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AN EFFICIENT ITERATIVE METHOD FOR SOLVING QUASIMONOTONE BILEVEL SPLIT VARIATIONAL INEQUALITY PROBLEM

M. S. LUKUMON[1], A. A. MEBAWONDU[1,2,*], A. E. OFEM[1], C. AGBONKHESE[1,3], F. AKUTSAH[2], O. K. NARAIN[2]

[1]Department of Mathematics, University of KwaZulu-Natal, Durban, South Africa

[2]Department of Computer Science and Mathematics, Mountain Top University, Ogun State, Nigeria

[3]Department of Digital and Computational Studies, Bates College, Lewiston, ME, USA

Abstract. In this paper, we introduce and study a modified inertial subgradient extragradient iterative method for solving bilevel split quasimonotone variational inequality problems in the framework of real Hilbert spaces. The method involves strongly monotone operators and quasimonotone operators as the cost operators. In addition, we obtain a strong convergence result of the proposed method under some standard conditions on the control parameters of the method. Our method does not require the prior knowledge of the operator norm or the coefficient of the underlying operator in the space of infinite dimensional real Hilbert spaces. Finally, we provide some numerical experiments to demonstrate the efficiency of our proposed methods in comparison with some existing methods. Our result generalizes and improves some well-known results in literature.

Keywords: split bilevel variational inequality problem; inertial term; iterative method; subgradient; extragradient.
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1. Introduction

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A : H_1 \to H_1$, $B : H_2 \to H_2$ are two operators and $T : H_1 \to H_2$ is a bounded linear

*Corresponding author
E-mail addresses: aamebawondu@mtu.edu.ng, dele@aims.ac.za
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operator. The split variational inequality problem (SVIP) as introduced and studied by Censor et al. in [5, 7] is defined as:

$$\text{Find } x^* \in C \text{ that solves } \langle Ax^*, x - x^* \rangle \geq 0 \ \forall\, x \in C \tag{1}$$

and

$$y^* = Tx^* \in Q \text{ that solves } \langle By^*, y - y^* \rangle \geq 0 \ \forall\, y \in Q. \tag{2}$$

We denote the solution set of the problems (1), (2) and (1)-(2) by $VI(C,A)$, $VI(Q,B)$ and Γ, respectively. The SVIP has wide applications in many fields such as phase retrieval, medical image reconstruction, signal processing, and radiation therapy treatment planning see ([3, 8, 9]) and the references therein. It is easy see that, the SVIP (1) -(2) is a combination of the classical variational inequality problems (VIP) introduced and studied independently by by Stampacchia [26] and Fichera [10, 11] and the well-known split feasibility problem (SFP) introduced and studied by Censor and Elfving in [9]: Find $x^* \in C$ such that

$$y^* = Tx^* \in Q. \tag{3}$$

The notion of SVIP has been studied extensively by many researchers, see [7, 19, 20, 25, 26] and the reference therein for details. In 2017, Anh et al., [1] introduced and studied the notion of a Bilevel Split Variational Inequality Problem (BSVIP) involving a strongly monotone operator and a pseudomonotone operator. The BSVIP is defined as follows:

$$\text{Find } x^* \in \Gamma \text{ such that } \langle Fx^*, x - x^* \rangle \geq 0 \ \forall\, x \in \Gamma, \tag{4}$$

where Γ is the solution set of a SVIP (1)-(2) and $F : H_1 \to H_1$ is a strongly monotone and L-Lipschitz continuous operator on H_1. It is easy to see that (4) is a generalization of SVIP (1)-(2). Over the years one of the interesting techniques for approximating the solution of any bilevel problem is the method of regularization or the use of the penalty function. These methods will only work if the underlying cost operators are monotone, else, the methods will not be applicable. For example, if the underlying operator is pseudomonotone, the regularization or the use of the penalty function will not be applicable, because, the sum of a strongly monotone operator and a pseudomonotone operator is not always certain to be a monotone operator or a pseudomonotone operator. Due to this drawback, researchers have employed the use of extragradient method (EM), subgradient extragradient method (SEGM), the projection contraction methods (PCM) and the Tseng's extragradient methods for solving any form of bilevel problems. In particular, Anh et al., [1] applied the SEGM method to solve BSVIP in the framework of Hilbert spaces. They defined their iterative method as follows.

$$
\begin{cases}
u_n = Tx_n, \\
v_n = P_Q(u_n - \mu_n F_2 u_n), \\
w_n = P_{Q_n}(u_n - \mu_n F_2 v_n), \\
\text{where } Q_n = \{w \in H_2 : \langle u_n - \mu_n F_2 u_n - v_n, w - v_n \rangle \leq 0\} \\
y_n = x_n + \delta_n T^*(w_n - u_n) \\
t_n = P_C(y_n - \lambda_n F_1 y_n) \\
z_n = P_{C_n}(y_n - \lambda F_1 t_n) \\
\text{where } C_n = \{y \in H_1 : \langle y_n - \lambda_n F_1 y_n - t_n, y - t_n \rangle \leq 0\} \\
x_{n+1} = \eta_n x_n + (1 - \eta_n) z_n - \alpha_n \mu F z_n,
\end{cases} \tag{5}
$$

where $F : H_1 \to H_1$ is β-strongly monotone and L-Lipschitz continuous on H_1, $F_1 : H_1 \to H_1$ is pseudomonotone and L_1-Lipschitz continuous on H_1, $F_2 : H_2 \to H_2$ is pseudomonotone and L_2-Lipschitz continuous on H_2, $\delta_n \subset [a,b]$ for some $a,b \in \left(0, \frac{1}{\|T\|+1}\right)$, $\lambda_n \subset [c,d]$ for some $c,d \in \left(0, \frac{1}{L_1}\right)$ and $\mu_n \subset [e,f]$ for some $e,f \in \left(0, \frac{1}{L_2}\right)$. They established that the sequence generated by the iterative method (5) converges strongly to a unique solution of the BSVIP (4). However, we have the following remarks regarding the iterative method (5):

Remark 1.1.
(1) The operators F_2 and F_1 are pseudomonotone and Lipschitz continuous. It is natural to ask if it is possible to weaken the operators F_1 and F_2.
(2) It will be very difficult or impossible to compute the value of δ_n, since, it depends on the value of the operator norm. That is $\delta_n \subset [a,b]$ for some $a,b \in \left(0, \frac{1}{\|T\|+1}\right)$.

(3) Similarly, it will be difficult or impossible to compute the values of λ_n and μ_n since $\lambda_n \subset [c,d]$ for some $c,d \in \left(0, \frac{1}{L_1}\right)$ and $\mu_n \subset [e,f]$ for some $e,f \in \left(0, \frac{1}{L_2}\right)$.

(4) Is it possible to improve the convergence rate of the iterative method.

Furthermore, Huy et al., [16] introduced and studied a modified Tseng's EM for solving BSVIP as follows:

$$
\begin{cases}
u_n = Tx_n, \\
v_n = P_Q(u_n - \mu_n F_2 u_n), \\
w_n = v_n - \mu_n(F_2 v_n - F_2 u_n), \\
y_n = x_n + \delta_n T^*(w_n - u_n) \\
z_n = P_C(y_n - \lambda_n F_1 y_n) \\
t_n = z_n - \lambda_n(F_1 z_n - F_1 y_n) \\
x_{n+1} = \eta_n x_n + (1-\eta_n)t_n - \alpha_n \mu F t_n,
\end{cases} \tag{6}
$$

where

$$
\mu_{n+1} = \begin{cases}
\min\left\{\frac{\|u_n - v_n\|}{\|F_2 u_n - F_2 v_n\|}, \mu_n\right\}, & \text{if } F_2 u_n \neq F_2 v_n \\
\\
\mu_n, & F_2 u_n = F_2 v_n,
\end{cases} \tag{7}
$$

$$
\delta_{n+1} = \begin{cases}
\frac{\|w_n - u_n\|}{2\|T^*(w_n - u_n)\|}, & \text{if } T^*(w_n - u_n) \neq 0 \\
\\
0, & T^*(w_n - u_n) = 0,
\end{cases} \tag{8}
$$

$F : H_1 \to H_1$ is β-strongly monotone and L-Lipschitz continuous on H_1, $F_1 : H_1 \to H_1$ is pseudomonotone and L_1-Lipschitz continuous on H_1, $F_2 : H_2 \to H_2$ is pseudomonotone and L_2-Lipschitz continuous on H_2. They established that the sequence generated by the iterative method (6) converges strongly to a unique solution of the BSVIP (4). It is easy to see that the iterative method (6) provides an affirmative solution to some of the concerns in the iterative method (5). However, the iterative method (6) can still be improved, if the following questions are considered:

Remark 1.2. (1) Is it possible to weaken the underlying operators F_2 and F_1 from pseudomonotone and Lipschitz continuous operators to quasiomonotone and Lipschitz continuous operators?

(2) Is it possible to construct an iterative method that outperforms the above listed methods?

