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SOME FIXED POINT THEOREMS FOR TRICYCLIC CONTRACTIONS IN DISLOCATED QUASI-B-METRIC SPACES

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Abstract. This paper centers on introducing dislocated quasi-b-metric spaces and proposing the concepts of Geraghty-type dqb-tricyclic Banach contraction and dqb-tricyclic-Kannan mapping. These innovations lead to successfully proving the existence of fixed-point theorems within these spaces. In doing so, the main theorem presented in this work extends and solidifies previous discoveries from recent literature into a unified framework.

Keywords: fixed point; dqb-tricyclic; Banach contraction convex; tricyclic contraction; Kannan mapping; b-metric spaces.

2020 AMS Subject Classification: 54H25, 47H09, 47H10.

1. Introduction

Fixed point theory is a branch of mathematics that deals with the study of fixed points of functions, particularly in the context of transformations and mappings. It provides a framework for understanding the existence, uniqueness, and properties of fixed points for various types of functions and spaces.

Fixed point theory has applications in various fields, including:

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Dynamical Systems: In the study of dynamical systems, fixed points often represent equilibrium points where the system remains unchanged over time. Understanding the stability and behavior of these fixed points is crucial for predicting the long-term behavior of the system.

Functional Analysis: Fixed point theory is closely related to functional analysis, a branch of mathematics that deals with vector spaces of functions. Many theorems and concepts from functional analysis are used to study fixed points in more general settings.

Topology: Topological methods are often employed in fixed point theory to establish the existence of fixed points. The Brouwer fixed point theorem and the Kakutani fixed point theorem are classic examples of results in this area.

Economics and Game Theory: Fixed points have applications in economics and game theory, where they can represent equilibrium points in models of economic interactions and strategic games.

Optimization: Optimization problems often involve finding points where a certain function remains unchanged. Fixed point theory can provide insights into the existence and properties of solutions to optimization problems.

The Banach Fixed Point Theorem, also known as the Banach Contraction Principle or the Contraction Mapping Theorem, was indeed introduced in 1922 by the Polish mathematician Stefan Banach [2] . This theorem is a fundamental result in fixed point theory and functional analysis. Here is the statement of the Banach Fixed Point Theorem:

Banach Fixed Point Theorem: Let (E,d) be a complete metric space, and $T : E \to E$ be a contraction mapping, meaning there exists a constant $0 \leq k < 1$ such that for all $x, y \in E$, $d(Tx,Ty) \leq kd(x,y)$. Then, T has a unique fixed point in E

The concept of a Kannan mapping was introduced by K. Kannan in 1969 [3] .. The Kannan Fixed Point Theorem is a generalization of the Banach Contraction Mapping Principle and provides conditions for the existence of fixed points for certain types of mappings in metric spaces.

Theorem 1. *Let (E,d) be a complete metric spaces and $T : E \to E$ be a Kannan contraction mapping, i.e.*

$$d(Tx,Ty) \leq k[d(x,Tx)+(y,Ty)]$$

for all $x, y \in E$, where $k \in \left[0, \frac{1}{2}\right)$ .Then T has a unique fixed point.

In 2003, In their paper, Kirk et al. [1] introduced the concept of cyclical contractive mappings and extended the Banach fixed point result to the class of cyclic mappings. They generalized the notion of contractive mappings to cyclical contractive mappings and proved that these mappings also have a unique fixed point.

Definition 2. *Let (E,d) be a metric space and let X and Y be two nonempty subsets of E.*

A mapping $T : X \cup Y \to X \cup Y$ is said to be a cyclic mapping provided that

$$T(X) \subseteq Y,\ T(Y) \subseteq X. \tag{1}$$

A point $x \in X \cup Y$ is called a best proximity point if $d(x,Tx) = d(X,Y)$ where $d(X,Y) = \inf(d(x,y) : x \in X, y \in Y)$ .In 2011, Erdal Karapınar and Inci M. Erhan [4] proved.

The following fixed point theorem for a cyclic map.

Definition 3. *Let A and B be non-empty subsets of a metric space (E,d). A cyclic map $T : X \cup Y \to X \cup Y$ is said to be a Kannan Type cyclic contraction if there exists $k \in \left(0, \frac{1}{2}\right)$ such that:*

$$d(Tx,Ty) \leq k\left[d(Tx,x) + d(Ty,y)\right],\ \forall x \in X, \forall y \in Y. \tag{2}$$

This expanded the class of mappings for which a fixed point can be guaranteed, and provided a new tool for studyingthe existence and uniqueness of fixed points in various mathematical contexts.

