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BEST PROXIMITY POINT THEOREMS FOR TRICYCLIC DIAMETRICALLY CONTRACTIVE MAPPINGS

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Abstract. In this article, we are referring to introducing a new class of mappings called tricyclic diametrically contractive mappings, which are defined as the union of triad nonempty subsets of a metric space. The authors provide necessary and sufficient conditions for the existence of the best proximity point for these mappings.

This new class of mappings extends the theory of traditional diametrically contractive mappings and provides a more general framework for studying fixed points and optimization problems in metric spaces. The best proximity point, which is a particular type of fixed point that is closest to a given point in the metric space, is an important tool for solving optimization problems.

The results provided in this article represent a significant contribution to the field of analysis and its applications. They are expected to have far-reaching implications for the study of fixed points and optimization problems in metric spaces. The new class of tricyclic diametrically contractive mappings and the conditions for the existence of the best proximity point are likely to be of great interest to researchers in mathematics and science, as well as practitioners in various fields that make use of optimization algorithms.

Keywords: fixed point; tricyclic diametrically; tricyclic contraction; pointwise tricyclic; triad normal structure.

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The fixed-point theory is indeed a versatile and powerful tool with applications spanning various branches of mathematics and beyond. Here are a few more areas where fixed-point theory finds applications: Numerical Analysis: In numerical methods, fixed-point iteration is often used to approximate solutions to equations and systems of equations. Methods like the Newton-Raphson method rely on finding fixed points of iterative mappings.

Topology: Fixed-point theorems are essential in topology, as they provide insights into the structure and properties of topological spaces. Brouwer's Fixed-Point Theorem, in particular, has profound implications for the topology of Euclidean spaces.

Economics and Game Theory: Fixed-point theorems are used in economics to prove the existence of equilibrium points in economic models. In game theory, fixed points play a crucial role in understanding strategies that remain unchanged in certain conditions.

Control Theory: Fixed-point theory is applied in control theory to analyze the stability and convergence properties of dynamical systems. It helps in understanding the behaviour of systems under repeated iterations.

Functional Analysis: In functional analysis, fixed-point theorems are used to study the properties of operators on Banach spaces. The Schauder Fixed-Point Theorem, for example, is a fundamental tool in this context.

Computer Science: Fixed-point theory is utilized in various areas of computer science, such as formal methods, program analysis, and verification. It provides a theoretical basis for understanding the convergence of algorithms and the behaviour of recursive functions.

Physics: In the study of dynamical systems and chaos theory, fixed points are critical in characterizing the long-term behaviour of a system. Understanding fixed points helps in predicting stability and chaotic behaviour in physical systems.

Functional Equations: Fixed-point theory is employed to study solutions to functional equations, which arise in diverse areas of mathematics and its applications. [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13]

1. Introduction

Diametrically contractive mapping is a mathematical function that maps points in a metric space to other points in the same space such that the distance between the mapped points is always less than or equal to the distance between the original points. This concept was introduced by Hong-Kun Xu [3]. is a mathematician and researcher in the field of analysis and its applications. The idea of diametrically contractive mappings is used in various areas of mathematics, including fixed point theory, optimization, and control theory.Xu Hong-Kun presented the idea of diametrically contractive in [3].

Definition 1. *Let (E,d) be a metric space. A mapping $T : E \to E$ is said to be diametrically contractive if*

$$\delta(TM) < \delta(M) \tag{1}$$

for all closed subsets M of E for which $0 < \delta(M) < \infty$.

the elements of a proper subclass of the class of contractive mappings that Istratescu [6] introduced are referred to as diametrically contractive mappings. In the context of Banach spaces, Xu [3] demonstrated the following theorem:

Dhompongsa and Yingtaweesittikul provided sufficient conditions for the existence of fixed points for multivalued diametrically contracted mappings.