The purpose of this paper is to introduce and study a modified iterative method for solving a BSVIP (4) in infinite dimensional real Hilbert spaces, in which the underlying cost operators are quaismonotone and Lipschitz continuous, and strongly monotone, and Lipschitz continuous. We propose a modified SGEM for solving the BSVIP with the following properties:

(1) In comparison with different iterative techniques for solving BSVIP (4), our proposed iterative method is made up of two different types of step-sizes.
(2) In comparison with different iterative techniques for solving BSVIP (4), the way our $P_C^{\prime}s$ are defined is a modification of what we have in the literature.
(3) In comparison with different iterative techniques for solving BSVIP (4), our proposed iterative method is designed in such a way that the underlying cost operators are quasimonotone, Lipschitz continuous, and sequentially weakly continuous, and strongly monotone and Lipschitz continuous.
(4) Our proposed iterative method does not depend on the knowledge of the bounded linear operator $\|T\|$ unlike the following iterative methods in which knowledge of the bounded linear operator is relevant for their implementation (see [7, 21, 28] and the references therein).
(5) The sequence generated by the proposed methods converges strongly to a unique solution of the BSVIP in real Hilbert spaces.
(6) Our proposed iterative technique includes inertial extrapolation steps. We emphasise that the inertial extrapolation step helps to improve the rate of convergence of an iterative method. The inertial steps remarkably increase the convergence speed of these algorithms when compared with others without extrapolation step (see [1, 16] and the references therein).

To the best of our knowledge, no literature has reported the BSVIP in which the cost operators are quasimonotone, Lipschitz continuous, and sequentially weakly continuous, and strongly monotone and Lipschitz continuous. In addition, our numerical experiments justify that our method is better than other methods in the literature for solving the BSVIP.

The rest of this paper is organized as follows: In Section 2, we recall some useful definitions and results that are relevant for our study. In Section 3, we present our proposed method. In Section 4, we establish strong convergence of our method and in Section 5, we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite dimensional Hilbert spaces. Lastly in Section 6, we give the conclusion of the paper.

2. Preliminaries

In this section, we begin by recalling some known and useful results which are needed in the sequel. Let H be a real Hilbert space. The set of fixed points of a nonlinear mapping $T: H \to H$ will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. We denotes strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. For any $x, y \in H$ and $\alpha \in [0,1]$, it is well-known that

$$\|x-y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2. \tag{9}$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2. \tag{10}$$

$$\|x-y\|^2 \leq \|x\|^2 + 2\langle y, x-y\rangle. \tag{11}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{12}$$

Definition 2.1. Let $T: H \to H$ be an operator. Then T is called

(a) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x-y\|,$$

for all $x, y \in H$. If $L = 1$, then T is called nonexpansive. If $y \in F(T)$, and

$$\|Tx - y\| \leq \|x-y\|,$$

for all $x \in H$. Then T is called quasinonexpansive.

(b) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0,\ \forall x, y \in H;$$

(c) pseudomonotone if

$$\langle Tx, y - x\rangle \geq 0 \Rightarrow \langle Ty, y - x\rangle \geq 0,\ \forall x, y \in H;$$

(d) α- strongly monotone if there exists $\alpha > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2,\ \forall\ x, y \in H;$$

(e) quasimonotone

$$\langle Tx, x - y\rangle > 0 \Rightarrow \langle Ty, x - y\rangle \geq 0\ \forall\ x, y \in H;$$

(f) sequentially weakly continuous if for each sequence $\{x_n\}$, we obtain $\{x_n\}$ converges weakly to x implies that Tx_n converges weakly to Tx.

Remark 2.2. It is well-known that α-strongly monotone $\Rightarrow$ monotone $\Rightarrow$ pseudomonotone $\Rightarrow$ quasimonotone. However, the converses are not generally true.

Let C be a nonempty, closed and convex subset of H. For any $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\|\ \forall y \in C.$$

The operator P_C is called the metric projection of H onto C. It is well-known that P_C is a nonexpansive mapping and that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{13}$$

for all $x, y \in H$. Furthermore, P_C is characterized by the property

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$$

and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{14}$$

for all $x \in H$ and $y \in C$.

Lemma 2.3. [13, 29] *Let C be a nonempty, closed and convex subset of a Hilbert space H and $A : H \to H$ ba L-Lipschitzian and quasimonotone operator. Suppose that $y \in C$ and for some $p \in C$, we have $\langle Ay, p-y\rangle \geq 0$, then at least on of the following hold*

$$\langle Ap, p-y\rangle \geq 0 \text{ or } \langle Ay, q-y\rangle \leq 0$$

for all $q \in C$.

Lemma 2.4. [24] *Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n d_n, n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}\{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then, $\lim_{k\to\infty} a_n = 0$.

Lemma 2.5. [1] *Let C be nonempty closed convex subset of a real Hilbert space H. For any $x \in H$ and $z \in C$, we have $z = P_C x$ if and only if $\langle x-z, y-z\rangle \leq 0 \ \forall\ y \in C$.*

Lemma 2.6. [1] *Let H be a Hilbert space and $F : H \to H$ be a τ-strongly monotone and L-Lipschitz continuous operator on H. Let $\alpha \in (0,1)$ and $\gamma \in (0, \frac{2\tau}{L^2})$. Then for any nonexpansive operator $T : H \to H$, we can associate $T^\gamma : H \to H$ defined by $T^\gamma x = (I - \alpha\gamma F)Tx$ for all $x \in H$. Then, T^γ is a contraction. That is*

$$\|T^\gamma x - T^\gamma y\| \leq (1-\alpha\nu)\|x-y\|$$

for all $x, y \in H$, where $\nu = 1 - \sqrt{1-\gamma(2\tau - \gamma L^2)} \in (0,1)$.

3. PROPOSED ALGORITHM

In this section, we present our proposed method for solving a bilevel split variational inequality problem in which the cost operators are quasimonotone and strongly monotone operators.

Assumption 3.1. Condition A.

(1) The feasible sets C and Q are nonempty set, closed and convex subsets of the real Hilbert spaces H_1 and H_2 respectively.

(2) $A : H_2 \to H_2$ and $B : H_1 \to H_1$ are quasimonotone, sequentially weakly continuous and Lipschitz continuous with Lipschitz constant L_1 and L_2 respectively.

(3) $F : H_1 \to H_1$ is ρ-strongly monotone and L_3- Lipschitz continuous on H_1 such that $\phi = 1 - \sqrt{1 - \omega(2\rho - \omega L_3^2)}$ where $\omega \in (0, \frac{2\rho}{L_3^2})$

(4) $T : H_1 \to H_2$ is a bounded linear operator.

(5) The solution set of problem BSVIP (4) is denoted by $\Omega \neq \emptyset$.

Condition B.

(1) $\alpha_n \subset (0,1)$, $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

(2) $\gamma, \kappa > 0, l, j, \delta_1, \delta_2 \in (0,1), \eta \in (0, \frac{2}{\delta_1}), \beta \in (\frac{\eta}{2}, \frac{1}{\delta_1}), \psi \in (0, \frac{2}{\delta_2}), \alpha \in (\frac{\psi}{2}, \frac{1}{\delta_2})$

We present the following iterative algorithm.

Algorithm 3.2. Initialization Step:

Step 1: Choose $x_0, x_1 \in H_1$, given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \le \theta_n \le \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|\}}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise,} \end{cases} \tag{15}$$

with θ been a positive constant and $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n = \circ(\alpha_n)$.

Step 2. Set

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$y_n = P_Q(Tw_n - \beta\lambda_n ATw_n), \tag{16}$$

where the step size λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda\|ATw_n - Ay_n\| \le \delta_1\|Tw_n - y_n\| \tag{17}$$

$$z_n = P_{\Phi_n}(Tw_n - \eta\tau_n\lambda_n Ay_n) \tag{18}$$

$\Phi_n = \{w \in H_2 : \langle Tw_n - \beta\lambda_n ATw_n - y_n, w - y_n\rangle \le 0\}$ and

$$\tau_n = \frac{(1-\beta\delta_1)\|Tw_n - y_n\|^2}{\|d_n\|^2},\ d_n = Tw_n - y_n - \beta\lambda_n(ATw_n - Ay_n). \tag{19}$$

Step 3. Compute

$$v_n = w_n + \gamma_n T^*(z_n - Tw_n), \tag{20}$$

$$u_n = P_C(v_n - \alpha\mu_n Bv_n), \tag{21}$$

where the step size μ_n is chosen to be the largest $\mu \in \{\kappa, \kappa j, \kappa j^2, \cdots\}$ satisfying

$$\mu\|Bv_n - Bu_n\| \le \delta_2\|v_n - u_n\| \tag{22}$$

and γ_n is chosen such that for small enough $\varepsilon > 0$, $\gamma_n \in \left[\varepsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \varepsilon\right]$ if $Tw_n \neq z_n$, otherwise $\gamma_n = \gamma$.

$$t_n = P_{\Psi_n}(v_n - \psi\nu_n\mu_n Bu_n), \tag{23}$$

where $\Psi_n = \{v \in H_1 : \langle v_n - \alpha\mu_n Bv_n - u_n, v - u_n\rangle \le 0\}$ and

and

$$\nu_n = \frac{(1-\alpha\delta_2)\|v_n - u_n\|^2}{\|b_n\|^2},\ b_n = v_n - u_n - \alpha\mu_n(Bv_n - Bu_n). \tag{24}$$

