_{p+1},\chi_p)\end{array}\right)\right)\\ &\leq \lim_{p\to\infty}\psi\left(\frac{G_b(x_p,x_{p+1},x_{p+1})+G_b(y_p,y_{p+1},y_{p+1})}{2}\right).\end{aligned}$$

By the definition of ψ, it follows that

$$\lim_{p\to\infty}4\kappa^3G_b(x_p,x_{p+1},x_{p+1})\leq\lim_{p\to\infty}\left(G_b(x_p,x_{p+1},x_{p+1})+G_b(y_p,y_{p+1},y_{p+1})\right). \tag{19}$$

Similarly, we can prove

$$\lim_{p\to\infty}4\kappa^3G_b(y_p,y_{p+1},y_{p+1})\leq\lim_{p\to\infty}\left(G_b(x_p,x_{p+1},x_{p+1})+G_b(y_p,y_{p+1},y_{p+1})\right). \tag{20}$$

Combining (19) and (20), we have $\lim_{p\to\infty}(4\kappa^3-1)\left(G_b(x_p,x_{p+1},x_{p+1})+G_b(y_p,y_{p+1},y_{p+1})\right)\leq 0$. So that $\lim_{p\to\infty}G_b(x_p,x_{p+1},x_{p+1})=0$ and $\lim_{p\to\infty}G_b(y_p,y_{p+1},y_{p+1})=0$.

Now we prove that $\{x_p\}$ is an G_b-Cauchy sequence in $(\mathscr{Q},G_b)$. On the contrary, suppose that $\{x_p\}$ is not G_b-Cauchy. There exists $\varepsilon>0$ and monotone increasing sequences of natural numbers $\{q_k\}$ and $\{p_k\}$ such that $p_k>q_k$,

$$\begin{aligned}&G_b(x_{q_k},x_{p_k},x_{p_k})\geq\varepsilon \quad G_b(y_{q_k},y_{p_k},y_{p_k})\geq\varepsilon\\ &G_b(x_{q_k},x_{p_{k-1}},x_{p_{k-1}})<\varepsilon \quad G_b(y_{q_k},y_{p_{k-1}},y_{p_{k-1}})<\varepsilon.\end{aligned} \tag{21}$$

From (21), we have $\varepsilon\leq G_b(x_{q_k},x_{p_k},x_{p_k})\leq\kappa\left(G_b(x_{q_k},x_{q_{k+1}},x_{q_{k+1}})+G_b(x_{q_{k+1}},x_{p_k},x_{p_k})\right)$. Letting $p\to\infty$ and applying ψ on both sides, we have that

$$\psi(2\kappa^3\varepsilon) \leq \lim_{p\to\infty}\psi\left(2\kappa^4G_b(x_{q_{k+1}},x_{p_k},x_{p_k})\right). \tag{22}$$

But

$$\begin{aligned}&\lim_{p\to\infty}\psi\left(2\kappa^4G_b(x_{q_{k+1}},x_{p_k},x_{p_k})\right)\\ \leq\ &\lim_{p\to\infty}\psi\left(2\kappa^4G_b(H_b(x_{q_{k+1}},y_{q_{k+1}},\chi_{q_{k+1}}),H_b(x_{p_k},y_{p_k},\chi_{p_k}),H_b(x_{p_k},y_{p_k},\chi_{p_k}))\right)\\ \leq\ &\lim_{p\to\infty}\psi\left(\frac{G_b(x_{q_{k+1}},x_{p_k},x_{p_k})+G_b(y_{q_{k+1}},y_{p_k},y_{p_k})}{2}\right).\end{aligned}$$

It follows that

$$\lim_{p\to\infty} 4\kappa^4 G_b(x_{q_{k+1}}, x_{p_k}, x_{p_k}) \le \lim_{p\to\infty} \left(G_b(x_{q_{k+1}}, x_{p_k}, x_{p_k}) + G_b(y_{q_{k+1}}, y_{p_k}, y_{p_k})\right). \tag{23}$$

