Available online at http://scik.org
Adv. Fixed Point Theory, 2023, 13:25
https://doi.org/10.28919/afpt/8257
ISSN: 1927-6303

FIXED POINT THEOREMS FOR $\psi-$CONTRACTION MAPPING IN FUZZY N-CONTROLLED METRIC SPACE

AMINE FAIZ, ADIL BAIZ*, JAMAL MOULINE, KHADIJA BOUZKOURA

Laboratory of Analysis, Modeling and Simulation Faculty of Sciences Ben M'Sik, Hassan II University, B.P. 7955 Casablanca, Morocco

Abstract. This manuscript consists of the idea of n-controlled metric space in fuzzy set theory to generalize a number of fuzzy metric spaces in the literature, for example, pentagonal, hexagonal, triple, and double controlled metric spaces and many other spaces in fuzzy environment. Various examples are given to explain definitions and results. We define open ball, convergence of a sequence and a Cauchy sequence in the context of fuzzy n-controlled metric space. We also prove, by means of an example, that a fuzzy n-controlled metric space is not Hausdorff. At the end of the article, an application is given to prove the uniqueness of the solution to fractional differential equations.

Keywords: fixed point; fuzzy metric space; $\psi-$contraction.

2020 AMS Subject Classification: 47H09, 47H10.

1. Introduction

The applications of fixed point theory is the key to prove the uniqueness of the solution of a scientific problem with the help of Banach fixed point theorem [1]. Researchers have implemented this famous theorem in other directions (see [3, 12, 9, 2, 15, 17, 18, 19, 14, 4, 5]) and obtained interesting results. There are many generalizations of [1]. For example, Edelstein

*Corresponding author
E-mail address: baizadil@gmail.com

Received October 02, 2023

[23], generalized the Banach theorem in 1961. Kannan, [4] proved Banach's theorems without using the completeness of the metric and continuity of the contraction, however, he obtained the same conclusion but different sufficient conditions. Similar results were proved by Chatterjea [17]. In 1974, Ciric [18], utilized the quasi contractive mappings that generalizes [1]. He also introduced multi-valued quasi contractions. Samet et. al [25] introduced a very interesting contraction, called $\alpha-\psi$-contraction, that enhanced and generalized numerous

results in the literature. In 2014, Jleli et. al [2] gave the generalized version of [1] by introducing the function that satisfies certain properties. Since all the above generalizations of [1] need to be continuous mappings, so Suzuki [19] gave the idea of Suzuki type mappings in which the contraction need not be continuous. Using F-contractions, which is given by Wardowski [20], and Suzuki contraction, the authors in [21], introduced generalized Suzuki F-contractions. Same authors have discussed the notion of Suzuki-type (α,β,γ_g)-generalized proximal contractions and proved some results. Recently, Saleem et al. [22], gave the idea of modified F-contractions, generalized Suzuki F-contractions and proved some interesting results. In 1965, Zadeh [24] generalized the definition of a crisp set by defining the fuzzy set that gives more efficient and accurate results. As fuzzy set addresses the uncertainty and give more accuracy compared to crisp set, researcher have used fuzzy sets in almost every branch of mathematics, see ([26, 27, 28]). Metric space in a fuzzy environment is the most studied topic. The first definition of metric space using fuzzy sets was given by Kramosil et. al [29] which is considered as the generalization of statistical metric spaces defined by Menger [30]. But in their definition, they did not discuss any topological aspects. The convergence of a sequence in fuzzy metric spaces was defined by Grabiec [31]. By discussing Cauchyness he proved the fuzzy version of the Banach theorem. As topological properties of metric spaces play a vital role so, George and Veeramani [32] generalized the definition given in [31] by discussing topology and proved that it is Hausdorff. Branciari [33] introduced generalized metric space which is known as rectangular metric space or b-Branciari space. He proved Banach-Caccippoli type fixed point results. In [34], the author has introduced a fuzzy version of b-metric space and generalized some spaces. The authors in [35] utilized the function to generalize the notion of

[34] by introducing an extended version of a fuzzy b-metric space and proved interesting results. Sezen [36] first used a controlled function to define the concept of controlled spaces in fuzzy sets theory. She utilized the sense of [29] and prove Banach fixed point results. Saleem et al. [37] used two functions and defined double controlled metric in a fuzzy environment which generalizes the results in [36]. Chugh et al. [38] gave the fuzzy version of [33] by giving the concept of rectangular fuzzy metric space. The notion of a rectangular b-metric space in fuzzy set theory is given by [39] to generalize the notion given in [38]. Recently, the concept of an extended rectangular metric space in a fuzzy environment is given by Saleem et al. [40] that generalize the results of [39] and [38]. They also proved that this space is not Hausdorff. The authors in [41] utilized three functions f ; g; h and gave the notions of fuzzy triple controlled metric spaces. They also showed, with the help of an example, that this space is not Hausdorff. The ideas of extended hexagonal b-metric and pentagonal controlled metric spaces in the fuzzy environments were given by Zubair et al. [42] and Hussain et al. [43] respectively and proved some fixed point results. In [44], the authors have introduced graphical fuzzy metric spaces and proved interesting results.

In this paper we define n-controlled metric space in fuzzy set theory that generalizes almost all the metric spaces discussed above. We prove some fixed point results and elaborate our results with examples. We use the sense of [32] to define this space. We will use ψ-contractive mapping in our main results that generalize some existing fixed point theorems in the literature. Each result and definition is supported by examples, further, we prove that this newly defined space is not Hausdorff.

2. Preliminaries

Definition 2.1. A binary operation $*: J\times J\to J, (J=[0,1])$ is known as continuous triangular norm, if for all $x,y,z,t\in[0,1]$, $*$ satisfy:

(i) $*(x,y)=*(y,x)$;

(ii) $*(x,*(y,z))=*(*(x,y),z)$;

(iii) $*$ is continuous;

(iv) $*(x,1)=x$ for every $x\in[0,1]$

(v) $*(x,z)\le *(y,t)$ whenever $x\le y$, $z\le t$.

Definition 2.2. Let $X \neq \emptyset$, a fuzzy set $P : X \times X \times \mathbb{R}^+ \to [0,1]$ is called fuzzy metric on X with $*$ as a (CTN), if for all $x,y,z \in X$ the following conditions holds:

(i) $P(x,y,t) > 0$; for all $t > 0$,

(ii) $P(x,y,t) = 1$; for all $t > 0$, if and only if $x = y$

(iii) $P(x,y,t) = P(y,x,t)$;

(iv) $P(x,z,t+t') \geq P(x,y,t) * P(y,z,t')$ for all $t,t' > 0$

(v) $P(x,y,.) : (0,\infty) \to [0,1]$ is continuous.