Step 4. Compute

$$x_{n+1} = t_n - \omega\alpha_n Ft_n \tag{25}$$

---

4. Convergence Analysis

Lemma 4.1. *Suppose the Assumptions 3.1 hold. Then the Armijo-like criteria* (17) *and* (22) *in Algorithm 3.2 are well defined. In addition, the step size γ_n is also well defined.*

Proof. The proof that γ_n is well define follows similar approach as in Lemma 3.1 of [18]. In addition, the proof that the Armijo-like criteria (17) and (22) are well defined follows a similar approach as in and Lemma 3.1 of [28], thus we omit it. □

Lemma 4.2. *Let $\{x_n\}$ be a sequence sequence generated by Algorithm 3.2 under the Assumption 3.1. Then $\{x_n\}$ is bounded.*

Proof: Let $p \in \Omega$, it follows that $Tp \in VI(A,Q) \subset Q$. Since $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, there exist $N_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le N_1$ for all $n \in \mathbb{N}$. Thus, using Algorithm 2.2, we have

$$
\begin{aligned}
\|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\
&\le \|x_n - p\| + \theta_n\|(x_n - x_{n-1})\| \\
&\le \|x_n - p\| + \alpha_n\frac{\theta_n}{\alpha_n}\|(x_n - x_{n-1})\| \\
&\le \|x_n - p\| + \alpha_n N_1. \qquad (26)
\end{aligned}
$$

Also using Algorithm 3.2 and (13)*, we have*

$$
\begin{aligned}
2\|z_n - Tp\|^2 &= 2\|P_{\Phi_n}(Tw_n - \eta\tau_n\lambda_n Ay_n) - P_{\Phi_n}(Tp)\|^2 \\
&\le 2\langle z_n - Tp, Tw_n - \eta\tau_n\lambda_n Ay_n - Tp\rangle \\
&= \|z_n - Tp\|^2 + \|Tw_n - \eta\tau_n\lambda_n Ay_n - Tp\|^2 - \|z_n - Tw_n + \eta\tau_n\lambda_n Ay_n\|^2 \\
&= \|z_n - Tp\|^2 + \|Tw_n - Tp\|^2 + \eta^2{\tau_n}^2{\lambda_n}^2\|Ay_n\|^2 - 2\langle Tw_n - Tp, \eta\tau_n\lambda_n Ay_n\rangle \\
&\quad - \|z_n - Tw_n\|^2 - \eta^2{\tau_n}^2{\lambda_n}^2\|Ay_n\|^2 - 2\langle z_n - Tw_n, \eta\tau_n\lambda_n Ay_n\rangle \\
&= \|z_n - Tp\|^2 + \|Tw_n - Tp\|^2 - \|z_n - Tw_n\|^2 - 2\langle z_n - Tp, \eta\tau_n\lambda_n Ay_n\rangle,
\end{aligned}
$$

which implies

$$
\|z_n - Tp\|^2 \le \|Tw_n - Tp\|^2 - \|z_n - Tw_n\|^2 - 2\langle z_n - Tp, \eta\tau_n\lambda_n Ay_n\rangle. \qquad (27)
$$

Now observe that

$$
\begin{aligned}
\|d_n\| &= \|Tw_n - y_n - \beta\lambda_n(ATw_n - Ay_n)\| \\
&\le \|Tw_n - y_n\| + \beta\lambda_n\|ATw_n - Ay_n\| \\
&\le \|Tw_n - y_n\| + \beta\delta_1\|Tw_n - y_n\| \\
&= (1 + \beta\delta_1)\|Tw_n - y_n\|. \qquad (28)
\end{aligned}
$$

Moreso, since $y_n \in Q$ and $Tp \in VI(Q,A)$, we have $\langle ATp, y_n - Tp\rangle \geq 0$ and using Lemma 2.3, we have $\langle Ay_n, y_n - Tp\rangle \geq 0$. Thus, we have

$$\langle Ay_n, z_n - y_n\rangle \leq \langle Ay_n, z_n - Tp\rangle \implies \langle Ay_n, z_n - Tp\rangle \geq \langle Ay_n, z_n - y_n\rangle. \tag{29}$$

Thus, we have

$$-2\eta\tau_n\lambda_n\langle Ay_n, z_n - Tp\rangle \leq -2\eta\tau_n\lambda_n\langle Ay_n, z_n - y_n\rangle. \tag{30}$$

More so, using the fact that $\langle Tw_n - \beta\lambda_n ATw_n - y_n, w - y_n\rangle \leq 0 \;\forall\; w \in Q \subset H_2$, we have

$$\begin{aligned}
\langle d_n, z_n - y_n\rangle &= \langle Tw_n - y_n - \beta\lambda_n(ATw_n - Ay_n), z_n - y_n\rangle \\
&= \langle Tw_n - y_n - \beta\lambda_n ATw_n, z_n - y_n\rangle + \beta\lambda_n\langle Ay_n, z_n - y_n\rangle \\
&\leq \beta\lambda_n\langle Ay_n, z_n - y_n\rangle. \qquad (31)
\end{aligned}$$

This implies that

$$\begin{aligned}
&\langle d_n, z_n - y_n\rangle \leq \beta\lambda_n\langle Ay_n, z_n - y_n\rangle \\
&-\beta\lambda_n\langle Ay_n, z_n - y_n\rangle \leq -\langle d_n, z_n - y_n\rangle \\
&-2\eta\tau_n\lambda_n\langle Ay_n, z_n - y_n\rangle \leq \frac{-2\eta\tau_n}{\beta}\langle d_n, z_n - y_n\rangle. \qquad (32)
\end{aligned}$$

Using (30) *and* (32)*, we have*

$$\begin{aligned}
-2\eta\tau_n\lambda_n\langle Ay_n, z_n - Tp\rangle &\leq -2\eta\tau_n\lambda_n\langle Ay_n, z_n - y_n\rangle \\
&\leq -\frac{2\eta\tau_n}{\beta}\langle d_n, z_n - y_n\rangle \\
&= -\frac{2\eta\tau_n}{\beta}\langle d_n, z_n - Tw_n + Tw_n - y_n\rangle \\
&= -\frac{2\eta\tau_n}{\beta}\langle d_n, Tw_n - y_n\rangle + \frac{2\eta\tau_n}{\beta}\langle d_n, Tw_n - z_n\rangle \qquad (33)
\end{aligned}$$

In addition, using (19)*, we have*

$$\begin{aligned}
\langle d_n, Tw_n - y_n\rangle &= \langle Tw_n - y_n - \beta\lambda_n(ATw_n - Ay_n), Tw_n - y_n\rangle \\
&\geq \|Tw_n - y_n\|^2 - \beta\lambda_n\|ATw_n - Ay_n\|\|Tw_n - y_n\| \\
&\geq \|Tw_n - y_n\|^2 - \beta\delta_1\|Tw_n - y_n\|^2
\end{aligned}$$

$$= (1-\beta\delta_1)\|\|Tw_n - y_n\|^2$$

$$= \tau_n\|d_n\|^2, \tag{34}$$

we then have that

$$-\frac{2\eta\tau_n}{\beta}\langle d_n, Tw_n - y_n\rangle \leq \frac{-2\eta\tau_n{}^2}{\beta}\|d_n\|^2 \tag{35}$$

and (34) *becomes*

$$-2\eta\tau_n\lambda_n\langle Ay_n, z_n - Tp\rangle \leq \frac{-2\eta\tau_n{}^2}{\beta}\|d_n\|^2 + \frac{2\eta\tau_n}{\beta}\langle d_n, Tw_n - z_n\rangle. \tag{36}$$

Furthermore, using (19)*, we have*

$$\begin{aligned}
&\frac{2\eta\tau_n}{\beta}\langle d_n, Tw_n - z_n\rangle \\
&= 2\langle\frac{\eta\tau_n}{\beta}d_n, Tw_n - z_n\rangle \\
&= \|Tw_n - z_n\|^2 + \frac{\eta^2}{\beta^2}\tau_n^2\|d_n\|^2 - \|Tw_n - z_n - \frac{\eta}{\beta}\tau_n d_n\|^2 \\
&= \|Tw_n - z_n\|^2 + \frac{\eta^2}{\beta^2}(1-\beta\delta_1)^2\frac{\|Tw_n - y_n\|^4}{\|d_n\|^4}\|d_n\|^2 - \|Tw_n - z_n - \frac{\eta}{\beta}\tau_n d_n\|^2 \\
&\leq \|Tw_n - z_n\|^2 + \frac{\eta^2}{\beta^2}(1-\beta\delta_1)^2\|Tw_n - y_n\|^4 . \frac{1}{(1+\beta\delta_1)^2\|Tw_n - y_n\|^2} - \|Tw_n - z_n - \frac{\eta}{\beta}\tau d_n\|^2 \\
&= \|Tw_n - z_n\|^2 + \frac{\eta^2(1-\beta\delta_1)^2}{\beta^2(1+\beta\delta_1)^2}\|Tw_n - y_n\|^2 - \|Tw_n - z_n - \frac{\eta}{\beta}\tau_n d_n\|^2. \qquad (37)
\end{aligned}$$