In 2017, Sabar, Bassou, and their colleagues[5] introduced the concept of ”tricyclic contractions” and established a result related to them. The term ”tricyclic mappings” likely refers to a certain type of mapping or transformation that they studied. The concept of tricyclic contractions might involve properties similar to the well-known contraction mappings but adapted to a specific context or structure.

The mention of ”a similar result” suggests that the authors found an analogous theorem or fixed point result, possibly akin to the Banach Fixed Point Theorem or another well-known fixed point theorem, but specialized for tricyclic contractions[5, 15].

Let (E,d) be a metric space and let X,Y and Z be nonempty subsets of E.

A mapping $T : X \cup Y \cup Z \to X \cup Y \cup Z$ is said to be a tricyclic mapping provided that

$$T(X) \subseteq Y,\ T(Y) \subseteq Z \text{ and } T(Z) \subseteq X \tag{3}$$

In [4], M.Aamri, T. Sabar,, A.Bassou established new fixed point theorems

Theorem 4. *Suppose that (X,Y,Z) is a nonempty and closed triad of subsets of a complete metric space (E,d) and $T : X \cup Y \cup Z \to X \cup Y \cup Z$ is tricyclic mapping for which there exists $k \in]0,1[$ such that $\Delta(Tx,Ty,Tz) \leq k\Delta(x,y;z)$ for all $(x,y,z) \in X \times Y \times C$.*

where the mapping

$$\Delta \quad : \quad X \times Y \times Z \to [0,+\infty) \tag{4}$$

$$\Delta(x,y;z) \quad \to \quad d(x,y)+d(y,z)+d(z,x) \tag{5}$$

Then $X \cap Y \cap Z$ is non empty and T has a unique fixed point in $X \cap Y \cap Z$.

In 2012, Shah and Hussain [6] introduced the concept of ”quasi-b-metric spaces” and investigated fixed point theorems within this framework. The term ”quasi-b-metric spaces” is a combination of concepts from quasi-metric spaces and b-metric spaces, both of which involve notions of distance and space. This concept likely combines properties of both types of spaces to explore a new class of mathematical structures.

Quasi-metric spaces relax the strict triangle inequality property of metric spaces, allowing for more flexible distance measures between points. This can be useful in contexts where the distance between points doesn’t necessarily satisfy the standard triangle inequality.

B-metric spaces, on the other hand, are a modification of metric spaces where distances are bounded by a constant. This concept can be particularly relevant in situations where distances have an upper bound.

By combining these ideas, Shah and Hussain introduced the concept of quasi-b-metric spaces, which presumably involves distance functions that are both relaxed in terms of triangle inequality and bounded by constants.

Definition 5. [7] *Let E be a nonempty set . Suppose that the mapping $d : E \times E \to [0,\infty)$ such that constant $s \geq 1$ satisfies the folowing conditions:*

(c_1) $d(x,y) = d(y,x) = 0$ *implies* $x = y$ *for all* $x, y \in E$;

(c_2) $d(x,y) \leq s[d(x,z) + d(z,y)]$ *for all* $x, y \in E$.

The pair (E,d) *is then called a dislocated quasi-b-metric space (or simply* $dqb-metric$*). The number* s *is called the coefficient of* (E,d).

Remark 6. [7] *It is evident that b-metric spaces, quasi-b-metric spaces, and b-metric-like spaces fall under the category of dislocated quasi-b-metric spaces. However, it should be noted that the reverse is not true.*

Example 7. *Let* $E = \mathbb{R}$ *and let*

$$b(x,y) = |x-y|^2 + \frac{|x|}{g} + \frac{|y|}{h}$$

where $g \neq h$

Then (E,b) *is a dislocated quasi-b-metric space with the coefficient* $s = 2$*, but since* $b(2,2) \neq 0$*, we have* (E,b) *is not a quasi-b-metric space, and since* $b(1,2) \neq b(2,1)$*, we have* (E,b) *is not a b-metric-like sapace. And* (E,b) *is not a dislocated quasi-metric space. Indeed; let* $x,y,z \in E$. *suppose that* $b(x;y) = 0$. *Then*

$$|x-y|^2 + \frac{|x|}{g} + \frac{|y|}{h} = 0$$

it implies that $|x-y|^2 = 0$*, and so* $x = y$.