The triplet $(X,P,*)$ is called a fuzzy metric space.

Definition 2.3. Let $X \neq \emptyset$, $f,g : X \times X \to [1,\infty)$ are two non-comparable functions. Then a fuzzy set $P_d : X \times X \times \mathbb{R}^+ \to [0,1]$ is fuzzy double controlled metric on X with $*$ as a (CTN), if for all $x,y,z \in X$ the following conditions holds:

(i) $P_d(x,y,t) > 0$; for all $t > 0$,

(ii) $P_d(x,y,t) = 1$; for all $t > 0$, if and only if $x = y$

(iii) $P_d(x,y,t) = P_d(y,x,t)$;

(iv) $P_d(x,z,t+t') \geq P_d(x,y,\frac{t}{f(x,y)}) * P_d(y,z,\frac{t'}{g(y,z)})$ for all $t,t' > 0$

(v) $P_d(x,y,.) : (0,\infty) \to [0,1]$ is continuous.

The triplet $(X,P_d,*)$ is called a fuzzy double controlled metric space.

Example 2.4. Let $X = \{1,2,3\}$ and $f,g : X \times X \to [1,\infty)$ be two non-comparable continuous functions given by $f(x,y) = x+y+1$ and $g(y,z) = y^2+z^2-1$. Define $P_d : X \times X \times \mathbb{R}^+ \to [0,1]$ as

$$P_d(x,y,t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}.$$

Then $(X,P_d,*)$ is fuzzy double controlled metric space with product t-norm.

Definition 2.5. Let $X \neq \emptyset$ and consider three functions $f,g,h : X \times X \to [1,\infty)$. Then a fuzzy set $P_T : X \times X \times \mathbb{R}^+ \to [0,1]$ is fuzzy triple controlled metric on X with $*$ as a (CTN), if for all $x,y \in X$ and all distinct $z,s \in X$ the following conditions are satisfied:

(i) $P_T(x,y,t) > 0$; for all $t > 0$,

(ii) $P_T(x,y,t) = 1$; for all $t > 0$, if and only if $x = y$

(iii) $P_T(x,y,t) = P_T(y,x,t)$;

(iv) $P_T(x,s,t_1+t_2+t_3) \geq P_T(x,y,\frac{t_1}{f(x,y)}) * P_T(y,z,\frac{t_2}{g(y,z)}) * P_T(z,s,\frac{t_3}{h(z,s)})$ for all $t_1,t_2,t_3 > 0$

(v) $P_T(x,y,.) : (0,\infty) \to [0,1]$ is continuous.

Then $(X,P_T,*)$ is called a fuzzy triple controlled metric space.

Example 2.6. Let $X = [0,1]$ and $P_T : X \times X \times \mathbb{R}^+ \to [0,1]$ be defined as

$$P_T(x,y,t) = \exp(-\frac{|x-y|}{t}) \quad forall \quad t > 0.$$

Further let $f,g,h : X \times X \to [0,\infty)$ be continuous functions defined by $f(x,y) = x+y+1$, $g(y,z) = y^2+z+1$. and $g(z,s) = z^2+s^2+1$.

Then $(X,P_T,*)$ is fuzzy triple controlled metric space.

Definition 2.7. Let $X \neq \emptyset$, $f_i : X \times X \to [1,\infty)$, $1 \leq i \leq 5$ be given functions. Then a fuzzy set $P_F : X \times X \times \mathbb{R}^+ \to [0,1]$ is fuzzy pentagonal controlled metric on X with $*$ as a (CTN), if for any distinct $x,y,z,s_1,s_2,s_3 \in X$ the following conditions are satisfied:

(i) $P_F(x,y,t) > 0$; for all $t > 0$,

(ii) $P_F(x,y,t) = 1$; for all $t > 0$, if and only if $x = y$

(iii) $P_F(x,y,t) = P_F(y,x,t)$;

(iv) $P_F(x,s,t_1+t_2+t_3+t_4+t_5) \geq P_F(x,y,\frac{t_1}{f_1(x,y)}) * P_F(y,z,\frac{t_2}{f_2(y,z)}) * P_F(z,s,\frac{t_3}{f_3(s_1,s_2)}) * P_F(s_1,s_2,\frac{t_4}{f_4(s_1,s_2)}) * P_F(s_2,s_3,\frac{t_5}{f_5(s_2,s_3)})$ for all $t_1,t_2,t_3,t_4,t_5 > 0$

(v) $P_F(x,y,.) : (0,\infty) \to [0,1]$ is continuous and $\lim_{t\to\infty} P_F(x,y,t) = 1$.

Then $(X,P_F,*)$ is called a fuzzy pentagonal controlled metric space.

3. Main Results

This section contains definitions, examples and theorems related to fuzzy n-controlled metric space. We will also deduce some important remarks that prove generalizations of many metric spaces in fuzzy set theory. We will define open ball and will prove that the newly defined space is not Hausdorff. Each result is elaborated with the help of examples. Now we give the definition of a fuzzy n-controlled metric space in the sense of [23]:

Definition 3.1. Let $X \neq \emptyset$, $f_i : X \times X \to [1,\infty)$, $1 \leq i \leq n$ be n non-comparable functions. A fuzzy set $P_Q : X \times X \times \mathbb{R}^+$, together with a (CTN) $*$, is called a fuzzy n-controlled metric, if for any distinct $s_1,s_2,s_3,...,s_{n+1} \in X$ the following conditions are satisfied:

(i) $P_Q(s_1,s_2,t) > 0$; for all $t > 0$,

(ii) $P_Q(s_1,s_2,t) = 1$; for all $t > 0$, if and only if $s_1 = s_2$

(iii) $P_Q(s_1,s_2,t) = P_F(s_2,s_1,t)$;

(iv) $P_Q(s_1,s_{n+1},t_1 + t_2 + ... + t_n) \geq P_Q(s_1,s_2,\frac{t_1}{f_1(s_1,s_2)}) * P_Q(s_2,s_3,\frac{t_2}{f_2(s_2,s_3)}) * .... * P_Q(s_n,s_{n+1},\frac{t_n}{f_n(s_n,s_{n+1})})$ for all $t_1,t_2,t_3,t_4,..,t_n > 0$

(v) $P_Q(s_1,s_2,.) : (0,\infty) \to [0,1]$ is continuous and.

for all distinct $s_1,s_2,s_3,...,s_{n+1} \in X$ The quadruple $(X,P_Q,f_n,*)$ is called a fuzzy n-controlled metric space (FnCMS).