Similarly, we can prove

$$\lim_{p\to\infty} 4\kappa^4 G_b(y_{q_{k+1}}, y_{p_k}, y_{p_k}) \le \lim_{p\to\infty} \left(G_b(x_{q_{k+1}}, x_{p_k}, x_{p_k}) + G_b(y_{q_{k+1}}, y_{p_k}, y_{p_k})\right). \tag{24}$$

Combining (23) and (24), we have $\lim_{p\to\infty}(4\kappa^4 - 1)\left(G_b(x_{q_{k+1}}, x_{p_k}, x_{p_k}) + G_b(y_{q_{k+1}}, y_{p_k}, y_{p_k})\right) \le 0$. So that $\lim_{p\to\infty} G_b(x_{q_{k+1}}, x_{p_k}, x_{p_k}) = 0$ and $\lim_{p\to\infty} G_b(y_{q_{k+1}}, y_{p_k}, y_{p_k}) = 0$. Hence from (22) and the definition of ψ, we have that $\varepsilon \le 0$, which is a contradiction. Hence $\{x_p\}$ is an G_b-Cauchy sequence in $(\mathscr{Q}, G_b)$, by similar arguments $\{y_p\}$ is also an G_b-Cauchy sequence in $(\mathscr{Q}, G_b)$, and by completeness of $(\mathscr{Q}, G_b)$, there exist $a, b \in \mathscr{Q}$ with

$$\lim_{p\to\infty} x_p = a = \lim_{p\to\infty} x_{p+1} \quad \lim_{p\to\infty} y_p = b = \lim_{p\to\infty} y_{n+1}$$

using Lemma (2.7) and condition (τ_1), we have

$$\begin{aligned}
&\psi\left(2\kappa^3 G_b(a, H_b(a,b,\chi), H_b(a,b,\chi))\right) \le \liminf_{p\to\infty} \psi\left(2\kappa^4 G_b(x_p, H_b(a,b,\chi), H_b(a,b,\chi))\right) \\
\le\ & \liminf_{p\to\infty} \psi\left(2\kappa^4 G_b(H_b(x_p, y_p, \chi_p), H_b(a,b,\chi), H_b(a,b,\chi))\right) \\
\le\ & \liminf_{p\to\infty} \psi\left(\frac{G_b(x_p,a,a) + G_b(y_p,b,b)}{2}\right) = 0.
\end{aligned}$$

It follows that $H_b(a,b,\chi) = a$. Similarly, we obtain that $H_b(b,a,\chi) = b$. Thus $\chi \in \mathscr{B}$. Hence $\mathscr{B}$ is closed in $[0,1]$. Let $\chi_0 \in \mathscr{B}$, then there exist $x_0, y_0 \in U$ with $x_0 = H_b(x_0, y_0, \chi_0)$,
$y_0 = H_b(y_0, x_0, \chi_0)$, Since U is open, then there exist $r > 0$ such that $B_{G_b}(x_0, r) \subseteq U$.
Choose $\chi \in (\chi_0 - \varepsilon, \chi_0 + \varepsilon)$ such that $|\chi - \chi_0| \le \frac{1}{M^p} < \frac{\varepsilon}{2}$,
then for $x \in \overline{B_{G_b}(x_0, r)} = \left\{x \in \mathscr{Q} / G_b(x, x_0, x_0) \le r + 2\kappa^2 G_b(x_0, x_0, x_0)\right\}$, and
$y \in \overline{B_{G_b}(y_0, r)} = \left\{y \in \mathscr{Q} / G_b(y, y_0, y_0) \le r + 2\kappa^2 G_b(y_0, y_0, y_0)\right\}$. Also