Theorem 2. *Let M be a weakly compact subset of a Banach space E and let $T : M \to M$ be a diametrically contractive mapping. Then T has a fixed point.*

We continue this discussion with two extensions of diametrically contractive mappings, introduced in 2019 by V. Sankar Raj & T. Piramatchi, We will continue to generalize this theory for tricyclic diametrically contractive mappings and we prove the existence of the best proximity point for tricyclic mapping

In this manuscript, we consider a metric space and intend to prove the existence of best proximity point for tricyclic diametrically contractive mappings,

2. Introduction

In this section, we give some basic definitions and concepts that are useful and related to the context of our results.

Let (E,d) be a metric space and let X,Y and Z be nonempty subsets of E.

A mapping $T : X\cup Y\cup Z \to X\cup Y\cup Z$ is said to be a tricyclic mapping provided that

$$T(X) \subseteq Y,\ T(Y) \subseteq Z \text{ and } T(Z) \subseteq X \tag{2}$$

In [4], T. Sabar, M.Aamri, A.Bassou established the following theorem which is an interesting extension of the Banach contraction principle.

Theorem 3. [4] *Suppose that (X,Y,Z) is a nonempty and closed triad of subsets of a complete metric space (E,d) and $T : X\cup Y\cup Z \to X\cup Y\cup Z$ is tricyclic mapping for which there exists $k \in]0,1[$ such that $\Delta(Tx,Ty,Tz) \le k\Delta(x,y,z)$ for all $(x,y,z) \in X\times Y\times Z$.*

where the mapping

$$\Delta \quad : \quad X\times X\times X \to [0,+\infty) \tag{3}$$

$$\Delta(x,y;z) \quad \to \quad d(x,y)+d(y,z)+d(z,x) \tag{4}$$

Then $X\cap Y\cap Z$ is non empty and T has a unique fixed point in $X\cap Y\cap Z$.

Definition 4. [4] *Let (E,d) be a metric space and let X,Y and Z be nonempty subsets of E.*

Let $T : X\cup Y\cup Z \to X\cup Y\cup Z$ be a tricyclic mapping. A point $x \in X\cup Y\cup Z$ is said to be the best proximity point for T if

$$\Delta\left(x,Tx,T^2x\right) = \delta(X,Y,Z) \tag{5}$$

Definition 5. [4] *Let (X,Y,Z) be a nonempty triad subsets of a metric space (E,d). Let $T :$ $X\cup Y\cup Z \to X\cup Y\cup Z$ be a tricyclic mapping. Then a sequence (x_n) in $X\cup Y\cup Z$ is said to be an approximate best proximity point sequence for T if*

$$\Delta\left(x_n,Tx_n,T^2x_n\right) \to \delta(X,Y,Z) \text{ as } n\to\infty \tag{6}$$

where $\delta(X,Y,Z) = \inf\{\Delta(x,y;z) : x\in X,\ y\in Y \text{ and } z\in Z\}$

Definition 6. [4] *A triad (X,Y,Z) of subsets of a normed linear space is said to be a proximal triad if for each $(x,y,z) \in X \times Y \times Z$ there exists $(x',y',z') \in X \times Y \times Z$ such that*

$$\Delta\left(x',y,z\right) = \Delta(x,y',z) = \Delta(x,y,z') = \delta\left(X,Y,Z\right)$$

The triad $(x,y,z) \in X \times Y \times Z$ is said to be proximal in (X,Y,Z) if $\Delta(x,y,z) = \delta\left(X,Y,Z\right)$ we set

$$\begin{aligned} X_0 &= \{x_1 \in X : \Delta(x_1,y_2,z_3) = \delta(X,Y,Z), \text{ for some } (y_2,z_3) \in Y \times Z\} \\ Y_0 &= \{y_1 \in Y : \Delta(x_3,y_1,z_2) = \delta(X,Y,Z), \text{ for some } (x_3,z_2) \in X \times Z\} \\ Z_0 &= \{z_1 \in Z : \Delta(x_2,y_3,z_1) = \delta(X,Y,Z), \text{ for some } (x_2,y_3) \in X \times Y\} \end{aligned}$$

Clearly, $\delta\left(X_0,Y_0,Z_0\right) = \delta\left(X,Y,Z\right)$

In [6], M.Edraoui, A. El Koufi, and M. Aamri introduced the concept of pointwise tricyclic contraction and studied the existence of the best proximity point on a triad of weakly compact convex subsets of a Banach space.