Example 3.2. Consider $X = \{1,2,3,4,5,6,7\}$ and $f_i : X \times X \to [0,\infty)$ $(1 \leq i \leq 6)$ be defined as $f_1(s_1,s_2) = s_1 + s_2 + 1$, $f_2(s_1,s_2) = s_1^2 + s_2 + 1$, $f_3(s_1,s_2) = s_2^2 + s_1 + 1$, $f_4(s_1,s_2) = s_1^2 + s_2^2 + 1$, $f_5(s_1,s_2) = s_1 + s_2^3 + 1$, $f_6(s_1,s_2) = s_1^3 + s_2^3 + 1$. Now define $P_Q : X \times X \times \mathbb{R}^+ \to [0,1]$ as

$$P_Q(s_1,s_2,t) = \frac{min\{s_1,s_2\} + t}{max\{s_1,s_2\} + t}.$$

Then with product t-norm $(X,P_Q,f_n,*)$ is a (FnCMS).Here we will prove only (iv) as (i)-(iii) and (v) are easy to prove.

Let $s_1 = 1$, $s_2 = 7$. then

$$\begin{aligned} P_Q(1,7,t_1+t_2+t_3+t_4+t_5+t_6) &= \frac{min\{1,7\} + t_1+t_2+t_3+t_4+t_5+t_6}{max\{1,7\} + t_1+t_2+t_3+t_4+t_5+t_6}. \\ &= \frac{1+t_1+t_2+t_3+t_4+t_5+t_6}{7+t_1+t_2+t_3+t_4+t_5+t_6} \end{aligned}$$

$$P_Q(1,2,\frac{t_1}{f_1(1,2)}) = \frac{min\{1,2\} + \frac{t_1}{f_1(1,2)}}{max\{1,2\} + \frac{t_1}{f_1(1,2)}} = \frac{1+\frac{t_1}{4}}{2+\frac{t_1}{4}} = \frac{4+t_1}{8+t_1}.$$

$$P_Q(2,3,\frac{t_2}{f_2(2,3)}) = \frac{min\{2,3\} + \frac{t_2}{f_2(2,3)}}{max\{2,3\} + \frac{t_2}{f_2(2,3)}} = \frac{2+\frac{t_2}{8}}{3+\frac{t_2}{8}} = \frac{16+t_2}{24+t_2}.$$

$$P_Q(3,4,\frac{t_3}{f_3(3,4)}) = \frac{min\{3,4\} + \frac{t_3}{f_3(3,4)}}{max\{3,4\} + \frac{t_3}{f_3(3,4)}} = \frac{3+\frac{t_3}{20}}{4+\frac{t_3}{20}} = \frac{60+t_3}{80+t_3}.$$

$$P_Q(4,5,\frac{t_4}{f_4(4,5)})=\frac{min\{4,5\}+\frac{t_4}{f_4(4,5)}}{max\{4,5\}+\frac{t_4}{f_4(4,5)}}=\frac{4+\frac{t_4}{42}}{5+\frac{t_4}{42}}=\frac{168+t_4}{210+t_4}.$$

$$P_Q(5,6,\frac{t_5}{f_5(5,6)})=\frac{min\{5,6\}+\frac{t_5}{f_5(5,6)}}{max\{5,6\}+\frac{t_5}{f_5(5,6)}}=\frac{5+\frac{t_5}{222}}{6+\frac{t_5}{222}}=\frac{1110+t_5}{1332+t_5}.$$

$$P_Q(6,7,\frac{t_6}{f_6(6,7)})=\frac{min\{6,7\}+\frac{t_6}{f_6(6,7)}}{max\{6,7\}+\frac{t_6}{f_6(6,7)}}=\frac{6+\frac{t_6}{560}}{7+\frac{t_6}{560}}=\frac{3360+t_6}{3920+t_6}.$$

Clearly,

$$P_Q(1,7,t_1+t_2+t_3+t_4+t_5+t_6)\geq P_Q(1,2,\frac{t_1}{f_1(1,2)})*P_Q(2,3,\frac{t_2}{f_2(2,3)})*P_Q(3,4,\frac{t_3}{f_3(3,4)})$$
$$*P_Q(4,5,\frac{t_4}{f_4(4,5)})*P_Q(5,6,\frac{t_5}{f_5(5,6)})*P_Q(6,7,\frac{t_6}{f_6(6,7)}).$$

Similarly, we can prove in other cases. Hence $(X,P_Q,f_n,*)$ is called a fuzzy n-controlled metric space, for $n=6$.In the same steps, we can prove higher values of n.

Definition 3.3. Let s_n be a sequence in (FnCMS) $(X,P_Q,f_n,*)$.

Then:

(1) s_n is convergent sequence,if for any $t>0$; there exists $s\in X$ satisfy

$$lim_{n\to\infty}P_Q(s_n,s,t)=1.$$

(2) s_n is Cauchy sequence,if for all $t>0$, $t>0$

$$lim_{n\to\infty}P_Q(s_{n+p},s_n,t)=1.$$

A (FnCMS) $(X,P_Q,f_n,*)$ is called complete (FnCMS), if every Cauchy sequence s_n converges to some $s\in X$.

Definition 3.4. Let $(X,P_Q,f_n,*)$ be a (FnCMS). then the open ball $B(s,r,t)$, is given by

$$B(s,r,t)=\{v\in X:P_Q(s,v,t)>1-r\}.$$

where s is the center and r is the radius of the ball.

In next example, we will prove a (FnCMS) need not to be Hausdorff.

