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BEST PROXIMITY POINT THEOREMS FOR GENERALIZED RATIONAL TYPE CONTRACTION CONDITIONS INVOLVING CONTROL FUNCTIONS ON COMPLEX VALUED METRIC SPACES

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Abstract. Fixed-point theory is being adopted in pure and applied mathematics to a great extent. In scenarios where the fixed point equation lacks a solution, the best approximation theorems and the best proximity pair theorems are used as alternatives. The existence of approximate solutions is guaranteed by the best approximation theorem, but these solutions are not optimal. The best proximity point theorems provide sufficient conditions that guarantee the existence of optimal approximate solutions. In addition, the most effective proximity point theorems serve as generalizations of the fixed point theorems. So we have introduced generalized rational type contraction conditions involving control functions on complex valued metric spaces to prove common best proximity point results for commute proximally non-self mappings under certain assumptions. Many existing results in the literature are extended, generalized, and improvised in the theorems presented in this paper. We have supported our findings with some examples.

Keywords: optimal approximate solution; common best proximity point; complex valued metric space; common fixed points.

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1. INTRODUCTION

One of the most popular fixed point theorems in analysis is Banach's contraction principle [6], which is known for its simplicity. This is because the contractive condition on the mapping is easy to understand and test because all that it needs is only a complete metric space and it has a constructive algorithm. It is used in almost all differential and integral equation theories. Several articles have been published on the improvisation of this principle and that resulted in the generalization. Majority of them were concerned with the metric space generalizations of the contractive condition.

Fixed point equations have no solution, if f is not a self-mapping. In such cases we need to identify an element x that is closer to fx by using the best approximation theorems and best proximity theorems. The best approximation theorem offers adequate criteria to prove that a point $x \in \mathscr{P}$, also known as a best approximate, exists and that $d_{\mathbb{C}}(x, fx) = dit(fx, \mathscr{P})$. For more details refer [13, 19, 21, 24, 26]. This theorem is helpful in finding an approximate solution to $fx = x$, but not a solution that is optimal. But finding an approximate and optimal solution can be achieved through best proximity point theorems which were recently developed by Sadiq Basha and Veeramani [7, 8]. When the mapping $f : \mathscr{P} \to \mathscr{Q}$ is non-self, the best proximity theorem offers adequate conditions that ensure the existence of an element x in $\mathscr{P}$ such that $d_{\mathbb{C}}(x, fx) = dit(\mathscr{P}, \mathscr{Q})$. It also addresses the issue of reducing it to the real-valued function $x \to d_{\mathbb{C}}(x, fx)$ and finding a solution. Also as it turns out to be a fixed point, if the mapping under consideration is a self-mapping, it naturally generalizes fixed point theorems.

Azam et al.[5] have introduced complex valued metric spaces to prove the existence of a common fixed point using rational type contraction condition. This novel concept has helped researchers to overcome the handicap of not being able to define rational expressions in cone metric spaces. See [5, 23, 25, 15]. The equations $fx = x$ and $gx = x$ are likely to have no common solution if $f, g : \mathscr{P} \to \mathscr{Q}$, are two non-self mappings. That is why, for a pair of non-self mappings, the common best proximity point problem has been introduced. The objective is to identify an element $x \in \mathscr{P}$ such that $d_{\mathbb{C}}(x, fx) = dit(\mathscr{P}, \mathscr{Q}) = d_{\mathbb{C}}(x, gx)$. This element is known as the common best proximity point of a pair of non-self mappings (f, g). There exists an extensive literature on common best proximity point. Refer [2, 4, 3, 12, 11, 1, 8, 9, 18, 7]. In this work, we have proved some new common best proximity point theorems for two non-self maps on a complex metric space by introducing the generalized rational type contraction conditions involving control functions. We have also obtained the best proximity point for non-self-maps between two subsets of a complex valued metric space using commute proximally mapping under some assumption. Our theorems on common best proximity points extend certain well-known findings from classical metric spaces to the complex-valued metric spaces.

2. Preliminaries

Let $\mathbb{C}$ be the set of all complex numbers and let $\varkappa_1, \varkappa_2 \in \mathbb{C}$. There exists a partial order relation between $\varkappa_1$ and $\varkappa_2$ iff $\mathfrak{Rea}(\varkappa_1) \leq \mathfrak{Rea}(\varkappa_2)$ and $\mathfrak{Ima}(\varkappa_1) \leq \mathfrak{Ima}(\varkappa_2)$ and we write $\varkappa_1 \preceq \varkappa_2$.

Definition 2.1. [5] *Let $\mathscr{X}$ be a nonempty set. The mapping $d_{\mathbb{C}} : \mathscr{X} \times \mathscr{X} \to \mathbb{C}$ is said to be a complex valued metric if the following conditions hold.*

(a_1) $0 \preceq d_{\mathbb{C}}(\varkappa_1, \varkappa_2), \forall \varkappa_1, \varkappa_2 \in \mathscr{X}$ *and* $d_{\mathbb{C}}(\varkappa_1, \varkappa_2) = 0 \Leftrightarrow \varkappa_1 = \varkappa_2$

(a_2) $d_{\mathbb{C}}(\varkappa_1, \varkappa_2) = d_{\mathbb{C}}(\varkappa_2, \varkappa_1), \forall \varkappa_1, \varkappa_2 \in \mathscr{X}$

(a_3) $d_{\mathbb{C}}(\varkappa_1, \varkappa_2) \preceq d_{\mathbb{C}}(\varkappa_1, \varkappa_3) + d_{\mathbb{C}}(\varkappa_3, \varkappa_2), \forall \varkappa_1, \varkappa_2, \varkappa_3 \in \mathscr{X}$.

Then $(\mathscr{X}, d_{\mathbb{C}})$ is called a complex valued metric space.