Thus, (36) *becomes*

$$\begin{aligned}
-2\eta\tau_n\lambda_n\langle Ay_n, z_n - Tp\rangle \leq \frac{-2\eta\tau_n{}^2}{\beta}\|d_n\|^2 + \|Tw_n - z_n\|^2 + \frac{\eta^2(1-\beta\delta_1)^2}{\beta^2(1+\beta\delta_1)^2}\|Tw_n - y_n\|^2 \\
- \|Tw_n - z_n - \frac{\eta}{\beta}\tau_n d_n\|^2. \qquad (38)
\end{aligned}$$

Hence, (27) *becomes*

$$\begin{aligned}
&\|z_n - Tp\|^2 \\
&\leq \|Tw_n - Tp\|^2 - \|z_n - Tw_n\| - \frac{2\eta}{\beta}\tau_n{}^2\|d\|^2 + \|Tw_n - z_n\| + \frac{\eta^2(1-\beta\delta_1)^2}{\beta^2(1+\beta\delta_1)^2}\|Tw_n - y_n\|^2 \\
&- \|Tw_n - z_n - \frac{\eta}{\beta}\tau_n d_n\|^2 \\
&= \|Tw_n - Tp\|^2 - \frac{2\eta\tau_n{}^2}{\beta}\frac{(1-\beta\delta_1)}{\tau_n}\|Tw_n - y_n\|^2 + \frac{\eta^2(1-\beta\delta_1)^2}{\beta^2(1+\beta\delta_1)^2}\|Tw_n - y_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&- \|Tw_n - z_n - \frac{\eta}{\beta}\tau_n d_n\|^2 \\
&= \|Tw_n - Tp\|^2 - \frac{\eta}{\beta^2}\frac{(1-\beta\delta_1)^2}{(1+\beta\delta_1)^2}\|Tw_n - y_n\|^2 - \|Tw_n - z_n - \frac{\eta}{\beta}\tau_n d_n\|^2 \\
&\leq \|Tw_n - Tp\|^2. \qquad (39)
\end{aligned}
$$

Which implies that

$$\|z_n - Tp\| \leq \|Tw_n - Tp\| \tag{40}$$

Furthermore, using Algorithm 3.2, step-size (γ_n) *and* (40)*, we have*

$$
\begin{aligned}
\|v_n - p\|^2 &= \|w_n + \gamma_n T^*(z_n - Tw_n) - p\| \\
&= \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 + 2\gamma_n\langle w_n - p, T^*(z_n - Tw_n)\rangle \\
&= \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 + 2\gamma_n\langle Tw_n - Tp, z_n - Tw_n\rangle \\
&= \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 + \gamma_n\|z_n - Tp\|^2 - \gamma_n\|Tw_n - Tp\|^2 - \gamma_n\|z_n - Tw_n\|^2 \\
&\leq \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 + \gamma_n\|Tw_n - Tp\|^2 - \gamma_n\|Tw_n - Tp\|^2 - \gamma_n\|z_n - Tw_n\|^2 \\
&\leq \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 - \gamma_n(\gamma_n + \varepsilon)\|T^*(z_n - Tw_n)\|^2 \\
&= \|w_n - p\|^2 - \gamma_n\varepsilon\|T^*(z_n - Tw_n)\|^2 \\
&\leq \|w_n - p\|^2, \qquad (41)
\end{aligned}
$$

which implies that

$$\|v_n - p\| \leq \|w_n - p\|. \tag{42}$$

Using a similar approach that we use to obtain the inequality (40)*, we have*

$$\|t_n - p\|^2 \leq \|v_n - p\|^2 - \frac{\psi(1-\alpha\delta_2)^2}{\alpha(1+\alpha\delta_2)^2}\|v_n - u_n\|^2 - \|v_n - t_n - \frac{\psi}{\alpha}\nu_n b_n\|^2. \tag{43}$$

Which implies that

$$\|t_n - p\| \leq \|v_n - p\| \tag{44}$$

In addition, using Algorithm 3.2, Lemma 2.6 and the fact that $\phi = 1 - \sqrt{1 - \omega(2\rho - \omega L_3^2)} \in (0,1)$, *we have*

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|t_n-\alpha_n\omega Ft_n-p\| \\
&= \|(1-\alpha_n\omega F)t_n-(1-\alpha_n\omega F)p-\alpha_n\omega Fp\| \\
&\leq (1-\phi\alpha_n)\|t_n-p\|+\alpha_n\omega\|Gp\| \\
&\leq (1-\phi\alpha_n)\|v_n-p\|+\omega\alpha_n\|Fp\| \\
&\leq (1-\phi\alpha_n)\|w_n-p\|+\omega\alpha_n\|Fp\| \\
&\leq (1-\phi\alpha_n)[\|x_n-p\|+\alpha_nN_1]+\omega\alpha_n\|F_2p\| \\
&\leq (1-\phi\alpha_n)\|x_n-p\|+\phi\alpha_n[\frac{N_1+\omega\|Fp\|}{\phi}] \\
&\leq \max\{\|x_n-p\|,\frac{[N_1+\omega\|Fp\|}{\phi}\} \\
&\vdots \qquad (45) \\
&\leq \max\{\|x_1-p\|,\frac{[N_1+\omega\|Fp\|}{\phi}\}. \qquad (46)
\end{aligned}
$$

Therefore, $\{x_n\}$ is bounded.

Lemma 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1 and suppose that there exists a subsequence $\{x_{n_k}$ of $\{x_n\}$ which converges weakly to $x^*\in H_1$ and* $\lim_{k\to\infty}\|w_{n_x}-v_{n_k}\|=0=\lim_{k\to\infty}\|t_{n_x}-v_{n_k}\|$.*Then, $x^*\in\Gamma$.*

Proof. We suppose that $z_{n_k}\neq Tw_{n_k}$. It is easy to see from (41) that

$$
\begin{aligned}
\|v_{n_k}-p\|^2 &\leq \|w_{n_k}-p\|^2-\gamma_{n_k}\varepsilon\|T^*(z_{n_k}-Tw_{n_k})\|^2 \\
&\leq \|w_{n_k}-p\|^2-\varepsilon^2\|T^*(z_{n_k}-Tw_{n_k})\|, \qquad (47)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\varepsilon^2\|T^*(z_{n_k}-Tw_{n_k})\|^2 \\
&\leq \|w_{n_k}-p\|^2-\|v_{n_k}-p\|^2 \\
&\leq (\|w_{n_k}-v_{n_k}\|+\|v_{n_k}-p\|)^2-\|v_{n_k}-p\|^2
\end{aligned}
$$

$$\leq \|w_{n_k}-v_{n_k}\|^2+2\|w_{n_k}-v_{n_k}\|\|v_{n_k}-p\|+\|v_{n_k}-p\|^2-\|v_{n_k}-p\|^2$$

$$= \|w_{n_k}-v_{n_k}\|^2+2\|w_{n_k}-v_{n_k}\|\|v_{n_k}-p\| \tag{48}$$

using our hypothesis, we have

$$\lim_{k\to\infty}\|T^*(z_{n_k}-Tw_{n_k})\|=0. \tag{49}$$

More so, we have

$$\|v_{n_k}-p\|^2\leq \|w_{n_k}-p\|^2+\gamma_n^2\|T^*(z_{n_k}-Tw_{n_k})\|^2-\gamma_n\|z_{n_k}-Tw_{n_k}\|^2, \tag{50}$$

which implies that

$$\gamma_{n_k}\|z_{n_k}-Tw_{n_k}\|^2\leq \|w_{n_k}-p\|^2-\|v_{n_k}-p\|^2+\gamma_{n_k}^2\|T^*(z_{n_k}-Tw_{n_k})\|^2$$

$$\leq \|w_{n_k}-v_{n_k}\|^2+2\|w_{n_k}-v_{n_k}\|\|v_{n_k}-p\|+\gamma_{n_k}^2\|T^*(z_{n_k}-Tw_{n_k})\|^2, \tag{51}$$

using our hypothesis, we have

$$\lim_{k\to\infty}\|z_{n_k}-Tw_{n_k}\|=0. \tag{52}$$

From (39), we have

$$\|z_{n_k}-Tp\|^2\leq \|Tw_n-Tp\|^2-\frac{\eta}{\beta^2}\frac{(1-\beta\delta_1)^2}{(1+\beta\delta_1)^2}\|Tw_n-y_n\|^2-\|Tw_n-z_n-\frac{\eta}{\beta}\tau_n d_n\|^2. \tag{53}$$

Now, observe that

$$\|z_{n_k}-Tp\|^2=\|z_{n_k}-Tw_{n_k}+Tw_{n_k}-Tp\|^2$$

$$=\|Tw_{n_k}-Tp-(Tw_{n_k}-z_{n_k})\|^2$$

$$=\|Tw_{n_k}-Tp\|^2-2\langle Tw_{n_k}-Tp, Tw_{n_k}-z_n\rangle+\|Tw_{n_k}-z_{n_k}\|^2$$

$$\geq \|Tw_{n_k}-Tp\|^2-2\|T(w_{n_k}-p)\|\|Tw_{n_k}-z_{n_k}\|+\|Tw_n-z_n\|^2$$

$$\geq \|Tw_{n_k}-Tp\|^2-2\|T\|\|w_{n_k}-p\|\|Tw_{n_k}-z_{n_k}\|+\|Tw_{n_k}-z_{n_k}\|^2, \tag{54}$$

this implies that

$$-\|z_{n_k}-Tp\|^2\leq -\|Tw_{n_k}-Tp\|^2+2\|T\|\|w_{n_k}-p\|\|Tw_{n_k}-z_{n_k}\|-\|Tw_{n_k}-z_{n_k}\|^2. \tag{55}$$