Example 3.5. Take the (FnCMS) of example (3.1) and define $B_1(1,0.4,5)$ with center $s_1 = 1$, radius $r_1 = 0.4$ and $t_1 = 5$ as

$$B_1(1,0.4,5) = \{s \in X : P_Q(1,s,5) > 0.6\}.$$

Let $1 \in X$ then $P_Q(1,1,5) = \dfrac{min\{1,1\}+5}{max\{1,1\}+5} = \dfrac{1+5}{1+5} = 1$, so $1 \in B_1(1,0.4,5)$.
Let $2 \in X$ then $P_Q(1,2,5) = \dfrac{min\{1,2\}+5}{max\{1,2\}+5} = \dfrac{1+5}{2+5} = 0.8571$, so $2 \in B_1(1,0.4,5)$.
Let $3 \in X$ then $P_Q(1,3,5) = \dfrac{min\{1,3\}+5}{max\{1,3\}+5} = \dfrac{1+5}{3+5} = 0.75$, so $3 \in B_1(1,0.4,5)$.
Let $4 \in X$ then $P_Q(1,4,5) = \dfrac{min\{1,4\}+5}{max\{1,4\}+5} = \dfrac{1+5}{4+5} = 0.6666$, so $4 \in B_1(1,0.4,5)$.
Let $5 \in X$ then $P_Q(1,5,5) = \dfrac{min\{1,5\}+5}{max\{1,5\}+5} = \dfrac{1+5}{5+5} = 0.6$, so $5 \notin B_1(1,0.4,5)$.
Let $6 \in X$ then $P_Q(1,6,5) = \dfrac{min\{1,6\}+5}{max\{1,6\}+5} = \dfrac{1+5}{6+5} = 0.5454$, so $6 \notin B_1(1,0.4,5)$.
Let $7 \in X$ then $P_Q(1,7,5) = \dfrac{min\{1,7\}+5}{max\{1,7\}+5} = \dfrac{1+5}{7+5} = 0.5$, so $7 \notin B_1(1,0.4,5)$.

$$B_2(2,0.2,5) = \{s \in X : P_Q(2,s,5) > 0.8\}.$$

Let $1 \in X$ then $P_Q(2,1,5) = \dfrac{min\{2,1\}+5}{max\{2,1\}+5} = \dfrac{1+5}{1+5} = 0.8571$, so $1 \in B_2(2,0.2,5)$.
Let $2 \in X$ then $P_Q(2,2,5) = \dfrac{min\{2,2\}+5}{max\{2,2\}+5} = \dfrac{2+5}{2+5} = 1$, so $2 \in B_2(2,0.2,5)$.
Let $3 \in X$ then $P_Q(2,3,5) = \dfrac{min\{2,3\}+5}{max\{2,3\}+5} = \dfrac{2+5}{3+5} = 0.875$, so $3 \in B_2(2,0.2,5)$.
Let $4 \in X$ then $P_Q(2,4,5) = \dfrac{min\{2,4\}+5}{max\{2,4\}+5} = \dfrac{2+5}{4+5} = 0.7777$, so $4 \in B_2(2,0.2,5)$.
Let $5 \in X$ then $P_Q(2,5,5) = \dfrac{min\{2,5\}+5}{max\{2,5\}+5} = \dfrac{2+5}{5+5} = 0.7$, so $5 \notin B_2(2,0.2,5)$.
Let $6 \in X$ then $P_Q(2,6,5) = \dfrac{min\{2,6\}+5}{max\{2,6\}+5} = \dfrac{2+5}{6+5} = 0.6363$, so $6 \notin B_2(2,0.2,5)$.
Let $7 \in X$ then $P_Q(2,7,5) = \dfrac{min\{2,7\}+5}{max\{2,7\}+5} = \dfrac{2+5}{7+5} = 0.5833$, so $7 \notin B_2(2,0.2,5)$.
Thus $B_2(2,0.2,5) = 1,2,3$. Clearly $B_1(1,0.4,5) \cap B_2(2,0.2,5) \neq \emptyset$.
Hence a (FnCMS) need not to be Hausdorff.

Remark 3.6. In the light of remark (3.1), a pentagonal, hexagonal, triple controlled, double controlled, b-extended and controlled rectangular, b-rectangular metric space and some other fuzzy metric spaces are also not Hausdorff.

Denote $\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+$, such that, ψ is non-decreasing, continuous, $\lim_{n\to\infty} \psi^n(t) = 0$, $\psi(t) < t$ for $t > 0$, where ψ^k is the $k-th$ iterate of ψ $\}$.

Definition 3.7. Let $(X, P_Q, f_n, *)$ be a (FnCMS). Then the mapping $T : X \to X$ is called a generalized ψ-fuzzy contractive mapping if for function $\psi \in \Psi$, we have

$$\frac{1}{P_Q(Ts_1, Ts_2, t)} - 1 \leq \psi(\frac{1}{P^*(s_1, s_2, t)} - 1). \tag{3.1}$$

for all $s_1, s_2 \in X$ and $t > 0$, where
$P^*(s_1, s_2, t) = min\{P_Q(s_1, s_2, t), P_Q(s_1, Ts_1, t), P_Q(s_2, Ts_2, t), \frac{2P_Q(s_1, Ts_2, t)P_Q(s_2, Ts_1, t)}{P_Q(s_1, Ts_2, t) + P_Q(s_2, Ts_1, t)}\}$
We now prove Banach fixed point theorem by using generalized ψ-fuzzy contraction.

Theorem 3.8. *Let $(X, P_Q, f_n, *)$ be a complete (FnCMS), and $T : X \to X$ be a generalized ψ-fuzzy contractive mapping, continuous, then T has a fixed point.*

Proof. Let s_n be a sequence such that $s_n = Ts_{n-1}$ Suppose that there exists $n_0 \in \mathbb{N}$ such that $s_{n_0} = Ts_{n_0}$. Then s_{n_0} is a fixed point of T and the prove is finished. Hence, we assume that $s_n \neq Ts_n$, we have

$$\begin{aligned}\frac{1}{P_Q(s_n, s_{n+1}, t)} - 1 &= \frac{1}{P_Q(Ts_{n-1}, Ts_n, t)} - 1 \\ &\leq \psi(\frac{1}{P^*(s_{n-1}, s_n, t)} - 1),\end{aligned}$$

where

$$\begin{aligned}P^*(s_{n-1}, s_n, t) = \min\{&P_Q(s_{n-1}, s_n, t), P_Q(s_{n-1}, Ts_{n-1}, t), \\ &P_Q(s_n, Ts_n, t), \frac{2P_Q(s_{n-1}, Ts_n, t)P_Q(s_n, Ts_{n-1}, t)}{P_Q(s_{n-1}, Ts_n, t) + P_Q(s_n, Ts_{n-1}, t)}\}\end{aligned}$$

On simplifying, we have

$$P^*(s_{n-1}, s_n, t) = \min\{P_Q(s_{n-1}, s_n, t), P_Q(s_n, s_{n+1}, t), \frac{2P_Q(s_{n-1}, s_{n+1}, t)}{P_Q(s_{n-1}, s_{n+1}, t) + 1}\}.$$

We have

$$\begin{aligned}\frac{2P_Q(s_{n-1},s_{n+1},t)}{P_Q(s_{n-1},s_{n+1},t)+1} &= \frac{2}{1+\dfrac{1}{P_Q(s_{n-1},s_{n+1},t)}}\\ &\geq \frac{2}{\dfrac{1}{P_Q(s_{n-1},s_n,t)}+\dfrac{1}{P_Q(s_n,s_{n+1},t)}}\\ &\geq \min\{P_Q(s_{n-1},s_n,t),P_Q(s_n,s_{n+1},t)\}.\end{aligned}$$

Then $P^*(s_{n-1},s_n,t)=\min\{P_Q(s_{n-1},s_n,t),P_Q(s_n,s_{n+1},t)\}$.