Lemma 2.1. [5] *let $\{x_m\}$ be a sequence in $\mathscr{X}$. Then $\{x_m\}$ converges to $x \Leftrightarrow |d_{\mathbb{C}}(x_m, x)| \to 0$ as $m \to \infty$.*

Lemma 2.2. [5] *let $\{x_m\}$ be a sequence in $\mathscr{X}$. Then $\{x_m\}$is a Cauchy sequence $\Leftrightarrow$ $|d_{\mathbb{C}}(x_m, x_{n+m})| \to 0$ as $m \to \infty$.*

According to Choudhury et al. [11], two nonempty subsets $\mathscr{P}$ and $\mathscr{Q}$ of a complex valued metric space $(\mathscr{X}, d_{\mathbb{C}})$, $\{d_{\mathbb{C}}(x,y) : x \in \mathscr{P}, y \in \mathscr{Q}\} \subseteq \mathbb{C}$ is always bounded below by $z_0 = 0 + i0$ and inf $\{d_{\mathbb{C}}(x,y) : x \in \mathscr{P}, y \in \mathscr{Q}\}$ exists. Let

$$dit(\mathscr{P}, \mathscr{Q}) = inf\{d_{\mathbb{C}}(x,y) : x \in \mathscr{P}, y \in \mathscr{Q}\}$$

$$\mathscr{P}_0 = \{x \in \mathscr{P} : d_{\mathbb{C}}(x,y) = dit(\mathscr{P}, \mathscr{Q}) \text{ for some } y \in \mathscr{Q}\}$$

$$\mathscr{Q}_0 = \{y \in \mathscr{Q} : d_{\mathbb{C}}(x,y) = dit(\mathscr{P}, \mathscr{Q}) \text{ for some } x \in \mathscr{P}\}$$

From the definitions above, it can be observed that for every $x \in \mathscr{P}_0$ $\exists y \in \mathscr{Q}_0$ such that $d_{\mathbb{C}}(x,y) = dit(\mathscr{P},\mathscr{Q})$ and conversely, for every $y \in \mathscr{Q}_0$ $\exists x \in \mathscr{P}_0$ such that $d_{\mathbb{C}}(x,y) = dit(\mathscr{P},\mathscr{Q})$.

Definition 2.2. [11] *If an element $x_1 \in \mathscr{P}$ satisfies the condition that*

$$d_{\mathbb{C}}(x_1, fx_1) = d_{\mathbb{C}}(x_1, gx_1) = dit(\mathscr{P},\mathscr{Q}),$$

then it is said to be a common best proximity point of the mappings $f,g : \mathscr{P} \to \mathscr{Q}$.

Definition 2.3. [11]*Let $(f,g) : \mathscr{P} \to \mathscr{Q}$ be non-self mappings.*

- *(a_1) The mappings (f,g) are said to commute proximally if it satisfies the condition $[d_{\mathbb{C}}(z_2, fz_1) = d_{\mathbb{C}}(z_3, gz_1) = d_{\mathbb{C}}(\mathscr{P},\mathscr{Q})] \implies fz_3 = gz_2, \forall z_1, z_2$ and $z_3 \in \mathscr{P}$.*
- *(a_2) The mappings (f,g) can be swapped proximally if $d_{\mathbb{C}}(z_1,z_2) = d_{\mathbb{C}}(z_1,z_3) = d_{\mathbb{C}}(\mathscr{P},\mathscr{Q})$ and $f(z_2) = g(z_3) \implies f(z_3) = g(z_2)$ $\forall z_2, z_3 \in \mathscr{P}$ and $z_1 \in \mathscr{Q}$.*

Definition 2.4. [11] *Let $(\mathscr{P},\mathscr{Q})$ represent a pair of nonempty subsets of a complex valued metric space $(\mathscr{X}, d_{\mathbb{C}})$ where $\mathscr{P}_0 \neq \emptyset$. Then the pair $(\mathscr{P},\mathscr{Q})$ is said to have weak p-property if and only if for any $z_1, z_2 \in \mathscr{P}_0$ and $w_1, w_2 \in \mathscr{Q}_0$, $d_{\mathbb{C}}(z_1, w_1) = dit(\mathscr{P},\mathscr{Q})$ and $d_{\mathbb{C}}(z_2, w_2) = dit(\mathscr{P},\mathscr{Q})$, implies that $d_{\mathbb{C}}(z_1,z_2) \preceq d_{\mathbb{C}}(w_1,w_2)$.*

3. Main Results

In this section, we first define the concept of generalized rational type contraction condition involving control functions in complex valued metric spaces for a pair of non-self mappings.

Definition 3.1. *Let $(\mathscr{X}, d_{\mathbb{C}})$ be a complex valued metric space. The mappings $f,g : \mathscr{P} \to \mathscr{Q}$ are said to be generalized rational type contraction condition involving control functions, if there exists $\alpha(gx_1), \beta(gx_1), \gamma(gx_1)$ for a suitable mapping $\alpha, \beta, \gamma : \mathscr{P} \to [0,1)$, which satisfies $\forall x_1, x_2 \in \mathscr{P}$, $\alpha(fx_1) \leq \alpha(gx_1), \beta(fx_1) \leq \beta(gx_1)$, $\gamma(fx_1) \leq \gamma(gx_1)$ with $\alpha(gx_1) + \beta(gx_1) + \gamma(gx_1) < 1$, $\alpha(fx_1) + \beta(fx_1) + \gamma(fx_1) < 1$ and*

$$d_{\mathbb{C}}(fx_1, fx_2) \preceq \alpha(gx_1) d_{\mathbb{C}}(gx_1, gx_2) + \frac{\beta(gx_1) d_{\mathbb{C}}(gx_1, fx_1) d_{\mathbb{C}}(gx_2, fx_2)}{1 + d_{\mathbb{C}}(gx_1, gx_2)} + \frac{\gamma(gx_1) d_{\mathbb{C}}(gx_2, fx_1) d_{\mathbb{C}}(gx_1, fx_2)}{1 + d_{\mathbb{C}}(gx_1, gx_2)}. \tag{1}$$

Theorem 3.1. *Assume that $\mathscr{P}$ and $\mathscr{Q}$ are non-empty closed subsets of a complete complex valued metric space, where $\mathscr{P}_0$ and $\mathscr{Q}_0$ are non-empty sets, and the pair $(\mathscr{P},\mathscr{Q})$ satisfies the weak p-property. Let $f,g:\mathscr{P}\to\mathscr{Q}$ be two non-self-continuous mappings which satisfies the following conditions.*

- *(a_1) $f(\mathscr{P}_0)\subseteq\mathscr{Q}_0$ and $f(\mathscr{P}_0)\subseteq g(\mathscr{P}_0)$,*
- *(a_2) The mappings (f,g) commute proximally,*
- *(a_3) f and g satisfy generalized rational type contraction condition involving control functions.*

Then $\exists$ an element $x\in\mathscr{P}$ such that $d_{\mathbb{C}}(x,fx)=dit(\mathscr{P},\mathscr{Q})=d_{\mathbb{C}}(x,gx)$.