Definition 7. [6] *Let (X,Y,Z) be a nonempty triad of subsets of a metric space (E,d). A mapping $T : X \cup Y \cup Z \to X \cup Y \cup Z$ is said to be a pointwise tricyclic contraction if*

it satisfies

i) $T(X) \subseteq Y$, $T(Y) \subseteq Z$ and $T(Z) \subseteq X$

ii) For each $(x,y,z) \in X \times Y \times C$ there exists $\alpha(x), \alpha(y), \alpha(z)$ in $(0,1)$ such that

$$\begin{aligned} \Delta(Tx,Ty,Tz) &\leq \alpha(x)\Delta(x,y;z) + (1-\alpha(x))\Delta(x,y;z) \textit{ for } (y,z) \in Y \times Z \\ \Delta(Tx,Ty,Tz) &\leq \alpha(y)\Delta(x,y;z) + (1-\alpha(y))\Delta(x,y;z) \textit{ for } (x,z) \in X \times Z \\ \Delta(Tx,Ty,Tz) &\leq \alpha(z)\Delta(x,y;z) + (1-\alpha(z))\Delta(x,y;z) \textit{ for } (x,y) \in X \times Y \end{aligned}$$

Definition 8. [6] *A convex triad $(K_1;K_2;K_3)$ in a Banach space is said to have a proximal triad normal structure if for any closed bounded and convex proximal triad $(H_{1;}H_{2;}H_3) \subseteq (K_1;K_2;K_3)$ for which $\Delta(H_{1;}H_{2;}H_3) = \Delta(K_1;K_2;K_3)$ and $\delta(H_{1;}H_{2;}H_3) > \Delta(H_{1;}H_{2;}H_3)$ there exists $(x_1,x_2,x_3) \in H_1 \times H_2 \times H_3$ such that*

$$\delta(x_{1;}H_{2;}H_3) < \delta(H_{1;}H_{2;}H_3), \ \delta(x_{2,}H_{1,}H_3) < \delta(H_{1;}H_{2;}H_3), \delta(x_{3,}H_{1,}H_2) < \delta(H_{1;}H_{2;}H_3)$$

Then $x_{1,}$ is a nondiametral point of $H_{1,}$ and $x_{2,}$ is a nondiametral point of $H_{2,}$ and $x_{3,}$ is a nondiametral point of $H_{3,}$

Theorem 9. [6] *Let X, Y and Z be nonempty weakly compact convex subsets in a Banach space and T is a pointwise tricyclic contraction mapping. Then T has a best proximity point.*

3. Tricyclic Diametrically Contractive Mappings

Now let's introduce the concept of tricyclic diametrically contractive mapping and prove the existence of best proximity points of such mappings.

Definition 10. *Let X,Y,Z be nonempty subsets of a normed linear space. A tricyclic mapping $T : X \cup Y \cup Z \to X \cup Y \cup Z$ is a diametrically contractive if for any closed triad $(H_1,H_2,H_3) \subseteq (X,Y,Z)$*

(1) $\delta\ (TH_1,TH_2,TH_3) \subset \delta\ (H_1,H_2,H_3)$, whenever $\delta\ (H_1,H_2,H_3) > \delta\,(X,Y,Z)$

(2) $\delta\ (TH_1,TH_2,TH_3) = \delta\ (H_1,H_2,H_3)$, whenever $\delta\ (H_1,H_2,H_3) = \delta\,(X,Y,Z)$