Adding (53) and (55), we have

$$\frac{\eta}{\beta^2}\frac{(1-\beta\delta_1)^2}{(1+\beta\delta_1)^2}\|Tw_n - y_n\|^2 + \|Tw_n - z_n - \frac{\eta}{\beta}\tau_n d_n\|^2$$
$$\leq 2\|T\|\|w_{n_k} - p\|\|Tw_{n_k} - z_{n_k}\| - \|Tw_{n_k} - z_{n_k}\|^2, \tag{56}$$

using (52), we have

$$\lim_{k\to\infty}\|Tw_{n_k} - y_{n_k}\| = 0 = \lim_{k\to\infty}\|Tw_n - z_n - \frac{\eta}{\beta}\tau_n d_n\|. \tag{57}$$

Since $y_{n_k} = P_Q(Tw_{n_k} - \beta\lambda_{n_k}ATw_{n_k})$, then from the characteristic of the metric projection, we have

$$\langle Tw_{n_k} - \beta\lambda_{n_k}ATw_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0 \;\forall\, x \in Q, \tag{58}$$

which implies that

$$\langle Tw_{n_k} - y_{n_k}, x - y_{n_k}\rangle - \beta\lambda_{n_k}\langle ATw_{n_k}, x - y_{n_k}\rangle \leq 0, \tag{59}$$

which implies that

$$\langle Tw_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq \beta\lambda_{n_k}\langle ATw_{n_k}, x - y_{n_k}\rangle \tag{60}$$
$$= \beta\lambda_{n_k}\langle ATw_{n_k}, Tw_{n_k} - y_{n_k}\rangle + \beta\lambda_{n_k}\langle ATw_{n_k}, x - Tw_{n_k}\rangle. \tag{61}$$

Since $\lambda_{n_k}, \beta > 0$, we have

$$\frac{1}{\beta\lambda_{n_k}}\langle Tw_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle ATw_{n_k}, y_{n_k} - Tw_{n_k}\rangle \leq \langle ATw_{n_k}, x - Tw_{n_k}\rangle. \tag{62}$$

Using (98), we have

$$0 \leq \liminf_{k\to\infty}\langle ATw_{n_k}, x - Tw_{n_k}\rangle \leq \limsup_{k\to\infty}\langle ATw_{n_k}, x - Tw_{n_k}\rangle. \tag{63}$$

Now, observe that

$$\langle Ay_{n_k}, x - y_{n_k}\rangle = \langle Ay_{n_k}, x - Tw_{n_k}\rangle + \langle Ay_{n_k}, Tw_{n_k} - y_{n_k}\rangle$$
$$= \langle Ay_{n_k} - ATw_{n_k}, x - Tw_{n_k}\rangle + \langle ATw_{n_k}, x - Tw_{n_k}\rangle + \langle Ay_{n_k}, Tw_{n_k} - y_{n_k}\rangle. \tag{64}$$

Since A is Lipschitz continuous on H_2 and (98)

$$\lim_{k\to\infty}\|ATw_{n_k} - Ay_{n_k}\| \leq L_1 \lim_{k\to\infty}\|Tw_{n_k} - y_{n_k}\| = 0. \tag{65}$$

Combining (63), (64) and (65), we have

$$0 \leq \liminf_{k\to\infty} \langle Ay_{n_k}, x - y_{n_k}\rangle \leq \limsup_{k\to\infty} \langle Ay_{n_k}, x - y_{n_k}\rangle. \tag{66}$$

In what follows, we now establish that $Tx^* \in VI(A,Q)$. To start with, we consider the case in which $\limsup_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle > 0$ for all $x \in Q$. Then there exists a subsequence $\{y_{n_{k_m}}\}$ of sequence $\{y_{n_k}\}$ such that $\limsup_{m\to\infty}\langle Ay_{n_{k_m}}, x - y_{n_{k_m}}\rangle > 0$ for all $x \in Q$. It follows that we can find N_0 such that

$$\langle Ay_{n_{k_m}}, x - y_{n_{k_m}}\rangle > 0 \ \forall m > N_0. \tag{67}$$

Since A is quasimonotone, it follows that

$$\langle Ax, x - y_{n_{k_m}}\rangle > 0 \ \forall m > N_0. \tag{68}$$

Now observe that

$$\|w_{n_{k_m}} - x_{n_{k_m}}\| = \alpha_{n_{k_m}} \frac{\theta_{n_{k_m}}}{\alpha_{n_{k_m}}} \|x_{n_{k_m}} - x_{n_{k_m}-1}\| \to 0 \text{ as } m \to \infty. \tag{69}$$

Since, the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is weakly convergent to a point $x^* \in H_1$. Again, since T is a bounded linear operator, we obtain that $\{Tw_{n_k}\}$ converges weakly to Tx^*. Hence, using the fact that $\lim_{n\to\infty} \|Tw_{n_{k_m}} - y_{n_{k_m}}\| = 0$, we have that $\{y_{n_{k_m}}\}$ also converges to Tx^*. Now passing the limit as $m \to \infty$ in (68), we have

$$\lim_{m\to\infty} \langle Ax, x - y_{n_{k_m}}\rangle = \langle Ax, x - Tx^*\rangle > 0. \tag{70}$$

Hence, $Tx^* \in VI(A,Q)$.

Secondly, we consider the case in which $\limsup_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle = 0$ for $x \in Q$. Let $\{\delta_k\}$ be a non-increasing positive sequence defined by

$$\delta_k = |\langle Ay_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}. \tag{71}$$

By our hypothesis, it is easy to see that

$$\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \langle Ay_{n_k}, x - y_{n_k}\rangle + \lim_{k\to\infty} \frac{1}{k+1} = 0. \tag{72}$$

By our hypothesis and (71), we have

$$\langle Ay_{n_k}, x - y_{n_k}\rangle + \delta_k > 0 \tag{73}$$

for each $k \geq 1$, since $\{y_{n_k}\} \subset Q$, it implies that $\{Ay_{n_k}\}$ is strictly non-zero and $\liminf_{k\to\infty} \|Ay_{n_k}\| = N_0 > 0$. We therefore deduce that

$$\|Ay_{n_k}\| > \frac{N_0}{2} \tag{74}$$

In addition, let $\{\varepsilon_{n_k}\}$ be a sequence defined by $\varepsilon_{n_k} = \frac{Ay_{n_k}}{\|Ay_{n_k}\|^2}$. It implies that

$$\langle Ay_{n_k}, \varepsilon_{n_k}\rangle = 1. \tag{75}$$

Combining (73) and (75), we have

$$\langle Ay_{n_k}, x + \delta_k \varepsilon_{n_k} - y_{n_k}\rangle > 0. \tag{76}$$

By quasimonotonicity of the operator A on H_2, we get that

$$\langle A(x + \delta_k \varepsilon_{n_k}), x + \delta_k \varepsilon_{n_k} - y_{n_k}\rangle \geq 0. \tag{77}$$

Now, observe that

(78)

$$\langle Ax, x + \delta_k \varepsilon_{n_k} - y_{n_k}\rangle = \langle Ax - A(x + \delta_k \varepsilon_{n_k}) + A(x + \delta_k \varepsilon_{n_k}), x + \delta_k \varepsilon_{n_k} - y_{n_k}\rangle$$

$$= \langle Ax - A(x + \delta_k \varepsilon_{n_k}), x + \delta_k \varepsilon_{n_k} - y_{n_k}\rangle + \langle A(x + \delta_k \varepsilon_{n_k}), x + \delta_k \varepsilon_{n_k} - y_{n_k}\rangle \tag{79}$$

Combining (77), (78) and applying the well known Cauchy Schwartz inequality, we have

$$\langle Ax, x + \delta_k \varepsilon_{n_k} - y_{n_k}\rangle \geq \langle Ax - A(x + \delta_k \varepsilon_{n_k}), x + \delta_k \varepsilon_{n_k} - y_n\rangle \tag{80}$$

$$\geq -\|Ax - A(x + \delta_k \varepsilon_{n_k})\| \|x + \delta_k \varepsilon_{n_k} - y_{n_k}\|. \tag{81}$$

Since A is Lipschitz continuous, we have

$$\langle Ax, x + \delta_k \varepsilon_{n_k} - y_{n_k}\rangle + L_1 \|\delta_k \varepsilon_{n_k}\| \|x + \delta_k \varepsilon_{n_k} - y_{n_k}\| \geq 0 \tag{82}$$

Combining (74) and (82) and using the definition of $\{\varepsilon_{n_k}\}$, we have

$$\langle Ax, x + \delta_k \varepsilon_{n_k} - y_{n_k}\rangle + \frac{2L_1}{N_0} \delta_k \|x + \delta_k \varepsilon_{n_k} - y_{n_k}\| \geq 0. \tag{83}$$

Since, the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is weakly convergent to a point $x^* \in H_1$. Again, since T is a bounded linear operator, we obtain that $\{Tw_{n_k}\}$ converges Tx^*. Hence, using the fact that $\lim_{n\to\infty} \|Tw_{n_k} - y_{n_k}\| = 0$, we have that $\{y_{n_k}\}$ also converges to Tx^*. Taking limit as $k \to \infty$, since $\delta_k \to 0$, we have

$$\lim_{k\to\infty} \left[\langle Ax, x + \delta_k \varepsilon_{n_k} - y_{n_k} \rangle + \frac{2L_1}{N_0} \delta_k \|x + \delta_k \varepsilon_{n_k} - y_{n_k}\| \right] = \langle Ax, x - Tx^* \rangle > 0. \tag{84}$$

Hence $Tx^* \in VI(A,Q)$.