Proof. Let x_0 be in $\mathscr{P}_0$. By our assumption (a_1), there exists $x_1\in\mathscr{P}_0$ such that $fx_0=gx_1$. Using an inductive approach $\exists\ \{x_m\}\in\mathscr{P}_0$ such that $fx_{m-1}=gx_m$ for every non negative integer m. Since $f(\mathscr{P}_0)\subseteq\mathscr{Q}_0$, we can conclude that there exists k_m in $\mathscr{P}_0$ such that $d_{\mathbb{C}}(fx_m,k_m)=dit(\mathscr{P},\mathscr{Q})$ for every $m\in\mathbb{N}$. It follows that, the selection of k_m and x_m are such that

$$d_{\mathbb{C}}(fx_m,k_m)=dit(\mathscr{P},\mathscr{Q}),\ d_{\mathbb{C}}(fx_{m+1},k_{m+1})=dit(\mathscr{P},\mathscr{Q})$$

for every positive integer m. Due to weak p-property of $(\mathscr{P},\mathscr{Q})$ and condition (a_3), we have

$$\begin{aligned}\alpha(fx_{m-1})&\leq\alpha(gx_{m-1})\\&=\alpha(fx_{m-2})\\&\leq\alpha(gx_{m-2})\\&=\alpha(fx_{m-3})....=\alpha(gx_0)\leq\alpha(fx_0).\end{aligned}$$

Similarly, we can find $\beta(fx_{m-1})\leq\beta(fx_0)$ and $\gamma(fx_{m-1})\leq\gamma(fx_0)$.

$$\begin{aligned}d_{\mathbb{C}}(k_m,k_{m+1})&\preceq d_{\mathbb{C}}(fx_m,fx_{m+1})\\&\preceq\alpha(gx_m)d_{\mathbb{C}}(gx_m,gx_{m+1})+\frac{\beta(gx_m)d_{\mathbb{C}}(gx_m,fx_m)d_{\mathbb{C}}(gx_{m+1},fx_{m+1})}{1+d_{\mathbb{C}}(gx_m,gx_{m+1})}\\&+\frac{\gamma(gx_m)d_{\mathbb{C}}(gx_{m+1},fx_m)d_{\mathbb{C}}(gx_m,fx_{m+1})}{1+d_{\mathbb{C}}(gx_m,gx_{m+1})}\end{aligned}$$

$$
\begin{aligned}
&= \alpha(fx_{m-1})d_{\mathbb{C}}(fx_{m-1},fx_m) + \frac{\beta(fx_{m-1})d_{\mathbb{C}}(fx_{m-1},fx_m)d_{\mathbb{C}}(fx_m,fx_{m+1})}{1+d_{\mathbb{C}}(fx_{m-1},fx_m)} \\
&\quad + \frac{\gamma(fx_{m-1})d_{\mathbb{C}}(fx_m,fx_m)d_{\mathbb{C}}(fx_{m-1},fx_{m+1})}{1+d_{\mathbb{C}}(fx_{m-1},fx_m)} \\
d_{\mathbb{C}}(k_m,k_{m+1}) &\preceq \alpha(fx_{m-1})d_{\mathbb{C}}(fx_{m-1},fx_m) + \beta(fx_{m-1})d_{\mathbb{C}}(fx_m,fx_{m+1}) \\
&\vdots \\
&\preceq \frac{\alpha(fx_0)}{1-\beta(fx_0)} d_{\mathbb{C}}(fx_{m-1},fx_m) \preceq c\, d_{\mathbb{C}}(fx_{m-1},fx_m).
\end{aligned}
$$

It follows that $\{k_m\}$ is Cauchy and it converges to k in $\mathscr{P}$, because $\mathscr{X}$ is complete and $\mathscr{P}$ is closed. Also, using $f(\mathscr{P}_0) \subseteq \mathscr{Q}_0$, we can conclude that there exists k_m in $\mathscr{P}$ such that $d_{\mathbb{C}}(fx_m,k_m) = dit(\mathscr{P},\mathscr{Q})$ for every $m \in \mathbb{N}$. As a result of the selection of x_m, we find

$$d_{\mathbb{C}}(gx_m,k_{m-1}) = d_{\mathbb{C}}(fx_{m-1},k_{m-1}) = dit(\mathscr{P},\mathscr{Q})$$

for every positive integer m. Since (f,g) commute proximally, we can say that $gk_m = fk_{m-1}$. By the continuity of mappings, it implies that $gk = fk$. By choosing $f(k) \in f(\mathscr{P}_0) \subseteq \mathscr{Q}_0$, we find that there exists $x \in \mathscr{P}_0$ such that

$$d_{\mathbb{C}}(x,gk) = dit(\mathscr{P},\mathscr{Q}) = d_{\mathbb{C}}(x,fk).$$

By (a_2), we have $fx = gx$. For $f(x) \in f(\mathscr{P}_0) \subseteq \mathscr{Q}_0$, there exists $\ell \in \mathscr{P}_0$ such that

$$d_{\mathbb{C}}(\ell,gx) = dit(\mathscr{P},\mathscr{Q}) = d_{\mathbb{C}}(\ell,fx).$$

From (1), we have

$$
\begin{aligned}
d_{\mathbb{C}}(fk,fx) &\preceq \alpha(gk)d_{\mathbb{C}}(gk,gx) + \frac{\beta(gk)d_{\mathbb{C}}(gk,fk)d_{\mathbb{C}}(gx,fx)}{1+d_{\mathbb{C}}(gk,Tx)} + \frac{\gamma(gk)d_{\mathbb{C}}(gx,fk)d_{\mathbb{C}}(gk,fx)}{1+d_{\mathbb{C}}(gk,Tx)} \\
&\preceq \alpha(fk)d_{\mathbb{C}}(fk,fx) + \frac{\beta(fk)d_{\mathbb{C}}(fk,fk)d_{\mathbb{C}}(fx,fx)}{1+d_{\mathbb{C}}(fk,fx)} + \frac{\gamma(fk)d_{\mathbb{C}}(fx,fk)d_{\mathbb{C}}(fk,fx)}{1+d_{\mathbb{C}}(fk,fx)} \\
&= (\alpha(fk)+\gamma(fk))d_{\mathbb{C}}(fk,fx).
\end{aligned}
$$