If $P^*(s_{n-1},s_n,t)=P_Q(s_n,s_{n+1},t)$. then as $\psi(t)<t$ we have

$$\frac{1}{P_Q(s_n,s_{n+1},t)}-1\leq\psi(\frac{1}{P_Q(s_n,s_{n+1},t)}-1)<\frac{1}{P_Q(s_n,s_{n+1},t)}-1,$$

which is a contradiction.

So $P^*(s_{n-1},s_n,t)=P_Q(s_{n-1},s_n,t)$. and

$$\frac{1}{P_Q(s_n,s_{n+1},t)}-1\leq\psi(\frac{1}{P_Q(s_{n-1},s_n,t)}-1)<\frac{1}{P_Q(s_{n-1},s_n,t)}-1,$$

Hence $P_Q(s_n,s_{n+1},t)>P_Q(s_{n-1},s_n,t)$. So, the sequence $\{P_Q(s_n,s_{n+1},t)\}$ is strictly increasing in $[0,1]$, for all $t>0$. Let for all $t>0$, $l(t)=\lim_{n\to\infty}P_Q(s_n,s_{n+1},t)$. We claim that $l(t)=1$, on contrary, assume $l(t_0)<1$, for some $t_0>0$, Taking limit on both sides, we have

$$\frac{1}{l(t_0)}-1\leq\psi(\frac{1}{l(t_0)}-1)<\frac{1}{l(t_0)}-1,$$

a contradiction. Thus, we have

$$\lim_{n\to\infty}P_Q(s_n,s_{n+1},t)=1,\quad t>0.$$

To prove Cauchyness of s_n, consider the cases as:

Case-1. When $p=2q+1$ (odd), then by writing $t=\dfrac{(2q+1)t}{2q+1}=\dfrac{1}{2q+1}+\dfrac{1}{2q+1}+...+\dfrac{1}{2q+1}$.

We have

$$\begin{aligned}P_Q(s_n,s_{n+2q+1},t)\geq P_Q(s_n,s_{n+1},\frac{\frac{t}{2q+1}}{f_1(s_n,s_{n+1})})*P_Q(s_{n+1},s_{n+2},\frac{\frac{t}{2q+1}}{f_2(s_{n+1},s_{n+2})})*....\\ ...*P_Q(s_{n+2q},s_{n+2q+1},\frac{\frac{t}{2q+1}}{f_n(s_{n+2q},s_{n+2q+1})}).\end{aligned}$$

Applying limit $n \to \infty$, we have

$$\lim_{n\to\infty} P_Q(s_n, s_{n+2q+1}, t) \geq 1 * 1 * .... * 1 = 1.$$

Case-2. When $p = 2q$ (even), then by writing $t = \frac{(2q)t}{2q} = \frac{1}{2q} + \frac{1}{2q} + ... + \frac{1}{2q}$. we have

$$P_Q(s_n, s_{n+2q}, t) \geq P_Q(s_n, s_{n+1}, \frac{\frac{t}{2q}}{f_1(s_n, s_{n+1})}) * P_Q(s_{n+1}, s_{n+2}, \frac{\frac{t}{2q}}{f_2(s_{n+1}, s_{n+2})}) * ...$$

$$.... * P_Q(s_{n+2q-1}, s_{n+2q}, \frac{\frac{t}{2q}}{f_n(s_{n+2q-1}, s_{n+2q})}).$$

Applying limit $n \to \infty$, we have

$$\lim_{n\to\infty} P_Q(s_n, s_{n+2q}, t) \geq 1 * 1 * .... * 1 = 1.$$

Hence in either case, $\lim_{n\to\infty} P_Q(s_n, s_{n+p}, t) = 1$,showing Cauchyness of s_n and converges to $s \in X$, so

$$\lim_{n\to\infty} P_Q(s_n, s, t) = 1.$$

Now as T is continuous, we get $Ts_n \to Ts$, for all $t > 0$, that is $s_n \to Ts$, the uniqueness of the limit implies that $Ts = s$. Then s is the fixed point of T. □

Example 3.9. Let

$$X_1 = \{\frac{p}{q} : p = 0, 1, 3, 9, ..., q = 1, 4, ..., 3k+1, ..\}.$$

$$X_2 = \{\frac{p}{q} : p = 1, 3, 9, ..., q = 2, 5, ..., 3k+2, ..\}.$$

and $X = X_1 \cup X_2$. Let $t_1 * t_2 = t_1 t_2$ for all $t_1, t_2 \in [0, 1]$ and $P_Q(s_1, s_2, t) = \frac{t}{t + |s_1 - s_2|^n}$ for all $s_1, s_2 \in X$ and $t > 0$.

Define $T : X \to X$ by

$$Ts = \begin{cases} \frac{3s}{11}, & s \in X_1, \\ \frac{s}{8}, & s \in X_2, \end{cases}$$

If $s_1, s_2 \in X_1$. then

$$\begin{aligned}\frac{1}{P_Q(Ts_1,Ts_2,t)}-1 &= \frac{|\frac{3s_1}{11}-\frac{3s_2}{11}|^n}{t} = \left(\frac{3}{11}\right)^n \frac{|s_1-s_2|^n}{t} = \left(\frac{3}{11}\right)^n \left(\frac{1}{P_Q(s_1,s_2,t)}-1\right)\\ &\leq \left(\frac{3}{11}\right)^n \left(\frac{1}{P^*(s_1,s_2,t)}-1\right).\end{aligned}$$

If $s_1, s_2 \in X_2$. then

$$\begin{aligned}\frac{1}{P_Q(Ts_1,Ts_2,t)}-1 &= \frac{|\frac{s_1}{8}-\frac{s_2}{8}|^n}{t} = \left(\frac{1}{8}\right)^n \frac{|s_1-s_2|^n}{t} = \left(\frac{1}{8}\right)^n \left(\frac{1}{P_Q(s_1,s_2,t)}-1\right)\\ &\leq \left(\frac{1}{8}\right)^n \left(\frac{1}{P^*(s_1,s_2,t)}-1\right).\end{aligned}$$