Next consider

$$\begin{aligned} b(x,y) &= |x-y|^2 + \frac{|x|}{g} + \frac{|y|}{h} \\ &\leq (|x-z| + |z-y|)^2 + \frac{|x|}{g} + \frac{|y|}{h} \\ &\leq |x-z|^2 + 2\,|x-z|\,.\,|z-y| + |z-y|^2 + \frac{|x|}{g} + \frac{|y|}{h} \\ &\leq 2(|x-z|^2 + |z-y|^2) + \frac{|x|}{g} + \frac{|z|}{g} + \frac{|z|}{h} + \frac{|y|}{h} \\ &\leq s[b(x,z) + b(z,y)] \end{aligned}$$

Where $s = 2$.

Definition 8. **1-:** *A sequence* $(\{x_n\})$ *in a dqb-metric space* (E,d) *dislocated quasi-b-converges (for short, dqb-converges) to* $x \in E$ *if*

$$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0$$

In this case x *is called a dqb-limit of* $(\{x_n\})$ *, and we write* $(x_n \to x$ *as* $n \to \infty)$

2-: *A sequence* $(\{x_n\})$ *in a dqb-metric space* (E,d) *is called Cauchy if*

$$\lim_{n,m\to\infty} d(x_n,x_m) = \lim_{n,m\to\infty} d(x_m,x_n) = 0$$

3-: *A dqb-metric space* (E,d) *is complete if every Cauchy sequence in it is dqb-convergent in* E.

Definition 9. *Let* X *and* Y *be nonempty subsets of a dislocated quasi-b-metric sapce* (E,d). *A cyclic map* $T : X\cup Y \to X\cup Y$ *is said to be a* $dqb-cyclic-banach$ *contraction for all* $x\in X, y\in Y$ *and* $s\geq 1$ *and* $sk\leq 1$.

Theorem 10. [1] *Ler* X *and* Y *be nonempty subsets of a complete* $dqb-metric$ *space* (E,d). *Let* T *be a cyclic mapping thar satisfies the condition of a* $dqb-cyclic-banach$ *contraction. Then* T *has a unique fixed point in* $X\cap Y$.

We are going to generalize this result to a tricyclic mapping. To achieve this, we provide the following definitions :

2. Main Results

We start our results by the generalization of the definition of $dqb-tricyclic-banach$ contraction notion, as follows.

Definition 11. *Let* X, Y *and* Z *be nonempty subsets of a* $dqb-metric$ *space* (E,d).

A tricylic mapping $T : X\cup Y\cup Z \to X\cup Y\cup Z$ *is said to be* $dqb-tricyclic-banach$ *contraction if there existe* $k\in[0,\frac{1}{2})$ *such that*

$$\Delta(Tx,Ty,Tz) \leq k.\Delta(x,y,z) \tag{6}$$

for all $x\in X, y\in Y, z\in Z$ *and* $s\geq 1$ *and* $sk<1$.

Theorem 12. *Let X, Y and Z be nonempty subsets of a complete $dqb-metric$ space (E,d). Let T be a tricyclic mapping thar satisfies the condition of a $dqb-tricyclic-banach$ contraction. Then T has a unique fiexed point in $X \cap Y \cap Z$.*

Proof. Let $x \in X \cup Y \cup Z$ we have

$$\begin{aligned} d(Tx,T^2x) &\leq \Delta(Tx,T^2x,T^3x) \leq k\Delta(x,Tx,T^2x) \\ \text{and } d(T^2x,Tx) &\leq \Delta(T^2x,Tx,T^3x) \leq k\Delta(Tx,x,T^2x) \end{aligned}$$

So

$$d(Tx,T^2x) \leq k\beta \tag{7}$$

and

$$d(T^2x,Tx) \leq k\beta \tag{8}$$

Where $\beta = \max\{\Delta(x,Tx,T^2x), \Delta(Tx,x,T^2x)\}$

Using(7) and (8) we have $d(T^2x,T^3x) \leq k^2\beta$ and $d(T^3x,T^2x) \leq k^2\beta$.