It is easy to see that every tricyclic diametrically contractive mapping is tricyclic contractive by taking $H_1 = \{x\}$, $H_2 = \{y\}$ and $H_3 = \{y\}$

Proposition 11. *Let X,Y,Z nonempty compact subsets of a metric space X and $T : X \cup Y \cup Z \to X \cup Y \cup Z$ be a tricyclic contractive mapping. Suppose T is a closed mapping. Then T is a diametrically contractive mapping.*

Proof. SupposeH_1,H_2,H_3 are nonempty closed subsets of X,Y,Z respectively such that δ $(H_1,H_2,H_3) > \delta\,(X,Y,Z)$. Since T is closed mapping, $TH_1,TH_2,TH_3$are nonempty compact subsets of H_2,H_3,H_1, respectively. Hence there exists $x_1 \in H_1, x_2 \in H_2, x_3 \in H_3$ such that $\Delta(Tx_1,Tx_2,Tx_3) = \delta\ (TH_1,TH_2,TH_3)$. If $\Delta(x_1,x_2,x_3) > \delta\,(X,Y,Z)$, then by tricyclic contractive condition,

$$\delta\,(TH_1,TH_2,TH_3) = \Delta(Tx_1,Tx_2,Tx_3) < \Delta(x_1,x_2,x_3) \leq \delta\,(H_1,H_2,H_3)$$

If $\Delta(x_1,x_2,x_3) = \Delta(X,Y,Z)$, then by assumption,

$$\delta\,(TH_1,TH_2,TH_3) = \Delta(Tx_1,Tx_2,Tx_3) = \Delta(x_1,x_2,x_3) = \Delta(X,Y,Z) < \delta\,(H_1,H_2,H_3)$$

Thus, we showed that $\delta(TH_1,TH_2,TH_3) < \delta(H_1,H_2,H_3)$, whenever $\delta(H_1,H_2,H_3) > \Delta(X,Y,Z)$. Suppose that $\delta(H_1,H_2,H_3) = \Delta(X,Y,Z)$.Then $\Delta(x_1,x_2,x_3) = \Delta(X,Y,Z)$, for all $x \in H_1, y \in H_2, z \in H_3$, By cyclic contractive, we have $\Delta(Tx,Ty,Tz) = \Delta(X,Y,Z)$,for all $x \in H_1, y \in H_2, z \in H_3$. This shows that $\delta(TH_1,TH_2,TH_3) = \Delta(X,Y,Z)$. Hence T is a tricyclic diametrically contractive. □

The following theorem provides sufficient conditions for the existence of the best proximity point for the given diametrically contractive mapping.

Theorem 12. *Let X,Y and Z be nonempty weakly compact subsets of a normed linear space X and $T : X \cup Y \cup Z \to X \cup Y \cup Z$ be a tricyclic diametrically contractive mapping. Then there exist $(x',y',z') \in X \times Y \times Z$ such that:*

$$\Delta\left(x',Tx',T^2x'\right) = \Delta\left(y',Ty',T^2y'\right) = \Delta\left(z',Tz',T^2z'\right) = \delta(X,Y,Z). \tag{7}$$

Proof. Define

$$\mathscr{F} = \left\{ H_1,H_2,H_3 \Bigg/ \begin{array}{c} \text{are non empty, weakly compact of } (X,Y,Z) \\ T(H_1) \subset H_2,\ T(H_2) \subset H_3 \text{and } T(H_3) \subset H_1 \\ \text{and } \delta(H_1,H_2,H_3) = \delta(X,Y,Z) \end{array} \right\}$$

Clearly, $\mathscr{F}$ is nonempty, since $(X_0,Y_0,Z_0) \subset \mathscr{F}$. Then $\mathscr{F}$ is a partially ordered set with respect to the set inclusion.