Now if $s_1 \in X_1$ and $s_2 \in X_2$

$$\frac{1}{P_Q(Ts_1,Ts_2,t)}-1 = \frac{|\frac{3s_1}{11}-\frac{s_2}{8}|^n}{t} = \left(\frac{3}{11}\right)^n \frac{|s_1-\frac{11}{24}s_2|^n}{t}.$$

So, if $s_1 > \frac{11}{24}s_2$, then

$$\begin{aligned}\frac{1}{P_Q(Ts_1,Ts_2,t)}-1 &= \frac{|\frac{3s_1}{11}-\frac{s_2}{8}|^n}{t} = \left(\frac{3}{11}\right)^n \frac{|s_1-\frac{11}{24}s_2|^n}{t} \leq \left(\frac{3}{11}\right)^n \frac{|s_1-\frac{1}{8}s_2|^n}{t}\\ &\leq \left(\frac{6}{11}\right)^n \left[\frac{1}{2}\left(\frac{1}{P_Q(s_1,Ts_2,t)}-1\right)\right]\\ &\leq \left(\frac{6}{11}\right)^n \left[\frac{1}{2}\left(\frac{1}{P_Q(s_1,Ts_2,t)}+1\right)-1\right]\\ &\leq \left(\frac{6}{11}\right)^n \left[\frac{1}{2}\left(\frac{1}{P_Q(s_1,Ts_2,t)}+\frac{1}{P_Q(s_2,Ts_1,t)}\right)-1\right]\\ &= \left(\frac{6}{11}\right)^n \left[\frac{1}{\frac{2P_Q(s_1,Ts_2,t)P_Q(s_2,Ts_1,t)}{P_Q(s_1,Ts_2,t)+P_Q(s_2,Ts_1,t)}}-1\right]\\ &\leq \left(\frac{6}{11}\right)^n \left(\frac{1}{P^*(s_1,s_2,t)}-1\right)\end{aligned}$$

and if $s_1 < \dfrac{11}{24}s_2$, then

$$\begin{aligned}\frac{1}{P_Q(Ts_1,Ts_2,t)}-1 &= \left(\frac{3}{11}\right)^n \frac{|\frac{11}{24}s_2 - s_1|^n}{t} \\ &\leq \left(\frac{3}{11}\right)^n \frac{|s_2 - s_1|^n}{t}\left(\frac{3}{11}\right)^n\left(\frac{1}{P_Q(s_1,s_2,t)}-1\right) \\ &\leq \left(\frac{6}{11}\right)^n\left(\frac{1}{P^*(s_1,s_2,t)}-1\right).\end{aligned}$$

We see that $\dfrac{1}{P_Q(Ts_1,Ts_2,t)}-1 \leq \left(\dfrac{6}{11}\right)^n\left(\dfrac{1}{P^*(s_1,s_2,t)}-1\right)$ for all $s_1,s_2 \in X$. Thus, T is a generalized *psi* fuzzy contractive mapping with $\psi(t) = \left(\dfrac{6}{11}\right)^n t$. Then T has a fixed point, i-e $s = 0$

Theorem 3.10. *Let $(X,P_Q,f_n,*)$ be a complete (FnCMS) with*

$$\lim_{t\to\infty} P_Q(s_1,s_2,t) = 1.$$

And $T : X \to X$ be a self-mapping on X satisfying:

$$P_Q(Ts_1,Ts_2,kt) \geq P_Q(s_1,s_2,t).$$

for all $s_1,s_2 \in X$, then T has a unique fixed point in X.

Proof. Let $s_0 \in X$ and the sequence $Ts_n = T^{n+1}s_0 = s_{n+1}$. After routine steps, we have

$$P_Q(s_n,s_{n+1},t) \geq P_Q(s_0,s_1,\frac{t}{k^n}). \tag{3.2}$$

Consider the sequence s_n in X then:

Case-1 When $p = 2q+1$ (odd), then by writing $t = \dfrac{(2q+1)t}{2q+1} = \dfrac{1}{2q+1} + \dfrac{1}{2q+1} + ... + \dfrac{1}{2q+1}$. we have

$$P_Q(s_n,s_{n+2q+1},t) \geq P_Q(s_n,s_{n+1},\frac{\frac{t}{2q+1}}{f_1(s_n,s_{n+1})}) * P_Q(s_{n+1},s_{n+2},\frac{\frac{t}{2q+1}}{f_2(s_{n+1},s_{n+2})}) * ....$$
$$... * P_Q(s_{n+2q},s_{n+2q+1},\frac{\frac{t}{2q+1}}{f_n(s_{n+2q},s_{n+2q+1})}).$$

Using(..), we have

$$P_Q(s_n,s_{n+2q+1},t) \geq P_Q(s_0,s_1,\frac{\frac{t}{2q+1}}{f_1(s_n,s_{n+1})k^n}) * P_Q(s_0,s_1,\frac{\frac{t}{2q+1}}{f_2(s_{n+1},s_{n+2})k^{n+1}}) * ....$$

$$... * P_Q(s_0,s_1,\frac{\frac{t}{2q+1}}{f_n(s_{n+2q},s_{n+2q+1})k^{n+2q}}).$$

$$\lim_{n\to\infty} P_Q(s_n,s_{n+2q+1},t) \geq 1*1*....*1 = 1.$$

Case-2. When $p=2q$ (even), then by writing $t=\frac{(2q)t}{2q}=\frac{1}{2q}+\frac{1}{2q}+......+\frac{1}{2q}$. we have

$$P_Q(s_n,s_{n+2q},t) \geq P_Q(s_n,s_{n+1},\frac{\frac{t}{2q}}{f_1(s_n,s_{n+1})}) * P_Q(s_{n+1},s_{n+2},\frac{\frac{t}{2q}}{f_2(s_{n+1},s_{n+2})}) * ...$$

$$.... * P_Q(s_{n+2q-1},s_{n+2q},\frac{\frac{t}{2q}}{f_n(s_{n+2q-1},s_{n+2q})}).$$

using (3.10), we have

$$P_Q(s_n,s_{n+2q},t) \geq P_Q(s_0,s_1,\frac{\frac{t}{2q}}{f_1(s_n,s_{n+1})k^n}) * P_Q(s_0,s_1,\frac{\frac{t}{2q}}{f_2(s_{n+1},s_{n+2})k^{n+1}}) * ...$$

$$.... * P_Q(s_0,s_1,\frac{\frac{t}{2q}}{f_n(s_{n+2q-1},s_{n+2q})k^{n+2q-1}}).$$

Applying limit $n\to\infty$, we have

$$\lim_{n\to\infty} P_Q(s_n,s_{n+2q},t) \geq 1*1*....*1 = 1.$$

Hence in either case, $\lim_{n\to\infty} P_Q(s_n,s_{n+p},t)=1$,showing Cauchyness of s_n and converges to $s\in X$, so

$$\lim_{n\to\infty} P_Q(s_n,s,t) = 1.$$

Next to show that s is the fixed point of T. Here again arises two cases:

Case-1 When $n=2q+1$ is odd, then by writing
$t=\frac{(2q+1)t}{2q+1}=\frac{1}{2q+1}+\frac{1}{2q+1}+......+\frac{1}{2q+1}$.