Using a similar approach, we have $x^* \in VI(B,C)$. Hence, we conclude that $x^* \in \Gamma$.

□

Theorem 4.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. Then, $\{x_n\}$ converges strongly to $p \in \Omega$, which is the unique solution of BSVIP* (4).

Proof. Let $p \in \Omega$. Using Algorithm 3.2, we have

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + 2\theta_n \langle x_n - p, x_n - x_{n-1} \rangle + \theta_n^2 \|x_n - x_{n-1}\|^2 \\
&\le \|x_n - p\|^2 + 2\theta_n \|x_n - x_{n-1}\| \|x_n - p\| + \theta_n^2 \|x_n - x_{n-1}\|^2 \\
&\le \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \theta_n \|x_n - x_{n-1}\|] \\
&= \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\|] \\
&\le \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \alpha_n N_1] \\
&\le \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| N_2. \qquad (85)
\end{aligned}$$

for some $N_2 > 0$.

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|t_n - \omega \alpha_n F t_n - p\|^2 \\
&= \|(1 - \alpha_n \omega F) t_n - (1 - \alpha_n \omega F) p - \alpha_n \omega F p\|^2 \\
&\le \|(1 - \alpha_n \omega F) t_n - (1 - \alpha_n \omega F) p\|^2 - 2\alpha_n \omega \langle Fp, x_{n+1} - p \rangle \\
&\le (1 - \phi \alpha_n)^2 \|t_n - p\|^2 + 2\alpha_n \omega \langle Fp, p - x_{n+1} \rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\phi\alpha_n)^2\|v_n - p\|^2 + 2\alpha_n\omega\langle Fp, p - x_{n+1}\rangle \\
&\leq (1-\phi\alpha_n)\|w_n - p\|^2 + 2\alpha_n\omega\langle Fp, p - x_{n+1}\rangle \\
&\leq (1-\phi\alpha_n)[\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2] + 2\alpha_n\omega\langle Fp, p - x_{n+1}\rangle \\
&= (1-\phi\alpha_n)\|x_n - p\|^2 + \phi\alpha_n\left[\frac{\theta_n}{\phi\alpha_n}\|x_n - x_{n-1}\|N_2 + 2\frac{\omega}{\phi}\langle Fp, p - x_{n+1}\rangle\right] \\
&= (1-\phi\alpha_n)\|x_n - p\|^2 + \phi\alpha_n\Psi_n \qquad (86)
\end{aligned}
$$

where $\Psi_n = \frac{\theta_n}{\phi\alpha_n}\|x_n - x_{n-1}\|N_2 + 2\frac{\omega}{\phi}\langle Fp, p - x_{n+1}\rangle$. According to Lemma 2.4, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty}\Psi_n \leq 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition:

$$\liminf_{k\to\infty}\{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\} \geq 0. \qquad (87)$$

To establish that $\limsup_{k\to\infty}\Psi_{n_k} \leq 0$, we suppose that for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ such that (87) holds. Then, According to Lemma 2.4, to conclude our proof, it is sufficient to establish that $\limsup_{n\to\infty}\Psi_n \leq 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition:

$$\liminf_{k\to\infty}\{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\} \geq 0. \qquad (88)$$

To establish that $\limsup_{k\to\infty}\Psi_{n_k} \leq 0$, we suppose that for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ such that (87) holds. Then,

$$
\begin{aligned}
&\liminf_{k\to\infty}\{\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2\} \\
&= \liminf_{k\to\infty}\{(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|)(\|x_{n_k+1} - p\| + \|x_{n_k} - p\|)\} \geq 0. \qquad (89)
\end{aligned}
$$

From (86) and using (43), (41) that

$$
\begin{aligned}
&\|x_{n_k+1} - p\|^2 \\
&\leq \|t_{n_k} - p\| + 2\alpha_{n_k}\omega\langle Fp, p - x_{n_k+1}\rangle \\
&\leq \|v_{n_k} - p\|^2 - \frac{\psi(1-\alpha\delta_2)^2}{\alpha(1+\alpha\delta_2)^2}\|v_{n_k} - u_{n_k}\|^2 - \|v_{n_k} - t_{n_k} - \frac{\psi}{\alpha}v_{n_k}b_{n_k}\|^2 + 2\alpha_{n_k}\omega\langle Fp, p - x_{n_k+1}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq |w_{n_k}-p\|^2-\gamma_{n_k}\varepsilon\|T^*(z_{n_k}-Tw_{n_k})\|^2-\frac{\psi(1-\alpha\delta_2)^2}{\alpha(1+\alpha\delta_2)^2}\|v_n-u_n\|^2\\
&-\|v_n-t_n-\frac{\psi}{\alpha}\nu_{n_k}b_{n_k}\|^2+2\alpha_{n_k}\omega\langle Fp,p-x_{n+1}\rangle\\
&\leq \|x_{n_k}-p\|^2+\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|N_2-\gamma_{n_k}\varepsilon\|T^*(z_{n_k}-Tw_{n_k})\|^2\\
&-\frac{\psi(1-\alpha\delta_2)^2}{\alpha(1+\alpha\delta_2)^2}\|v_{n_k}-u_{n_k}\|^2-\|v_{n_k}-t_{n_k}-\frac{\psi}{\alpha}\nu_{n_k}b_{n_k}\|^2+2\alpha_{n_k}\omega\langle Fp,p-x_{n_k+1}\rangle,
\end{aligned} \tag{90}
$$

which implies that

$$
\begin{aligned}
&\limsup_{k\to\infty}\left(\gamma_{n_k}\varepsilon\|T^*(z_{n_k}-Tw_{n_k})\|^2+\frac{\psi(1-\alpha\delta_2)^2}{\alpha(1+\alpha\delta_2)^2}\|v_{n_k}-u_{n_k}\|^2+\|v_{n_k}-t_{n_k}-\frac{\psi}{\alpha}\nu_{n_k}b_{n_k}\|^2\right)\\
&\leq \limsup_{k\to\infty}\left[\|x_{n_k}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2+\alpha_{n_k}N_3+2\alpha_{n_k}\omega\langle Fp,p-x_{n+1}\rangle-\|x_{n_k+1}-p\|^2\right]\\
&\leq -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0.
\end{aligned}
$$

Thus, we have

$$
\lim_{k\to\infty}\|T^*(z_{n_k}-Tw_{n_k})\|=\lim_{k\to\infty}\|v_{n_k}-u_{n_k}\|=\lim_{k\to\infty}\|v_{n_k}-t_{n_k}-\frac{\psi}{\alpha}\nu_{n_k}b_{n_k}\|=0. \tag{91}
$$

Thus, using argument as in (90), (91) and Lemma 4.1, we have

$$
\lim_{k\to\infty}\|z_{n_k}-Tw_{n_k}\|=0. \tag{92}
$$

Also, using s similar approach as in (28), we obtain $\|b_n\|\geq(1-\delta_2\alpha)\|u_{n_k}-v_{n_k}\|$. Using, this fact, we obtain

$$
\begin{aligned}
\|v_{n_k}-t_{n_k}\| &= \|v_{n_k}-t_{n_k}-\frac{\psi}{\alpha}\nu_{n_k}b_{n_k}+\frac{\psi}{\alpha}\nu_{n_k}b_{n_k}\|\\
&\leq \|v_{n_k}-t_{n_k}-\frac{\psi}{\alpha}\nu_{n_k}b_{n_k}\|+\frac{\psi}{\alpha}\nu_{n_k}\|b_{n_k}\|\\
&= \|v_{n_k}-t_{n_k}-\frac{\psi}{\alpha}\nu_{n_k}b_{n_k}\|+\frac{\psi}{\alpha}(1-\alpha\delta_2)\frac{\|u_{n_k}-v_{n_k}\|^2}{\|b_{n_k}\|}\\
&\leq \|v_{n_k}-t_{n_k}-\frac{\psi}{\alpha}\nu_{n_k}b_{n_k}\|+\frac{\psi}{\alpha}\|u_{n_k}-v_{n_k}\|.
\end{aligned} \tag{93}
$$

Using (91), we have

$$
\lim_{k\to\infty}\|v_{n_k}-t_{n_k}\|=0. \tag{94}
$$

Using the triangular inequality and (94), we have

$$\lim_{k\to\infty}\|t_{n_k}-u_{n_k}\| \leq \lim_{k\to\infty}\|t_{n_k}-v_{n_k}\| + \lim_{k\to\infty}\|v_{n_k}-u_{n_k}\| = 0 \tag{95}$$