That is, $(H_1,H_2,H_3) \le (K_1,K_2,K_3)$ if and only if $H_1 \subseteq K_1$, $H_2 \subseteq K_2$ and $H_3 \subseteq K_3$.

It is easy to see that every chain in $\mathscr{F}$ has a lower bound. Hence by Zorn's lemma, there is a weakly compact triad $(K_1,K_2,K_3) \in \mathscr{F}$ which is minimal in $\mathscr{F}$.

Then (K_1,K_2,K_3) is a proximal triad. Otherwise, $((K_1)_0,(K_2)_0,(K_3)_0)$ would be a proper subset of (K_1,K_2,K_3) and is in $\mathscr{F}$ a contradiction to the minimality of (K_1,K_2,K_3).

Let $H_1 := \overline{T(K_3)}^w$ and $H_2 := \overline{T(K_1)}^w$ and $H_3 := \overline{T(K_2)}^w$.

Clearly $(H_1,H_2,H_3) \le (K_1,K_2,K_3)$. Now, $T(H_1) \subseteq T(K_1) \subseteq \overline{T(K_1)}^w = H_2$.

Similarly, $T(H_2) \subseteq H_3$ and $T(H_3) \subseteq H_1$.

Now, let us show that $\delta(H_1,H_2,H_3) = \delta(X,Y,Z)$.

In fact, we show that (H_1,H_2,H_3) is a proximal triad. Let $u \in H_1 = \overline{T(K_2)}^w$.

Then there is a sequence $w_n = Ty_n \in T(K_2)$ such that $w_n \xrightarrow{w} w$.

Since K_2, K_3 is proximal there is a sequence $x_n \in K_1, z_n \in K_3$ such that $\Delta(x_n, y_n, z_n) = \delta(X,Y,Z)$.

Consider the sequence $\{u_n\}$and $\{v_n\}$ defined by $u_n := Tz_n \in TK_3 \subseteq X$.

Since Y is weakly compact, without loss of generality, we assume that $u_n \xrightarrow{w} u$, for some $u \in \overline{T(K_3)}^w = H_1$.

From the weak lower semi-continuity of the norm, we have

$$\begin{aligned} \delta(X,Y,Z) &\leq \delta(H_1,H_2,H_3) \leq \Delta(u,v,w) \leq \liminf \Delta(Tx_n, Ty_n, Tz_n) \\ &= \liminf \Delta(x_n, y_n, z_n) = \delta(X,Y,Z) \end{aligned}$$

Hence $\delta(H_1,H_2,H_3) = \delta(X,Y,Z)$.Thus $(H_1,H_2,H_3) \subseteq \mathscr{F}$. By the minimality of (K_1,K_2,K_3), we conclude that $H_1 = K_1$, $H_2 = K_2$ and $H_3 = K_3$.

Then,

(8)

$$\delta(K_1,K_2,K_3) = \delta(H_1,H_2,H_3) = \delta\left(\overline{T(K_1)}^w, \overline{T(K_2)}^w, \overline{T(K_3)}^w\right) = \delta(T(K_1), T(K_2), T(K_2)).$$

By the definition of T, we conclude that K_1, K_2 and K_3 are singleton sets.

Say

(9) $$K_1 = \{x'\}, K_2 = \{y'\} \text{ and } K_3 = \{z'\}.$$

Since $TK_1 \subseteq K_2$, $TK_2 \subseteq K_3$ and $TK_3 \subseteq K_1$ and $\delta(K_1,K_2,K_3) = \delta(X,Y,Z)$,

We arrived that:

(10) $$\Delta\left(x', Tx', T^2x'\right) = \Delta\left(y', Ty', T^2y'\right) = \Delta\left(z', Tz', T^2z'\right) = \delta(X,Y,Z).$$

From the above theorem, we obtained the following fixed point theorem of Xu for a diametrically contractive mapping, by taking $X = Y = Z$ $\square$

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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