We have

$$P_Q(s,Ts,t) \geq P_Q(s,s_n,\frac{\frac{t}{2q+1}}{f_1(s,s_n)}) * P_Q(s_n,s_{n+1},\frac{\frac{t}{2q+1}}{f_2(s_n,s_{n+1})}) * ....$$

$$... * P_Q(s_{n+1},s_{n+2},\frac{\frac{t}{2q+1}}{f_2(s_{n+1},s_{n+2})}) * ....$$

$$... * P_Q(s_{n+2q+1},Ts,\frac{\frac{t}{2q+1}}{f_n(s_{n+2q+1},Ts)})$$

$$\geq P_Q(s,s_n,\frac{\frac{t}{2q+1}}{f_1(s,s_n)}) * P_Q(Ts_{n-1},Ts_n,\frac{\frac{t}{2q+1}}{f_2(s_n,s_{n+1})}) * ....$$

$$... * P_Q(Ts_n,Ts_{n+1},\frac{\frac{t}{2q+1}}{f_2(s_{n+1},s_{n+2})}) * ....$$

$$... * P_Q(Ts_{n+2q},Ts,\frac{\frac{t}{2q+1}}{f_n(s_{n+2q+1},Ts)})$$

$$\geq P_Q(s,s_n,\frac{\frac{t}{2q+1}}{f_1(s,s_n)}) * P_Q(s_{n-1},s_n,\frac{\frac{t}{2q+1}}{f_2(s_n,s_{n+1})k}) * ....$$

$$... * P_Q(s_{n+2q},s,\frac{\frac{t}{2q+1}}{f_n(s_{n+2q+1},Ts)k})$$

$$\to 1 * 1 * .... * 1 = 1,$$

$$as \quad n \to \infty.$$

Case-2 When $n = 2q$ is odd, then by writing $t = \frac{(2q)t}{2q} = \frac{1}{2q} + \frac{1}{2q} + ... + \frac{1}{2q}$.

We have

$$P_Q(s,Ts,t) \geq P_Q(s,s_n,\frac{\frac{t}{2q}}{f_1(s,s_n)}) * P_Q(s_n,s_{n+1},\frac{\frac{t}{2q}}{f_2(s_n,s_{n+1})}) * ....$$

$$... * P_Q(s_{n+1},s_{n+2},\frac{\frac{t}{2q}}{f_2(s_{n+1},s_{n+2})}) * ....$$

$$... * P_Q(s_{n+2q},Ts,\frac{\frac{t}{2q}}{f_n(s_{n+2q},Ts)})$$

$$\geq P_Q(s,s_n,\frac{\frac{t}{2q}}{f_1(s,s_n)})*P_Q(Ts_{n-1},Ts_n,\frac{\frac{t}{2q}}{f_2(s_n,s_{n+1})})*....$$

$$..*P_Q(Ts_n,Ts_{n+1},\frac{\frac{t}{2q}}{f_2(s_{n+1},s_{n+2})})*....$$

$$...*P_Q(Ts_{n+2q-1},Ts,\frac{\frac{t}{2q}}{f_n(s_{n+2q},Ts)})$$

$$\geq P_Q(s,s_n,\frac{\frac{t}{2q}}{f_1(s,s_n)})*P_Q(s_{n-1},s_n,\frac{\frac{t}{2q}}{f_2(s_n,s_{n+1})k})*....$$

$$...*P_Q(s_{n+2q-1},s,\frac{\frac{t}{2q}}{f_n(s_{n+2q},Ts)k})$$

$$\to 1*1*....*1=1,$$

$$as \quad n\to\infty$$

hence in either case, s is the fixed point of T.

Uniqueness: Assume $Tz=z$ for any other $z\in X$, then

$$P_Q(s,z,t)=P_Q(Ts,Tz,t)\geq P_Q(s,z,\frac{t}{k}),$$

which shows the uniqueness of s. □

Example 3.11. Let $X=[0,1]$, Define a (FnCMS) $(X,P_Q,f_n,*)$ as

$$P_Q(s_1,s_2,t)=exp^{-\frac{(s_1-s_2)^n}{t}}$$

with product t-norm. Further let $T:X\to X$ be defined as $Ts=1-\frac{s}{3}$.

Now

$$P_Q(Ts_1,Ts_2,kt)=exp^{-\frac{(Ts_1-Ts_2)^n}{kt}}$$

$$=exp^{-\frac{(1-\frac{s_1}{3}-1+\frac{s_2}{3})^n}{kt}}$$

$$= exp^{-\dfrac{(s_1-s_2)^n}{3^n kt}}$$

$$\geq exp^{-\dfrac{(s_1-s_2)^n}{t}}$$

$$= P_Q(s_1,s_2,t).$$

By Theorem (3.2), T has a unique fixed point, here $s = \frac{3}{4}$.

4. Application to Fractional Differential Equations

Fractional calculus has brought many significant improvements in scientific research. It deals with the variable derivative that gives more accuracy and helps to make models of mathematical problems. Whereas an ordinary derivative was not so good in this regard because it deals with integer order derivatives.