Using similarly approach as in 3.2 and (91), we have

$$\lim_{k\to\infty}\|v_{n_k}-w_{n_k}\| = \lim_{k\to\infty}\|w_{n_k}+\gamma_{n_k}T^*(z_{n_k}-Tw_{n_k})-w_{n_k}\| = \gamma_{n_k}\lim_{k\to\infty}\|T^*(z_{n_k}-Tw_{n_k})\| = 0. \tag{96}$$

From (53) and (55), we have

$$\begin{aligned}
&\frac{\eta}{\beta^2}\frac{(1-\beta\delta_1)^2}{(1+\beta\delta_1)^2}\|Tw_n-y_n\|^2 + \|Tw_n-z_n-\frac{\eta}{\beta}\tau_n d_n\|^2 \\
&\leq 2\|T\|\|w_{n_k}-p\|\|Tw_{n_k}-z_{n_k}\| - \|Tw_{n_k}-z_{n_k}\|^2, \qquad (97)
\end{aligned}$$

using (92), we have

$$\lim_{k\to\infty}\|Tw_{n_k}-y_{n_k}\| = 0 = \lim_{k\to\infty}\|Tw_{n_k}-z_{n_k}-\frac{\eta}{\beta}\tau_{n_k}d_{n_k}\|. \tag{98}$$

Also, using s similar approach as in (28), we obtain $\|d_n\| \geq (1-\delta_1\beta)\|Tw_{n_k}-y_{n_k}\|$. Using, this fact, we obtain

$$\begin{aligned}
\|Tw_{n_k}-z_{n_k}\| &= \|Tw_{n_k}-z_{n_k}-\frac{\eta}{\beta}\tau_{n_k}d_{n_k}+\frac{\eta}{\beta}\tau_{n_k}d_{n_k}\| \\
&\leq \|Tw_{n_k}-z_{n_k}-\frac{\eta}{\beta}\tau_{n_k}d_{n_k}\| + \frac{\eta}{\beta}\tau_{n_k}\|d_{n_k}\| \\
&\leq \|Tw_{n_k}-z_{n_k}-\frac{\eta}{\beta}\tau_{n_k}d_{n_k}\| + \frac{\eta}{\beta}(1-\delta_1\beta)\frac{\|Tw_{n_k}-y_{n_k}\|^2}{\|d_{n_k}\|} \\
&\leq \|Tw_{n_k}-z_{n_k}-\frac{\eta}{\beta}\tau_{n_k}d_{n_k}\| + \frac{\eta}{\beta}\|Tw_{n_k}-y_{n_k}\|. \qquad (99)
\end{aligned}$$

Using, (98), we have

$$\lim_{k\to\infty}\|Tw_{n_k}-z_{n_k}\| = 0. \tag{100}$$

In addition, we have

$$\lim_{k\to\infty}\|y_{n_k}-z_{n_k}\| \leq \lim_{k\to\infty}\|y_{n_k}-Tw_{n_k}\| + \lim_{k\to\infty}\|Tw_{n_k}-z_{n_k}\| = 0 \tag{101}$$

It is easy to see that, as $k\to\infty$, we have

$$\|w_{n_k}-x_{n_k}\| = \theta_{n_k}\|x_{n_k}-x_{n_k-1}\| = \alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\| \to 0. \tag{102}$$

In addition, we have that

$$\|v_{n_k}-x_{n_k}\| \leq \|w_{n_k}-x_{n_k}\|+\gamma_n\|T^*(z_{n_k}-Tw_{n_k})\| \to 0 \text{ as } k\to\infty. \tag{103}$$

$$\|w_{n_k}-v_{n_k}\| \leq \|w_{n_k}-x_{n_k}\|+\|x_{n_k}-v_{n_k}\| \to 0 \text{ as } k\to\infty. \tag{104}$$

$$\|t_{n_k}-x_{n_k}\| \leq \|t_{n_k}-v_{n_k}\|+\|v_{n_k}-x_{n_k}\| \to 0 \text{ as } k\to\infty. \tag{105}$$

$$\|t_{n_k}-w_{n_k}\| \leq \|t_{n_k}-x_{n_k}\|+\|x_{n_k}-w_{n_k}\| \to 0 \text{ as } k\to\infty. \tag{106}$$

$$\|u_{n_k}-x_{n_k}\| \leq \|u_{n_k}-v_{n_k}\|+\|v_{n_k}-x_{n_k}\| \to 0 \text{ as } k\to\infty. \tag{107}$$

$$\|x_{n_k+1}-t_{n_k}\| \leq \|t_{n_k}-\alpha_{n_k}\omega Ft_{n_k}-t_{n_k}\| = \alpha_{n_k}\|\omega Ft_{n_k}\| \ \to 0 \text{ as } k\to\infty. \tag{108}$$

$$\|x_{n_k+1}-x_{n_k}\| \leq \|x_{n_k+1}-t_{n_k}\|+\|t_{n_k}-x_{n_k}\| \ \to 0 \text{ as } k\to\infty. \tag{109}$$

Now, since $\{x_{n_k}\}$ is bounded, then, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to $x^*\in H$. In addition, using (105) and the boundedness of $\{t_{n_k}\}$, there exists there exists a subsequence $\{t_{n_{k_j}}\}$ of $\{t_{n_k}\}$ such that $\{t_{n_{k_j}}\}$ converges weakly to $x^*\in H_1$. Hence, by (94), (107) and Lemma 4.3, we obtain that $x^*\in\Omega$. Furthermore, since $x_{n_{k_j}}$ converges weakly to x^*, we obtain

$$\limsup_{k\to\infty}\langle Fp, p-x_{n_k}\rangle = \lim_{j\to\infty}\langle Fp, p-x_{n_{k_j}}\rangle = \langle Fp, p-x^*\rangle. \tag{110}$$

Hence, since p is a unique solution of Ω, it follows that

$$\limsup_{k\to\infty}\langle Fp, p-x_{n_k}\rangle = \langle Fp, p-x^*\rangle \leq 0, \tag{111}$$

we have obtain from (111) and (109)

$$\limsup_{k\to\infty}\langle Fp, p-x_{n_k+1}\rangle \leq 0. \tag{112}$$

Using our assumption and (111), we have that $\lim_{k\to\infty}\Psi_{n_k} = \lim_{k\to\infty}\Big(\frac{\theta_n}{\phi\alpha_n}\|x_n-x_{n-1}\|N_2+2\frac{\omega}{\phi}\langle Fp, p-x_{n+1}\rangle\Big) \leq 0$. Thus, From Lemma 2.4, we have that $\lim_{n\to\infty}\|x_n-p\|=0$. □

5. NUMERICAL EXAMPLE

In this section, we will give some numerical examples which will show the applicability and the efficiency of our proposed iterative technique in comparison to Algorithm 5, and Algorithm 6.

Example 5.1. Let $H_1 = H_2 = L_2([0,1])$ be equipped with the inner product

$$\langle x,y\rangle = \int_0^1 x(t)y(t)dt \ \forall\, x,y \in L_2([0,1]) \text{ and } \|x\|^2 := \int_0^1 |x(t)|^2 dt \ \forall x,y, \in L_2([0,1]).$$

Let $A;B;F,T : L_2([0,1]) \to L_2([0,1])$ be defined by

$$\begin{aligned} Ax(t) &= e^{\|x\|} \in_0^t x(s)ds, \ \forall\, x \in L_2([0,1]), \\ Fx(t) &= \frac{x(t)}{2}, \ t \in [0,1], \\ Tx(s) &= \int_0^1 K(s,t)x(t)dt \ \forall\, x \in L_2([0,1]), \\ Bx(t) &= f(x)Nx(t) \ \forall\, x \in Q, \end{aligned}$$

where $f : Q \to \mathbb{R}$ is defined as $f(x) = \frac{1}{1+\|x\|}, N : L_2([0,1]) \to L_2([0,1])$ is defined as $Nx(t) = \int_0^t x(s)ds$. Then, A,B are pseudomonotone and Lipschitz continuous but not monotone on $L_2([0,1])$, see [**?**]. It is easy to see that T is a bounded linear operator with $T^* = \int_0^1 K(t,s)x(t)dt \ \forall\, x \in L_2([0,1])$ and F strongly monotone on $L_2([0,1])$ (we use this example due to Remark 2.2). Let C be defined by $C = Q = \{x \in L_2 : \langle a,x\rangle = b\}$ where $a \neq 0$ and $b = 2$. Thus, we have

$$P_C(\bar{x}) = P_Q(\bar{x}) = \max\left\{0, \frac{b - \langle a,\bar{x}\rangle}{\|a\|^2}\right\} a + \bar{x}.$$

We note that F_1 and F_2 of Algorithm 5, and Algorithm 6 are equal to A and B in our Algorithm 3.2. We choose $\alpha_n = \frac{2}{200n+5}, \gamma = 1.1, l = 0.5, \kappa = 1.2, \delta_1 = 0.5, \delta_2 = 0.4, \beta = 0.7, \alpha = 0.6, \eta = 1.4, \psi = 1.3, \gamma_n > 0$ for all $n \in \mathbb{N}$. Also if we consider $\varepsilon = \|x_n - x_{n_1}\| \leq 10^{-5}$ as the stopping criterion and choose the following as starting points:

Case I: $x_0(t) = t^2 + t + 2, \ x_1(t) = t - 2;$

Case II: $x_0(t) = t + e^{2t} + 1, \ x_1(t) = 3t^3 + 3t^2;$

Case III: $x_0(t) = t^4 + e^(3t^2) + 2, \ x_1(t) = \sin(2t);$

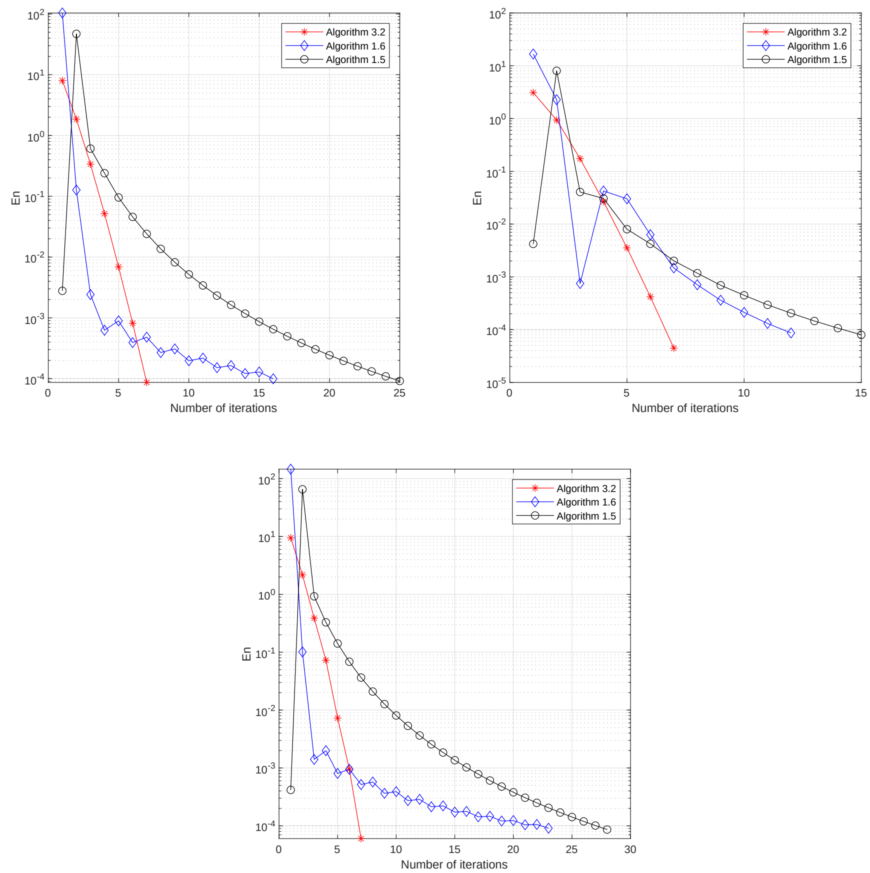

FIGURE 1. Example 5.1, **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom).

Example 5.2. [18, 23] Let $H_1 = H_2 = l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \cdots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $\|x\| = (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}$ for all $x \in l_2(\mathbb{R})$. Suppose the operators $T, A, B : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ are defined by

$$Tx = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),\ x \in l_2(\mathbb{R})$$

$$Ax = (7 - \|x\|)x \ \forall\ x \in l_2(\mathbb{R}),$$

$$Bx = (5 - \|x\|)x \ \forall\ x \in l_2(\mathbb{R}),$$

$$Fx = x - x_0.$$

It is easy to see that T is a bounded linear operator with the adjoint operator $T^*y = (0, y_1, \frac{y_2}{2}, \frac{y_3}{3}, \cdots)$ $y \in l_2(\mathbb{R})$ and A, B are quasimonotone, Lipschitzain continuous and weakly sequentially continuous on $l_2(\mathbb{R})$, and F is strongly monotone as seen in [23]. Let $C = Q =$

$\{x \in l_2(\mathbb{R}) : \|x\| \leq 3\}$. Clearly, C and Q are nonempty, closed and convex subsets of $l_2(\mathbb{R})$. Hence, we have

$$P_C(x) = P_Q(x) = \begin{cases} x & \text{if } \|x\| \leq 3, \\ \frac{3x}{\|x\|}, & \text{if otherwise.} \end{cases} \tag{113}$$

We note that F_1 and F_2 of Algorithm 5, and Algorithm 6 are equal to A and B in our Algorithm 3.2. We choose $\alpha_n = \frac{2}{200n+5}, \gamma = 1.1, l = 0.5, \kappa = 1.2, \delta_1 = 0.5, \delta_2 = 0.4, \beta = 0.7, \alpha = 0.6, \eta = 1.4, \psi = 1.3, \gamma_n > 0$ for all $n \in \mathbb{N}$. Also if we consider $\varepsilon = \|x_n - x_{n_1}\| \leq 10^{-5}$ as the stopping criterion and choose the following as starting points:

Case I: $x_0 = (1, 1, 1, \cdots)$, $x_1 = (0.1, 0.1, 0.1, \cdots)$;

Case II: $x_0 = (1, 2, 3, 4, \cdots)$, $x_1 = (2, 2, 2, \cdots)$;

Case III: $x_0 = (0.3, 0.6, 0.9, \cdots)$, $x_1 = (2, 4, 6, \cdots)$;

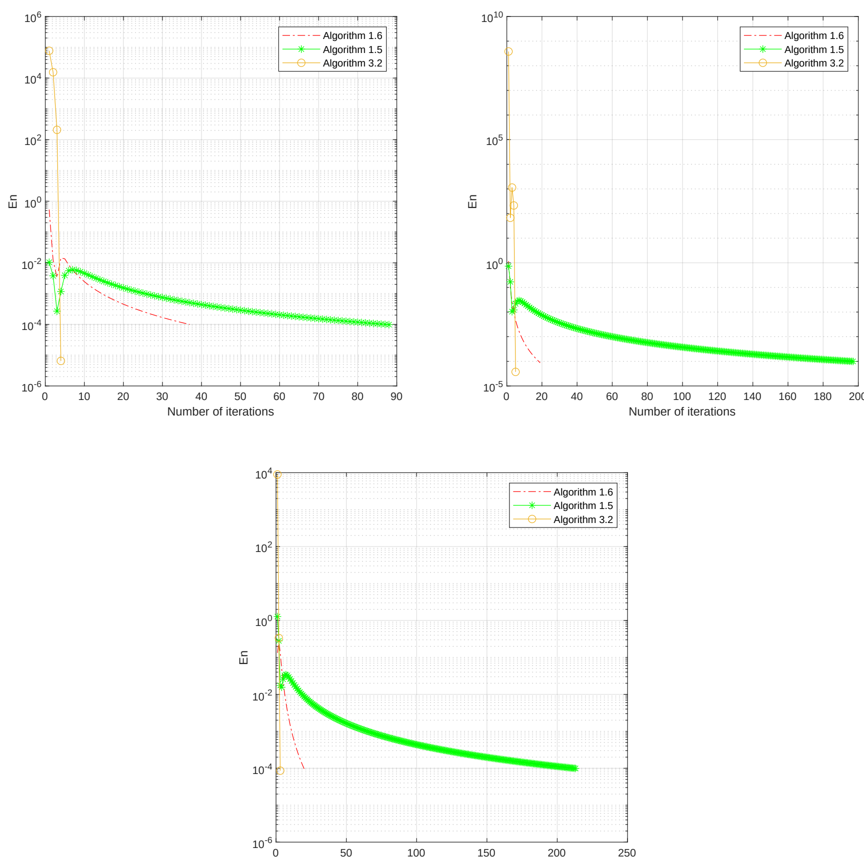

FIGURE 2. Example 5.2, **Case I** (top left); **Case II** (top right); **Case III** (bottom).

6. Conclusion

A modified inertial subgradient extragradient inertial extrapolation iterative method (with two different types of step-sizes) is introduced and studied for solving the BSVIP (4) in infinite dimensional real Hilbert spaces when the cost operators are quasimonotone, and strongly monotone and Lipschitz continuous. In addition, we established that the proposed iterative method converges strongly to the solution set of BSVIP (4). Our method uses stepsizes that are generated at each iteration by some simple computations, which allow them to be easily implemented without the prior knowledge of the operator norm or the coefficient of an underlying operator. In addition, we present some examples and numerical experiment to show the efficiency and implementation of our method in the framework of infinite and finite dimensional Hilbert spaces. We emphasize that one of the novelty of this work is in the use of a weaker operator (quasimonotone), modified inertial term, modification of the PC's introduced, and the method of proof of the strong convergence of our iterative algorithm to the solution of the problem (4).

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Availability of Data and Materials

Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Author Contributions

The authors acknowledge and agree with the content, accuracy and integrity of the manuscript and take absolute accountability for the same. All authors read and approved the final manuscript.

Conflict of Interests

The authors declare that there is no conflict of interests.

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