For all $n \in \mathbb{N}$, we get $d(T^{n+1}x,T^nx) \leq k^n\beta$ *and* $d(T^nx,T^{n+1}x) \leq k^n\beta$.

Let $m, n \in \mathbb{N}$ with $m > n$, using the triangular inequality, we get:

$$\begin{aligned} d(T^mx,T^nx) &\leq sd(T^mx,T^{n+1}x) + sd(T^{n+1}x,T^nx) \\ &\leq s^2d(T^mx,T^{n+2}x) + s^2d(T^{n+2}x+T^{n+1}x) + sd(T^{n+1}x+T^nx) \\ &\leq s^{m-n}d(T^mx,T^{m-1}x) + s^{m-n-1}d(T^{m-1}x,T^{m-2}x) + ... + sd(T^{n+1}x+T^nx) \\ &\leq (s^{m-n}k^{m-1} + s^{m-n-1}k^{m-2} + ... + sk^n)\beta \\ &\leq ((sk)^{m-n}k^{n-1} + (sk)^{m-n-1}k^{n-1} + ... + (sk)k^{n-1})\beta \\ &\leq \sum_{i=n}^{m}(sk)^ik^{n-1}\beta \\ &= \frac{1-(sk)^{m-n}}{1-sk}k^{n-1}\beta \quad \text{with } (sk<1) \\ &\leq \frac{1}{1-sk}\beta(k^{n-1}) \end{aligned}$$

For $n \to \infty$, we get $d(T^mx,T^nx) \to 0$.

By the same way for all $n,m \in \mathbb{N}$ with $m > n$ we have

$$d(T^n x, T^m x) \leq \frac{1}{1-sk}\beta(k^{n-1}).$$

Take $n \to \infty$, we get $d(T^n x, T^m x) \to 0$.Thus $T^n x$ is a cauchy sequence.

Since (E,d) is complet, we have $\{T^n x\}$ converge to some $z \in X$.

nevertheless, an infinite number of terms of the sequence $(T^n x)$ lie in X, as in Y and Z; as a result, $z \in X \cap Y \cap Z$.

Now, we will prove that z is the fixed point $Tz = z$.

We have

$$\begin{aligned} d(T^n x, Tz) &\leq \Delta(T^n x, T^{n-1}x, Tz) \\ &\leq k\Delta(T^{n-1}x, T^{n-2}x, z) \\ &\leq k[d(T^{n-1}x, T^{n-2}x) + d(T^{n-2}x, z) + d(z, T^{n-1}x)] \end{aligned}$$

let $n \to \infty$, we get $d(z, Tz) = 0$.

The same we get

$$d(Tz, T^n x) \leq k[d(z, T^{n-1}x) + d(T^{n-1}x, T^{n-2}x) + d(T^{n-2}x, z)]$$

let $n \to \infty$, we get $d(Tz, z) = 0$.

Hense $d(z,Tz) = d(Tz,z) = 0$ implies that $Tz = z$ where z is a fixed point of T.

To establish the uniqueness of a fixed point, consider another fixed point of T denoted as $z\prime$, such that $Tz' = z'$.

We have

$$\begin{aligned} d(z', z) &= d(Tz', Tz) \\ &\leq \Delta(Tz', Tz, Tz) \\ &\leq k\Delta(z', z, z) \\ &= k[d(z', z) + d(z, z) + d(z, z')] \\ &= k[d(z', z) + d(z, z')] \end{aligned}$$

By the same way we get that

$$d((z,z') \leq k[d(z',z)+d(z;z')]$$

From that we get

$$d(z',z)+d((z,z') \leq 2k[d(z',z)+d(z;z')]$$

implies that

$$d(z',z)+d((z,z') = 0 \Longrightarrow d(z',z) = d(z,z') = 0$$

and that prove the uniqueness of the fixed point of T. □

Example 13. *Let $X=\mathbb{R}^3$ and let $d(x,y)=\|x-y\|^2+3\|x\|^2+2\|y\|^2$ such that $\|\|$,it's the euclidean norme.*

Then $(E,d)$is a dislocated quasi-b-metric space with the coefficient $s=2$.

Suppose $A=[0;1]\times\{0\}\times\{0\}; B=\{0\}\times[0;1]\times\{0\}$ and $C=\{0\}\times\{0\}\times[0;1]$.