$\implies fk = fx$ and also $gk = gx$. Therefore, we have, $d_{\mathbb{C}}(x,fk) = dit(\mathscr{P},\mathscr{Q}) = d_{\mathbb{C}}(\ell,fx)$. Using the weak p-property of $(\mathscr{P},\mathscr{Q})$ we have,

$$d_{\mathbb{C}}(x,\ell) \preceq d_{\mathbb{C}}(fx,fk) = 0$$

which means that $x=\ell$. Thus $d_{\mathbb{C}}(x,fx)=dit(\mathscr{P},\mathscr{Q})=d_{\mathbb{C}}(x,gx)$. Let x_1 be another common best proximity point of f and g such that $d_{\mathbb{C}}(x_1,fx_1)=dit(\mathscr{P},\mathscr{Q})=d_{\mathbb{C}}(x_1,gx_1)$. By (a_2), we have $fx=gx$ and $fx_1=gx_1$. Using (1),

$$\begin{aligned}
d_{\mathbb{C}}(fx,fx_1) &\preceq \alpha(gx)d_{\mathbb{C}}(gx,gx_1)+\frac{\beta(gx)d_{\mathbb{C}}(gx,fx)d_{\mathbb{C}}(gx_1,fx_1)}{1+d_{\mathbb{C}}(gx,gx_1)} \\
&\quad+\frac{\gamma(gx)d_{\mathbb{C}}(gx_1,fx)d_{\mathbb{C}}(gx,fx_1)}{1+d_{\mathbb{C}}(gx,gx_1)}=\alpha(fx)d_{\mathbb{C}}(fx,fx_1) \\
&\quad+\frac{\beta(fx)d_{\mathbb{C}}(fx,fx)d_{\mathbb{C}}(fx_1,fx_1)}{1+d_{\mathbb{C}}(fx,fx_1)}+\frac{\gamma(fx)d_{\mathbb{C}}(fx_1,fx)d_{\mathbb{C}}(fx,fx_1)}{1+d_{\mathbb{C}}(fx,fx_1)} \\
&\preceq (\alpha(fx)+\gamma(fx))d_{\mathbb{C}}(fx,fx_1),
\end{aligned}$$

$\implies fx=fx_1$. Therefore, we have $d_{\mathbb{C}}(x,fx)=dit(\mathscr{P},\mathscr{Q})=d_{\mathbb{C}}(x_1,fx_1)$. The weak p-property of $(\mathscr{P},\mathscr{Q})$, leads us to the conclusion that $d_{\mathbb{C}}(x,x_1)\preceq d_{\mathbb{C}}(fx,fx_1)=0$ which in turn implies that $x=x_1$. Hence, $\exists$ an element $x\in\mathscr{P}$ such that $d_{\mathbb{C}}(x,fx)=dit(\mathscr{P},\mathscr{Q})$ and $d_{\mathbb{C}}(x,gx)=dit(\mathscr{P},\mathscr{Q})$. □

Example 1. *Let* $\mathscr{X}=\mathbb{C}$ *and* $d:\mathscr{X}\times\mathscr{X}\to\mathbb{C}$ *be defined as* $d_{\mathbb{C}}(x_1+iy_1,x_2+iy_2)=|x_1-x_2|+i|y_1-y_2|$. *Consider*

$$\mathscr{P}=\{z\in\mathbb{C}:\mathfrak{Rea}(z)=1,\mathfrak{Ima}(z)=y,\ \forall y\in[0,1]\}\ \textit{and}$$
$$\mathscr{Q}=\{z\in\mathbb{C}:\mathfrak{Rea}(z)=0,\mathfrak{Ima}(z)=y\ ,\forall y\in[0,1]\}.$$

Let $(f,g):\mathscr{P}\to\mathscr{Q}$ *be defined as* $f(z)=(1-x)+i\frac{2y}{5}$ *and* $g(z)=\frac{5(1-x)}{2}+iy$ *and* $\alpha,\beta,\gamma:\mathscr{P}\to[0,1)$ *are defined as* $\alpha(z)=\frac{2\mathfrak{Rea}(z)+2}{5}$, $\beta(z)=0$ *and* $\gamma(z)=0$. *Clearly, we observe the following.*

- (a_1) $\alpha(fz)\le\alpha(gz),\beta(fz)\le\beta(gz)$ *and* $\gamma(fz)\le\gamma(gz)$ *with* $\alpha(gz)+\beta(gz)+\gamma(gz)<1$ *and* $\alpha(fz)+\beta(fz)+\gamma(fz)<1$.
- (a_2) *If* $d_{\mathbb{C}}(z_1,f(z_3))=d_{\mathbb{C}}(z_2,g(z_3))=dit(\mathscr{P},\mathscr{Q})=1$ *for some* z_1,z_2 *and* $z_3\in\mathscr{P}$, *then we obtain that* $\mathfrak{Rea}(z_1)=\mathfrak{Rea}(z_2)=\mathfrak{Rea}(z_3)=1,\mathfrak{Ima}(z_1)=\frac{2}{5}\mathfrak{Ima}(z_3),\mathfrak{Ima}(z_2)=\mathfrak{Ima}(z_3)$ *and* $f(z_2)=g(z_1)$. *Therefore the pair (f, g) commute proximally.*
- (a_3) *Clearly the pair* $(\mathscr{P},\mathscr{Q})$ *has weak p-property.*
- (a_4) *Since* $dit(\mathscr{P},\mathscr{Q})=1+0i$, $\mathscr{P}_0=\{z\in\mathbb{C}:\mathfrak{Rea}(z)=1,\mathfrak{Ima}(z)=y\}$ *and* $\mathscr{Q}_0=\{z\in\mathbb{C}:\mathfrak{Rea}(z)=0,\mathfrak{Ima}(z)=y\}$, *using (1) , we have the following.*

$$\begin{aligned} d_{\mathbb{C}}(fz_1, fz_2) &= |x_1 - x_2| + i\frac{2}{5}|y_1 - y_2| \\ &= \frac{2}{5}\Big(\frac{5}{2}|x_1 - x_2| + i|y_1 - y_2|\Big) \\ &\preceq \frac{7 - 5\mathfrak{Rea}(z)}{5}\Big(d_{\mathbb{C}}(gz_1, gz_2)\Big), \forall z_1, z_2. \end{aligned}$$

Therefore, it is easy to check all the other conditions of Theorem 3.1. Hence z = 1+i0 is a common best proximity point of (f, g).