The main idea of fractional derivatives and integrals is usually associated with Liouville. However, mathematicians had already studied derivatives containing fractional order. Fractional calculus was the subject of Leibnitz's study. Later, Euler also made a contribution to it. Liouville, Reimann, Abel, Litnikov, Hadamard, Weyl, and many other mathematicians from past and present have made significant improvements in the study of fractional calculus and now it is a symbolic topic in mathematics. This section is devoted to prove the uniqueness of the solution of the following fractional differential equation consisting of Caputo fractional derivative

$$D^{\delta}_{0+}\nu(\xi)+g(\xi,\nu(\xi))=0, \quad 0<\xi<1, \tag{4.1}$$

where, $1<\delta<2$, $\xi(0)+\xi'(0)=0$, $\xi(1)+\xi'(1)=0$ are the boundary conditions with $g : [0,1]\times[0,\infty)\to[0,\infty)$ being continuous. Define a complete (FnCMS) $(X,P_Q,f_n,*)$ on $X = C([0,1],\mathbb{R})$ as

$$P_Q(\xi,\mu,t)=\exp(-\frac{\sup_{\nu\in[0,1]}|\xi(\nu)-\mu(\nu)|^n}{t}$$

for all $\nu,\mu\in X$, $t>0$, where $t_1*t_2=t_1t_2$. Note that $\nu\in X$ solves (4.1) whenever $\nu\in X$ is the solution of

$$\nu(\xi)=\frac{1}{\Gamma(\delta)}\int_0^1(1-\zeta)^{\delta-1}(1-\xi)g(\zeta,\nu(\zeta))d\zeta+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-\zeta)^{\delta-2}(1-\xi)g(\zeta,\nu(\zeta))d\zeta$$
$$+\frac{1}{\Gamma(\delta)}\int_0^\xi(\xi-\zeta)^{\delta-1}g(\xi,\nu(\xi))d\xi$$

Theorem 4.1. *Consider the operator $H:X\to X$ as*

$$H\nu(\xi)=\frac{1}{\Gamma(\delta)}\int_0^1(1-\zeta)^{\delta-1}(1-\xi)g(\zeta,\nu(\zeta))d\zeta+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-\zeta)^{\delta-2}(1-\xi)g(\zeta,\nu(\zeta))d\zeta$$
$$+\frac{1}{\Gamma(\delta)}\int_0^\xi(\xi-\zeta)^{\delta-1}g(\xi,\nu(\xi))d\xi$$

suppose the conditions:

(i) *for all $\nu,\mu\in X$, $g:[0,1]\times[0,\infty)\to[0,\infty)$, satisfies*

$$|g(\zeta,\xi(\zeta))-g(\zeta,\mu(\zeta))|\leq k^{\frac{1}{n}}|\xi(\zeta)-\mu(\zeta)|,$$

(ii)

$$\sup_{\xi\in(0,1)}\left|\frac{1-\xi}{\Gamma(\delta+1)}+\frac{1-\xi}{\Gamma(\delta)}+\frac{\xi^\delta}{\Gamma(\delta+1)}\right|^n=\eta<1,$$

holds. Then equation (4.1) has a unique solution.

Proof. Let $\nu,\mu\in X$ and consider

$$\begin{aligned}|H\nu(\xi)-H\mu(\xi)|^n=&\left|\frac{1}{\Gamma(\delta)}\int_0^1(1-\zeta)^{\delta-1}(1-\xi)(g(\zeta,\nu(\zeta))-g(\zeta,\mu(\zeta)))d\zeta\right.\\&+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-\zeta)^{\delta-2}(1-\xi)(g(\zeta,\nu(\zeta))-g(\zeta,\mu(\zeta)))d\zeta\\&\left.+\frac{1}{\Gamma(\delta)}\int_0^\delta(\xi-\zeta)^{\delta-1}(g(\zeta,\nu(\zeta))-g(\zeta,\mu(\zeta)))d\zeta\right|^n\\\leq&\left(\frac{1}{\Gamma(\delta)}\int_0^1(1-\zeta)^{\delta-1}(1-\xi)|g(\zeta,\nu(\zeta))-g(\zeta,\mu(\zeta))|d\zeta\right.\\&+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-\zeta)^{\delta-2}(1-\xi)|g(\zeta,\nu(\zeta))-g(\zeta,\mu(\zeta))|d\zeta\\&\left.+\frac{1}{\Gamma(\delta)}\int_0^\delta(\xi-\zeta)^{\delta-1}|g(\zeta,\nu(\zeta))-g(\zeta,\mu(\zeta))|d\zeta\right)^n\end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{1}{\Gamma(\delta)}\int_0^1 (1-\zeta)^{\delta-1}(1-\xi)k^{\frac{1}{n}}|\nu(\zeta)-\mu(\zeta)|d\zeta \right.\\
&\quad + \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\zeta)^{\delta-2}(1-\xi)k^{\frac{1}{n}}|\nu(\zeta)-\mu(\zeta)|d\zeta\\
&\quad \left. + \frac{1}{\Gamma(\delta)}\int_0^\delta (\xi-\zeta)^{\delta-1}k^{\frac{1}{n}}|\nu(\zeta)-\mu(\zeta)|d\zeta\right)^n\\
&= k|\nu(\xi)-\mu(\xi)|^n\left(\frac{1}{\Gamma(\delta)}\int_0^1 (1-\zeta)^{\delta-1}(1-\xi)d\zeta\right.\\
&\quad \left. + \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\zeta)^{\delta-2}(1-\xi)d\zeta + \frac{1}{\Gamma(\delta)}\int_0^\delta (\xi-\zeta)^{\delta-1}d\zeta\right)^n\\
&= k|\nu(\xi)-\mu(\xi)|^n\left(\frac{1-\xi}{\Gamma(\delta+1)}+\frac{1-\xi}{\Gamma(\delta)}+\frac{\xi^\delta}{\Gamma(\delta+1)}\right)^n\\
&\leq k|\nu(\xi)-\mu(\xi)|^n\sup_{\xi\in(0,1)}\left(\frac{1-\xi}{\Gamma(\delta+1)}+\frac{1-\xi}{\Gamma(\delta)}+\frac{\xi^\delta}{\Gamma(\delta+1)}\right)^n\\
&= \eta k|\nu(\xi)-\mu(\xi)|^n\\
&\leq k|\nu(\xi)-\mu(\xi)|^n
\end{aligned}$$

so, we have

$$|H\nu(\xi)-H\mu(\xi)|^n \leq k|\nu(\xi)-\mu(\xi)|^n,$$

i-e

$$-\frac{\sup_{\xi\in[0,1]}|H\nu(\xi)-H\mu(\xi)|^n}{kt} \geq -\frac{\sup_{\xi\in[0,1]}|\nu(\xi)-\mu(\xi)|^n}{t},$$

$$\exp\left(-\frac{\sup_{\xi\in[0,1]}|H\nu(\xi)-H\mu(\xi)|^n}{kt}\right) \geq \exp\left(-\frac{\sup_{\xi\in[0,1]}|\nu(\xi)-\mu(\xi)|^n}{t}\right),$$

thus, we have

$$P_Q(H\nu(\xi),H\mu(\xi),kt) \geq P_Q(\nu(\xi),\mu(\xi),t),$$

from Theorem 3.2, the equation (4.1) has a unique solution. □

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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