Let $T : X\cup Y\cup Z \to X\cup Y\cup Z$ defined by

$$T(x,y,z)=\begin{cases} (0,\frac{x}{3},0) \text{ ,if } (x,y,z)\in X, \\ (0,0,\frac{y}{3}), \text{ if } (x,y,z)\in Y, \\ (\frac{z}{3},0,0) \text{ ,if } (x,y,z)\in Z, \end{cases}$$

Clearly, T is tricyclic on $X\cup Y\cup Z$: Consider $(x;0;0)\in X$, $(0;y;0)\in Y$ and $(0;0;z)\in Z$, we have

$$\begin{aligned} \Delta(T(x,0,0),T(0,y,0),T(0,0,z)) &= \Delta((0,\frac{x}{3},0),T(0,0,\frac{y}{3}),T(\frac{z}{3},0,0)) \\ &= \frac{5}{9}(x^2+y^2+z^2) \end{aligned}$$

in other hand we have

$$H((x,0,0),(0,y,0),(0,0,z)) = 5(x^2+y^2+z^2)$$

Thus

$$\Delta(T(x,0,0),T(0,y,0),T(0,0,z)) = \frac{1}{9}\Delta((x,0,0),(0,y,0),(0,0,z))$$

It now follows from the theorem that T has a unique fixed point $(0,0,0)$.

We introduce a new dqb-tricyclic-Kannan in the following way.

Definition 14. *Let X, Y and Z be nonempty subsets of a $dqb-metric$ space (E,d).*

A tricylic mapping $T : X \cup Y \cup Z \to X \cup Y \cup Z$ is said to be dqb-tricyclic-Kannan contraction if there existe $k \in [0,1)$ such that

$$\Delta(Tx,Ty,Tz) \le k.\left[d(Tx,x) + d(Ty,y) + d(Tz,z)\right] \tag{9}$$

for all $x,y,z \in A \times B \times C$ and $s \ge 1$ and $(s+2)k < 1$.

Theorem 15. *Ler X, Y and Z be nonempty subsets of a complete $dqb-metric$ space (E,d). Let T be a dqb-tricyclic-Kannan contraction . Then T has a unique fiexed point in $X \cap Y \cap Z$.*

Proof. Let $x \in X \cup Y \cup Z$ we have

$$d(T^3x,T^2x) \le \Delta(T^3x,T^2x,Tx) \le k.\left[d\left(T^3x,T^2x\right) + d\left(T^2x,Tx\right) + d(Tx,x)\right]$$

Then,

$$d(T^3x,T^2x) \le k.\left[d\left(T^3x,T^2x\right) + d\left(T^2x,Tx\right) + d(Tx,x)\right]$$

which implies

$$d(T^3x,T^2x) \le \frac{k}{1-k}\left[d\left(T^2x,Tx\right) + d(Tx,x)\right] \tag{10}$$

$$d\left(T^2x,Tx\right) \le \Delta(T^3x,T^2x,Tx) \tag{11}$$

$$\le k.\left[d\left(T^3x,T^2x\right) + d\left(T^2x,Tx\right) + d(Tx,x)\right] \tag{12}$$

we have

$$d\left(T^2x,Tx\right) \le \frac{k}{1-k}\left[d\left(T^3x,T^2x\right) + d(Tx,x)\right] \tag{13}$$

□

$$d\left(T^2x,Tx\right) \le \frac{k}{1-k}\left[\frac{k}{1-k}\left[d\left(Tx,T^2x\right) + d(Tx,x)\right] + d(Tx,x)\right] \tag{14}$$

$$d\left(T^2x,Tx\right) \le \frac{k}{1-k}\left[\frac{k}{1-k}\left[d\left(Tx,T^2x\right) + d(Tx,x)\right] + d(Tx,x)\right] \tag{15}$$

which implies

$$d\left(T^2x,Tx\right) \le \frac{k}{1-2k}d(Tx,x) \tag{16}$$

Proof. And from (16) we have

$$d(Tx,T^2x) \leq \Delta(Tx,T^2x,T^2x) \leq k.\left[d(Tx,x)+d\left(T^2x,Tx\right)+\left(T^2x,Tx\right)\right]$$