Theorem 3.2. *Assume that $\mathscr{P}$ and $\mathscr{Q}$ are non-empty closed subsets of a complete complex valued metric space, where $\mathscr{P}_0$ and $\mathscr{Q}_0$ are non-empty sets, and the pair $(\mathscr{P}, \mathscr{Q})$ satisfies the weak p-property. Let $f, g : \mathscr{P} \to \mathscr{Q}$ be two nonself-continuous mappings that satisfy following.*

(a_1) There is a non negative real numbers α, β, γ with $\alpha + \beta + \gamma < 1$ such that

$$d_{\mathbb{C}}(fx_1, fx_2) \preceq \alpha d_{\mathbb{C}}(gx_1, gx_2) + \frac{\beta d_{\mathbb{C}}(gx_1, fx_1) d_{\mathbb{C}}(gx_2, fx_2)}{1 + d_{\mathbb{C}}(gx_1, gx_2)} + \frac{\gamma d_{\mathbb{C}}(gx_2, fx_1) d_{\mathbb{C}}(gx_1, fx_2)}{1 + d_{\mathbb{C}}(gx_1, gx_2)}, \ \forall x_1, x_2 \in \mathscr{P}. \tag{2}$$

(a_2) The mappings (f, g) commute proximally.

(a_3) $f(\mathscr{P}_0) \subseteq \mathscr{Q}_0$ and $f(\mathscr{P}_0) \subseteq g(\mathscr{P}_0)$.

Then $\exists\, x \in \mathscr{P}$ such that $d_{\mathbb{C}}(x, fx) = dit(\mathscr{P}, \mathscr{Q})$ and $d_{\mathbb{C}}(x, gx) = dit(\mathscr{P}, \mathscr{Q})$.

Example 2. *Let $\mathscr{X} = \mathbb{C}$ and let $d : \mathscr{X} \times \mathscr{X} \to \mathbb{C}$ be given by*

$$d_{\mathbb{C}}(x_1 + iy_1, x_2 + iy_2) = |x_1 - x_2| + i|y_1 - y_2|.$$

Let $\mathscr{P} = \{z \in \mathbb{C} : \mathfrak{Rea}(z) \geq 1, \mathfrak{Ima}(z) = y\}$ and $\mathscr{Q} = \{z \in \mathbb{C} : \mathfrak{Rea}(z) \leq 0, \mathfrak{Ima}(z) = y\}$. Let $(f, g) : \mathscr{P} \to \mathscr{Q}$ be defined by $f(z) = \frac{1-x}{7} + 2iy$ and $g(z) = \frac{1-x}{2} + 7iy$. Usual computation reveals the following.

(a_1) *If $d_{\mathbb{C}}(z_1, f(z_3)) = d_{\mathbb{C}}(z_2, g(z_3)) = dit(\mathscr{P}, \mathscr{Q}) = 1$ for some $z_1, z_2, z_3 \in \mathscr{P}$, then we can deduce that $\mathfrak{Rea}(z_1) = \mathfrak{Rea}(z_2) = \mathfrak{Rea}(z_3) = 1, \mathfrak{Ima}(z_1) = 2\mathfrak{Ima}(z_3), \mathfrak{Ima}(z_2) = 7\mathfrak{Ima}(z_3)$ and $f(z_2) = g(z_1)$. Thus (f, g) commute proximally.*

(a_2) *Clearly the pair $(\mathscr{P}, \mathscr{Q})$ has weak p-property.*

(a_3) *Since* $dit(\mathscr{P},\mathscr{Q}) = 1+0i$, $\mathscr{P}_0 = \{z \in \mathbb{C} : \mathfrak{Rea}(z) = 1, \mathfrak{Ima}(z) = y\}$ *and* $\mathscr{Q}_0 = \{z \in \mathbb{C} : \mathfrak{Rea}(z) = 0, \mathfrak{Ima}(z) = y\}$. *By (2)* $\forall\, z_1, z_2$, *we have*

$$\begin{aligned} d_{\mathbb{C}}(fz_1, fz_2) &= \frac{1}{7}|x_1 - x_2| + 2i|y_1 - y_2| \\ &= \frac{2}{7}\Big(\frac{1}{2}|x_1 - x_2| + 7i|y_1 - y_2|\Big) \\ &= \frac{2}{7}\Big(d_{\mathbb{C}}(gz_1, gz_2)\Big) \end{aligned}$$

All the other conditions of Theorem 3.2 can be easily checked with $\alpha = \frac{2}{7}$ *and* $\beta, \gamma \in [0,1)$. *Hence z = 1 + i 0 is a common best proximity point of (f, g).*

Following Theorem becomes an extension, generalization and complement of the findings of Jungck [16] and others in complex valued metric space, if we replace (f,g) by commuting self mappings in Theorem 3.2.

Theorem 3.3. *Assume that* $f, g : \mathscr{P} \to \mathscr{P}$ *to be two continuous mappings on complete complex valued metric space satisfying the following conditions.*

(a_1) $\exists \alpha, \beta, \gamma \in \mathbb{R}^+ \cup \{0\}$ *with* $\alpha + \beta + \gamma < 1$ *such that*

$$d_{\mathbb{C}}(fx_1, fx_2) \precsim \alpha d_{\mathbb{C}}(gx_1, gx_2) + \frac{\beta d_{\mathbb{C}}(gx_1, fx_1) d_{\mathbb{C}}(gx_2, fx_2)}{1 + d_{\mathbb{C}}(gx_1, gx_2)} + \frac{\gamma d_{\mathbb{C}}(gx_2, fx_1) d_{\mathbb{C}}(gx_1, fx_2)}{1 + d_{\mathbb{C}}(gx_1, gx_2)}, \ \forall x_1, x_2 \in \mathscr{P}. \tag{3}$$

(a_2) The mappings (f, g) are commutes.

(a_3) $f(\mathscr{P}) \subseteq g(\mathscr{P})$.

Then f and g have a unique common fixed point.

We define another generalized rational type contraction condition involving control functions.