$$\begin{aligned}
& G_b\left(H_b(x,y,\chi), x_0, x_0\right) = G_b\left(H_b(x,y,\chi), H_b(x_0,y_0,\chi_0), H_b(x_0,y_0,\chi_0)\right) \\
\le\ & \kappa\left(G_b\left(H_b(x,y,\chi), H_b(x,y,\chi_0), H_b(x,y,\chi_0)\right) + G_b\left(H_b(x,y,\chi_0), H_b(x_0,y_0,\chi_0), H_b(x_0,y_0,\chi_0)\right)\right) \\
\le\ & \kappa M|\chi - \chi_0| + \kappa G_b\left(H_b(x,y,\chi_0), H_b(x_0,y_0,\chi_0), H_b(x_0,y_0,\chi_0)\right) \\
\le\ & \frac{\kappa}{M^{p-1}} + \kappa G_b\left(H_b(x,y,\chi_0), H_b(x_0,y_0,\chi_0), H_b(x_0,y_0,\chi_0)\right).
\end{aligned}$$

Letting $p\to\infty$, we obtain $G_b\left(H(x,y,\chi),x_0,x_0\right)\leq \kappa G_b\left(H_b(x,y,\chi_0),H_b(x_0,y_0,\chi_0),H_b(x_0,y_0,\chi_0)\right).$
Since ψ is continuous and non-decreasing, we have

$$\begin{aligned}\psi\left(G_b\left(H(x,y,\chi),x_0,x_0\right)\right) &\leq \psi\left(2\kappa^3 G_b\left(H(x,y,\chi),x_0,x_0\right)\right)\\ &\leq \psi\left(2\kappa^4 G_b\left(H_b(x,y,\chi_0),H_b(x_0,y_0,\chi_0),H_b(x_0,y_0,\chi_0)\right)\right)\\ &\leq \psi\left(\frac{G_b(x,x_0,x_0)+G_b(y,y_0,y_0)}{2}\right).\end{aligned}$$

Since ψ is non-decreasing, we have

$$\begin{aligned}G_b\left(H(x,y,\chi),x_0,x_0\right) &\leq \frac{G_b(x,x_0,x_0)+G_b(y,y_0,y_0)}{2}\\ &\leq r+\kappa^2 G_b(x_0,x_0,x_0)+\kappa^2 G_b(y_0,y_0,y_0).\end{aligned}$$

Similarly, we can prove $G_b\left(H(y,x,\chi),y_0,y_0\right)\leq r+\kappa^2 G_b(x_0,x_0,x_0)+\kappa^2 G_b(y_0,y_0,y_0)$.
Thus we have,

$$G_b\left(H(x,y,\chi),x_0,x_0\right)+G_b\left(H(y,x,\chi),y_0,y_0\right) \leq r+\kappa^2 G_b(x_0,x_0,x_0)+\kappa^2 G_b(y_0,y_0,y_0).$$

Thus for each fixed $\chi\in(\chi_0-\varepsilon,\chi_0+\varepsilon)$, $H_b(.,\chi):\overline{B_{G_b}(x_0,r)}\to\overline{B_{G_b}(x_0,r)}$, $H_b(.,\chi):\overline{B_{G_b}(y_0,r)}\to\overline{B_{G_b}(y_0,r)}$. Since also (τ_1) holds and ψ is continuous and non-decreasing, then all conditions of Theorem (3.3.1) are satisfied. Thus we conclude that $H_b(.,\chi)$ has a coupled fixed point in $\overline{U}^2$. But this must be in U^2 since (τ_0) holds. Thus, $\chi\in\mathscr{B}$ for any $\chi\in(\chi_0-\varepsilon,\chi_0+\varepsilon)$. Hence $(\chi_0-\varepsilon,\chi_0+\varepsilon)\subseteq\mathscr{B}$. Clearly $\mathscr{B}$ is open in [0, 1].
For the reverse implication, we use the same strategy.

4. Conclusions

In this paper we conclude some applications to homotopy theory and integral equations as well as matrix equations by using coupled fixed point theorems for two mappings via generalized (ψ,ϕ)- contractive condition in partially ordered G_b-metric spaces.

Author Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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