Then,

$$\begin{aligned} d(Tx,T^2x) &\leq k.\left[d(Tx,x)+\frac{k}{1-2k}d(Tx,x)+\frac{k}{1-2k}d(Tx,x)\right] \\ &\leq k.\left[d(Tx,x)+\frac{2k}{1-2k}d(Tx,x)\right] \\ &\leq k\left[\frac{1}{1-2k}\right]d(Tx,x) \\ &\leq \frac{k}{1-2k}d(Tx,x) \end{aligned}$$

Then

(17) $$d\left(T^2x,Tx\right) \leq td(Tx,x)\ ,d\left(Tx,T^2x\right) \leq td(Tx,x) \text{ where } t=\frac{k}{1-2k} \text{ and } t\in(0,1)$$

□

Inductively, using this process for all $n \in \mathbb{N}$ we have

(18) $$d\left(T^{n+1}x,T^nx\right) \leq t^n d(Tx,x), \text{ for all } n\geq 1$$

and

(19) $$d\left(T^nx,T^{n+1}x\right) \leq t^n d(Tx,x), \text{ for all } n\geq 1$$

Let $n,m \in \mathbb{N}$ with $m>n$, by using the triangular inequality, we have

$$\begin{aligned} d(T^mx,T^nx) &\leq sd(T^mx,T^{n+1}x)+sd(T^{n+1}x+T^nx) \\ &\leq s^2d(T^mx,T^{n+2}x)+s^2d(T^{n+2}x+T^{n+1}x)+sd(T^{n+1}x+T^nx) \\ &\leq s^{m-n}d(T^mx,T^{m-1}x)+s^{m-n-1}d(T^{m-1}x,T^{m-2}x)+......+sd(T^{n+1}x+T^nx) \\ &\leq (s^{m-n}t^{m-1}d(Tx,x)+s^{m-n-1}t^{m-2}d(Tx,x)+......+st^n d(Tx,x)) \\ &\leq ((st)^{m-n}t^{n-1}+(st)^{m-n-1}t^{n-1}+.......+(st)t^{n-1})d(Tx,x) \\ &\leq \sum_{i=n}^{m}(st)^i t^{n-1}d(Tx,x) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1-(st)^{m-n}}{1-st} t^{n-1} d(Tx,x) \\
&\leq \frac{1}{1-st} t^{n-1} d(Tx,x)
\end{aligned}
$$

we conclude

$$d(T^m x, T^n x) \leq \eta t^{n-1} d(Tx,x) \tag{20}$$

for some $\eta > \frac{1}{1-st}$,by evaluating the limit as $n \to \infty$ in the preceding inequality, we get $d(T^m x, T^n x) \to 0$.

Similarly, $n,m \in \mathbb{N}$ with $m > n$,, by using the triangular inequality, we have

$$d(T^n x, T^m x) \leq \eta t^{n-1} d(Tx,x) \tag{21}$$

Take $n \to \infty$ we get $d(T^m x, T^n x) \to 0$. Thus $T^n x$ is a Cauchy sequence.

by the completeness of E, , we have $\{(T^n x)\}$ converges to some $z \in X \cup Y \cup Z$.

Notice that $\left\{T^{3n}x\right\}$ is a sequence in X, $\left\{T^{3n-1}x\right\}$ is a sequence in Y and $\left\{T^{3n-2}x\right\}$ is a sequence in Z and that both sequences tend to the same limit z. Regardinthe fact that X, Y and Z are closed, we conclude $z \in X \cup Y \cup Z$, and then $X \cap Y \cap Z \neq \emptyset$

Now, we will demonstrate that $Tz = z$. by (9),

$$
\begin{aligned}
d(Tz,z) &= \lim_{n\to\infty} d\left(Tz, T^{3n}x\right) \leq \lim_{n\to\infty} \Delta\left(T^{3n}x, T^{3n-1}x, Tz\right) && (22)\\
&\leq \lim_{n\to\infty} k\left[d\left(T^{3n}x, T^{3n-1}x\right) + d\left(T^{3n-1}x, T^{3n-2}x\right) + d(Tz,z)\right] && (23)\\
&\leq kd(Tz,z)
\end{aligned}
$$

and consequently we have $d(Tz,z) = 0$, which implies $Tz = z$.