Definition 3.2. *Let* $(\mathscr{X}, d_{\mathbb{C}})$ *be a complex valued metric space. The mappings* $f, g : \mathscr{P} \to \mathscr{Q}$ *are said to satisfy* $\star$*-generalized rational type contraction condition involving control functions, if there exists* $\alpha(gx_1), \beta(gx_1), \gamma(gx_1), \lambda(gx_1), \mu(gx_1)$ *for suitable mappings* $\alpha, \beta, \gamma, \lambda, \mu : \mathscr{P} \to [0,1)$, *that satisfy*

$$\alpha(fx_1) \leq \alpha(gx_1),\ \beta(fx_1) \leq \beta(gx_1),\ \gamma(fx_1) \leq \gamma(gx_1) \text{ and } \lambda(fx_1) \leq \lambda(gx_1),\ \mu(fx_1) \leq \mu(gx_1)$$
$$\text{with } \alpha(fx_1) + 2\beta(fx_1) + 2\gamma(fx_1) + \lambda(fx_1) + \mu(fx_1) < 1 \text{ as}$$
$$\alpha(gx_1) + 2\beta(gx_1) + 2\gamma(gx_1) + \lambda(gx_1) + \mu(gx_1) < 1$$

and

$$\begin{aligned} d_{\mathbb{C}}(fx_1, fx_2) \preceq\ & \alpha(gx_1) d_{\mathbb{C}}(gx_1, gx_2) + \beta(gx_1)[d_{\mathbb{C}}(gx_1, fx_1) + d_{\mathbb{C}}(gx_2, fx_2)] \\ & + \gamma(gx_1)[d_{\mathbb{C}}(gx_1, fx_2) + d_{\mathbb{C}}(gx_2, fx_1)] + \lambda(gx_1) \frac{(1 + d_{\mathbb{C}}(gx_1, fx_1)) d_{\mathbb{C}}(gx_2, fx_2)}{1 + d_{\mathbb{C}}(gx_1, gx_2)} \\ & + \mu(gx_1) \frac{(1 + d_{\mathbb{C}}(gx_1, gx_2)) d_{\mathbb{C}}(gx_1, fx_1)}{1 + d_{\mathbb{C}}(gx_1, gx_2) + d_{\mathbb{C}}(gx_2, fx_1)},\ \forall x_1, x_2 \in \mathscr{P}. \end{aligned} \tag{4}$$

Theorem 3.4. *Assume that $\mathscr{P}$ and $\mathscr{Q}$ are non-empty closed subsets of a complete complex valued metric space $(\mathscr{X}, d_{\mathbb{C}})$, where $\mathscr{P}_0$ and $\mathscr{Q}_0$ are non-empty sets, and the pair $(\mathscr{P}, \mathscr{Q})$ satisfies the weak p-property. Let $f, g : \mathscr{P} \to \mathscr{Q}$ be two non-self-continuous mappings which satisfy the following conditions.*

- *(a_1) $f(\mathscr{P}_0) \subseteq \mathscr{Q}_0$ and $f(\mathscr{P}_0) \subseteq g(\mathscr{P}_0)$,*
- *(a_2) The mappings (f, g) commute proximally,*
- *(a_3) f and g satisfy $\star$-generalized rational type contraction condition involving control functions.*

Then $\exists$ an element $x \in \mathscr{P}$ such that $d_{\mathbb{C}}(x, fx) = dit(\mathscr{P}, \mathscr{Q})$ and $d_{\mathbb{C}}(x, gx) = dit(\mathscr{P}, \mathscr{Q})$.

Proof. Let $x_0 \in \mathscr{P}_0$. By (a_1), $\exists\, x_1 \in \mathscr{P}_0 \ni fx_0 = gx_1$. Using an inductive approach $\exists\, \{x_m\} \in \mathscr{P}_0$ $\ni fx_{m-1} = gx_m\ \forall m \in \mathscr{Z}^+$. As $f(\mathscr{P}_0) \subseteq \mathscr{Q}_0$, we can conclude that

$$\exists\, k_m \in \mathscr{P}_0 \ni d_{\mathbb{C}}(fx_m, k_m) = dit(\mathscr{P}, \mathscr{Q}), \forall m \in \mathbb{N}.$$

It follows that, the selection of k_m and x_m are such that

$$d_{\mathbb{C}}(fx_m, k_m) = dit(\mathscr{P}, \mathscr{Q}),\ d_{\mathbb{C}}(fx_{m+1}, k_{m+1}) = dit(\mathscr{P}, \mathscr{Q}), \forall m \in \mathrm{Z}^+.$$

Since $(\mathscr{P}, \mathscr{Q})$ satisfies weak p-property and the condition (a_3), we have

$$\begin{aligned} \alpha(fx_{m-1}) &\leq \alpha(gx_{m-1}) = \alpha(fx_{m-2}) \\ &\leq \alpha(gx_{m-2}) \\ &= \alpha(fx_{m-3}) .... = \alpha(gx_0) \leq \alpha(fx_0). \end{aligned}$$

Similarly, we have,

$$\beta(fx_{m-1}) \leq \beta(fx_0), \gamma(fx_{m-1}) \leq \gamma(fx_0), \lambda(fx_{m-1}) \leq \lambda(fx_0) \text{ and } \mu(fx_{m-1}) \leq \mu(fx_0).$$

$$\begin{aligned}
d_{\mathbb{C}}(k_m, k_{m+1}) &\preceq d_{\mathbb{C}}(fx_m, fx_{m+1}) \\
&\preceq \alpha(fx_m) d_{\mathbb{C}}(gx_m, gx_{m+1}) + \beta(fx_m)[d_{\mathbb{C}}(gx_m, fx_m) + d_{\mathbb{C}}(gx_{m+1}, fx_{m+1})] \\
&+ \gamma(fx_m)[d_{\mathbb{C}}(gx_m, fx_{m+1}) + d_{\mathbb{C}}(gx_{m+1}, fx_m)] \\
&+ \lambda(fx_m) \frac{(1 + d_{\mathbb{C}}(gx_m, fx_m)) d_{\mathbb{C}}(gx_{m+1}, fx_{m+1})}{1 + d_{\mathbb{C}}(gx_m, gx_{m+1})} \\
&+ \mu(fx_m) \frac{(1 + d_{\mathbb{C}}(gx_m, gx_{m+1})) d_{\mathbb{C}}(gx_m, fx_m)}{1 + d_{\mathbb{C}}(gx_m, gx_{m+1}) + d_{\mathbb{C}}(gx_{m+1}, fx_m)} \\
&= \alpha(fx_{m-1}) d_{\mathbb{C}}(fx_{m-1}, fx_m) + \beta(fx_{m-1})[d_{\mathbb{C}}(fx_{m-1}, fx_m) + d_{\mathbb{C}}(fx_m, fx_{m+1})] \\
&+ \gamma(fx_{m-1})[d_{\mathbb{C}}(fx_{m-1}, fx_{m+1}) + d_{\mathbb{C}}(fx_m, fx_m)] \\
&+ \lambda(fx_{m-1}) \frac{(1 + d_{\mathbb{C}}(fx_{m-1}, fx_m)) d_{\mathbb{C}}(fx_m, fx_{m+1})}{1 + d_{\mathbb{C}}(fx_{m-1}, fx_m)} \\
&+ \mu(fx_{m-1}) \frac{(1 + d_{\mathbb{C}}(fx_{m-1}, fx_m)) d_{\mathbb{C}}(fx_{m-1}, fx_m)}{1 + d_{\mathbb{C}}(fx_{m-1}, fx_m) + d_{\mathbb{C}}(fx_m, fx_m)} \\
&\vdots \\
&\preceq \Big(\frac{\alpha(fx_0) + \beta(fx_0) + \gamma(fx_0) + \mu(fx_0)}{1 - (\beta(fx_0) + \gamma(fx_0) + \lambda(fx_0))} \Big) d_{\mathbb{C}}(fx_{m-1}, fx_m) \\
&= c \, d_{\mathbb{C}}(fx_{m-1}, fx_m),
\end{aligned}$$

where $c = \dfrac{\alpha(fx_0) + \beta(fx_0) + \gamma(fx_0) + \mu(fx_0)}{1 - (\beta(fx_0) + \gamma(fx_0) + \lambda(fx_0))}$. Given that $\mathscr{X}$ is complete and $\mathscr{P}$ is closed, it follows that k_m is Cauchy and converges to some k in $\mathscr{P}$. Also using $f(\mathscr{P}_0) \subseteq \mathscr{Q}_0$, we can conclude that $\exists\, k_m \in \mathscr{P} \ni d_{\mathbb{C}}(fx_m, k_m) = dit(\mathscr{P}, \mathscr{Q})\ \forall, m \in \mathbb{Z}^+$. The selection of x_m results in

$$d_{\mathbb{C}}(gx_m, k_{m-1}) = d_{\mathbb{C}}(fx_{m-1}, k_{m-1}) = dit(\mathscr{P}, \mathscr{Q})\ \forall, m \in \mathbb{Z}^+.$$

Since (f, g) commute proximally, we can say that $gk_m = fk_{m-1}$. By the continuity of mappings, it implies that $gk = fk$. Using $f(k) \in f(\mathscr{P}_0) \subseteq \mathscr{Q}_0$, $\exists\, x \in \mathscr{P}_0$ such that $d_{\mathbb{C}}(x, gk) = dit(\mathscr{P}, \mathscr{Q}) = d_{\mathbb{C}}(x, fk)$. By (a_2), we have $fx = gx$. Again, using $f(x) \in f(\mathscr{P}_0) \subseteq \mathscr{Q}_0$, $\exists\, \ell \in \mathscr{P}_0 \ni d_{\mathbb{C}}(\ell, gx) = dit(\mathscr{P}, \mathscr{Q}) = d_{\mathbb{C}}(\ell, fx)$.

From (4), we have

$$\begin{aligned}
d_{\mathbb{C}}(fk,fx) &\preceq \alpha(gk)d_{\mathbb{C}}(gk,gx)+\beta(gk)[d_{\mathbb{C}}(gk,fk)+d_{\mathbb{C}}(gx,fx)] \\
&+\gamma(gk)[d_{\mathbb{C}}(gk,fx)+d_{\mathbb{C}}(gx,fk)]+\lambda(gk)\frac{(1+d_{\mathbb{C}}(gk,fk))d_{\mathbb{C}}(gx,fx)}{1+d_{\mathbb{C}}(gk,gx)} \\
&+\mu(gk)\frac{(1+d_{\mathbb{C}}(gk,gx))d_{\mathbb{C}}(gk,fk)}{1+d_{\mathbb{C}}(gk,gx)+d_{\mathbb{C}}(gx,fk)} \\
&\preceq \alpha(fk)d_{\mathbb{C}}(fk,fx)+\beta(fk)[d_{\mathbb{C}}(fk,fk)+d_{\mathbb{C}}(fx,fx)] \\
&+\gamma(fk)[d_{\mathbb{C}}(fk,fx)+d_{\mathbb{C}}(fx,fk)]+\lambda(fk)\frac{(1+d_{\mathbb{C}}(fk,fk))d_{\mathbb{C}}(fx,fx)}{1+d_{\mathbb{C}}(fk,fx)} \\
&+\mu(fk)\frac{(1+d_{\mathbb{C}}(fk,fx))d_{\mathbb{C}}(fk,fk)}{1+d_{\mathbb{C}}(fk,fx)+d_{\mathbb{C}}(fx,fk)} \preceq (\alpha(fk)+2\gamma(fk))d_{\mathbb{C}}(fk,fx).
\end{aligned}$$

$\implies fk=fx$. Therefore, we have, $d_{\mathbb{C}}(x,fk)=dit(\mathscr{P},\mathscr{Q})=d_{\mathbb{C}}(\ell,fx)$. The pair $(\mathscr{P},\mathscr{Q})$ has weak p-property. So

$$d_{\mathbb{C}}(x,\ell)\preceq d_{\mathbb{C}}(fx,fk)=0 \implies x=\ell.$$

Thus $d_{\mathbb{C}}(x,fx)=dit(\mathscr{P},\mathscr{Q})=d_{\mathbb{C}}(x,gx)$. Assume that x_1 is another common best proximity point of f and $g \ni d_{\mathbb{C}}(x_1,fx_1)=dit(\mathscr{P},\mathscr{Q})=d_{\mathbb{C}}(x_1,gx_1)$. By (a_2), we have $fx=gx$ and $fx_1=gx_1$.