To prove the uniqueness of the fixed point z, assume that there exists $v \in X \cup Y \cup Z$ such that $z \neq \bar{z}$ and $T\bar{z} = \bar{z}$. Taking into account that T is tricyclic we get $\bar{z} \in X \cap Y \cap Z$. we have

$$
\begin{aligned}
d(\bar{z},z) &= d(T\bar{z}, Tz) \leq H(T\bar{z}, T\bar{z}, Tz) && (24)\\
&\leq k\left[d(T\bar{z},\bar{z}) + d(T\bar{z},\bar{z}) + d(Tz,z)\right] && (25)\\
&\leq k\left[d(\bar{z},\bar{z}) + d(\bar{z},v) + d(z,z)\right] = 0 && (26)
\end{aligned}
$$

which yields that $d(z,\bar{z}) = 0$. We conclude that $z = \bar{z}$ and hence z is the unique fixed point of T.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] W.A. Kirk, P.S. Srinavasan, P. Veeramani, Fixed points for mapping satisfying cyclical contractive conditions, Fixed Point Theory. 4 (2003), 79–89.

[2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. 3 (1922), 133–181.

[3] R. Kannan, Some results on fixed points–II, Amer. Math. Mon. 76 (1969), 405–408. https://doi.org/10.1080/00029890.1969.12000228.

[4] E. Karapınar, I.M. Erhan, Best proximity point on different type contractions, Appl. Math. Inform. Sci. 5 (2011), 558–569.

[5] T. Sabar, M. Aamri, A. Bassou, Best proximity points for tricyclic contractions, Adv. Fixed Point Theory, 7 (2017), 512–523.

[6] M.H. Shah, N. Hussain, Nonlinear contractions in partially ordered quasi b-metric spaces, Commun. Korean Math. Soc. 27 (2012), 117–128. https://doi.org/10.4134/CKMS.2012.27.1.117.

[7] C. Klin-eam, C. Suanoom, Dislocated quasi-b-metric spaces and fixed point theorems for cyclic contractions, Fixed Point Theory Appl. 2015 (2015), 74. https://doi.org/10.1186/s13663-015-0325-2.

[8] C. Zhu, C. Chen, X. Zhang, Some results in quasi-b-metric-like spaces, J. Inequal. Appl. 2014 (2014), 437. https://doi.org/10.1186/1029-242x-2014-437.

[9] K. Zoto, P.S. Kumari, E. Hoxha, Some fixed point theorems and cyclic contractions in dislocated and dislocated quasi-metric spaces, Amer. J. Numer. Anal. 2 (2014), 79–84.

[10] M. Edraoui, A. El koufi, S. Semami, Fixed points results for various types of interpolative cyclic contraction, Appl. Gen. Topol. 24 (2023), 247–252. https://doi.org/10.4995/agt.2023.19515.

[11] C.T. Aage, J.N. Salunke, Some results of fixed point theorem in dislocated quasi-metric spaces, Bull. Marathwada Math. Soc. 9 (2008), 1–5.

[12] A. Muraliraj, R.J. Hussain. New results on fixed point theorems in dislocated quasimetric spaces. Int. J. Math. Sci. Appl. 3 (2013), 321–326.

[13] M.U. Rahman, M. Sarwar, Dislocated quasi b-metric space and fixed point theorems, Electron. J. Math. Anal. Appl. 4 (2016), 16–24.

[14] M.U. Rahman, M. Sarwar, Fixed point theorems in generalized types of b-dislocated metric spaces, Electron. J. Math. Anal. Appl. 5 (2017), 10–17.

[15] E. Mohamed, E.K. Amine, A. Mohamed, Best proximity point theorems for proximal pointwise tricyclic contraction, Adv. Fixed Point Theory, 13 (2023), 22. https://doi.org/10.28919/afpt/8194.

[16] Y.E. Bekri, M. Edraoui, J. Mouline, et al. Interpolative Cirić-Reich-Rus-Type contraction in G-metric spaces, J. Survey Fisheries Sci. 10 (2023), 17–20.

[17] Y.E. Bekri, M. Edraoui, J. Mouline, et al. Cyclic coupled fixed point via interpolative Kannan type contractions, Math. Statistician Eng. Appl. 72 (2023), 24–30.

[18] E. Mohamed, A. Mohamed, L. Samih, Relatively cyclic and noncyclic *P*-contractions in locally *K*-convex space, Axioms. 8 (2019), 96. https://doi.org/10.3390/axioms8030096.