$$\begin{aligned}
d_{\mathbb{C}}(fx,fx_1) &\preceq \alpha(gx)d_{\mathbb{C}}(gx,gx_1)+\beta(gx)[d_{\mathbb{C}}(gx,fx)+d_{\mathbb{C}}(gx_1,fx_1)] \\
&+\gamma(gx)[d_{\mathbb{C}}(gx,fx_1)+d_{\mathbb{C}}(gx_1,fx)]+\lambda(gx)\frac{(1+d_{\mathbb{C}}(gx,fx))d_{\mathbb{C}}(gx_1,fx_1)}{1+d_{\mathbb{C}}(gx,gx_1)} \\
&+\mu(gx)\frac{(1+d_{\mathbb{C}}(gx,gx_1))d_{\mathbb{C}}(gx,fx)}{1+d_{\mathbb{C}}(gx,gx_1)+d_{\mathbb{C}}(gx_1,fx)} \\
&\preceq (\alpha(fx)+2\beta(fx))d_{\mathbb{C}}(fx,fx_1),
\end{aligned}$$

$\implies fx=fx_1$. Therefore, we have $d_{\mathbb{C}}(x,fx)=dit(\mathscr{P},\mathscr{Q})=d_{\mathbb{C}}(x_1,fx_1)$. Weak p-property of the pair $(\mathscr{P},\mathscr{Q})$ implies $d_{\mathbb{C}}(x,x_1)\preceq d_{\mathbb{C}}(fx,fx_1)=0$ and so $x=x_1$. Hence, $\exists$ an element $x\in\mathscr{P}$ such that $d_{\mathbb{C}}(x,fx)=dit(\mathscr{P},\mathscr{Q})$ and $d_{\mathbb{C}}(x,gx)=dit(\mathscr{P},\mathscr{Q})$. □

Theorem 3.5. *Assume that $\mathscr{P}$ and $\mathscr{Q}$ are non-empty closed subsets of a complete complex valued metric space $(\mathscr{X},d_{\mathbb{C}})$, where $\mathscr{P}_0$ and $\mathscr{Q}_0$ are non-empty sets, and the pair $(\mathscr{P},\mathscr{Q})$*

satisfies the weak p-property. Let $f,g:\mathscr{P}\to\mathscr{Q}$ be two non self continuous mappings that satisfy the following.

(a_1) There is $\alpha,\beta,\gamma,\lambda,\mu,\in\mathbb{R}^+\cup\{0\}$, $\alpha+2\beta+2\gamma+\lambda+\mu<1\ni$

$$\begin{aligned} d_{\mathbb{C}}(fx_1,fx_2) &\preceq \alpha d_{\mathbb{C}}(gx_1,gx_2)+\beta[d_{\mathbb{C}}(gx_1,fx_1)+d_{\mathbb{C}}(gx_2,fx_2)] \\ &\quad +\gamma[d_{\mathbb{C}}(gx_1,fx_2)+d_{\mathbb{C}}(gx_2,fx_1)]+\lambda\frac{(1+d_{\mathbb{C}}(gx_1,fx_1))d_{\mathbb{C}}(gx_2,fx_2)}{1+d_{\mathbb{C}}(gx_1,gx_2)} \\ &\quad +\mu\frac{(1+d_{\mathbb{C}}(gx_1,gx_2))d_{\mathbb{C}}(gx_1,fx_1)}{1+d_{\mathbb{C}}(gx_1,gx_2)+d_{\mathbb{C}}(gx_2,fx_1)},\ \forall x_1,x_2\in\mathscr{P}, \end{aligned} \tag{5}$$

(a_2) The mappings (f, g) are commute proximally,

(a_3) $f(\mathscr{P}_0)\subseteq\mathscr{Q}_0$ with $f(\mathscr{P}_0)\subseteq g(\mathscr{P}_0)$.

Then f and g have a common best proximity point.

Example 3. *Let $d:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ be defined as $d_{\mathbb{C}}(x_1+iy_1,x_2+iy_2)=|x_1-x_2|+i|y_1-y_2|$. Let $\mathscr{P}=\{z\in\mathbb{C}:\mathfrak{Rea}(z)\geq 1,\mathfrak{Ima}(z)=y\}$ and $\mathscr{Q}=\{z\in\mathbb{C}:\mathfrak{Rea}(z)\leq 0,\mathfrak{Ima}(z)=y\}$. Let $(f,g):\mathscr{P}\to\mathscr{Q}$ be two nonself maps which are defined by $f(z)=\frac{1-x}{5}+2iy$ and $g(z)=\frac{1-x}{2}+5iy$. As in Example 2, we can check for commute proximality, and condition (5) with $\alpha=\frac{2}{5},0\leq\beta,\gamma<\frac{1}{10},0\leq\lambda,\mu<\frac{1}{5}$. Thus it has a common best proximity point.*

Theorem 3.5 implies the subsequent theorem, which generalizes and completes the findings of Hardy [14], Jungck[16], Reich[20], Reich[22], Chatterjea[10], Kannan[17], Aghayan et al. [1] and others in complex valued metric spaces if we assume that f and g to be self-maps.

Theorem 3.6. *Assume $f,g:\mathscr{P}\to\mathscr{P}$ are two continuous mappings on complete complex valued metric space $(\mathscr{X},d_{\mathbb{C}})$ satisfying the following conditions.*

(a_1) There is $\alpha,\beta,\gamma,\lambda,\mu,\delta\in\mathbb{R}^+\cup\{0\}$ with $\alpha+2\beta+2\gamma+\lambda+\mu+\delta<1\ni$

$$\begin{aligned} d_{\mathbb{C}}(fx_1,fx_2) &\preceq \alpha d_{\mathbb{C}}(gx_1,gx_2)+\beta[d_{\mathbb{C}}(gx_1,fx_1)+d_{\mathbb{C}}(gx_2,fx_2)] \\ &\quad +\gamma[d_{\mathbb{C}}(gx_1,fx_2)+d_{\mathbb{C}}(gx_2,fx_1)]+\lambda\frac{(1+d_{\mathbb{C}}(gx_1,fx_1))d_{\mathbb{C}}(gx_2,fx_2)}{1+d_{\mathbb{C}}(gx_1,gx_2)} \\ &\quad +\mu\frac{(1+d_{\mathbb{C}}(gx_1,gx_2))d_{\mathbb{C}}(gx_1,fx_1)}{1+d_{\mathbb{C}}(gx_1,gx_2)+d_{\mathbb{C}}(gx_2,fx_1)},\ \forall x_1,x_2\in\mathscr{P}, \end{aligned} \tag{6}$$

(a_2) The mappings (f, g) are commutes,

(a_3) $f(\mathscr{P}) \subseteq g(\mathscr{P})$.

Then f and g have a unique common fixed point.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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