FIXED POINTS OF KANNAN AND REICH INTERPOLATIVE CONTRACTIONS IN CONTROLLED METRIC SPACES
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Abstract: In this paper, we introduce \((\lambda, \alpha)\)-interpolative Kannan contraction, \((\lambda, \alpha, \beta)\)-interpolative Kannan contraction and \((\lambda, \alpha, \beta, \gamma)\)-interpolative Reich contraction. Also, we establish some fixed-point theorems in complete controlled metric spaces. Additionally, these theorems expand and apply a number of intriguing findings from metric fixed-point theory to the controlled metric setting.

Keywords: fixed-point; iterative method; interpolative; contraction; controlled metric space.

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1. INTRODUCTION AND PRELIMINARIES

The first fixed point theorem for rational contraction conditions in metric space was established by Dass and Gupta [26].

Theorem 1.1 (see [26]). Let \((X, d)\) be a complete metric space, and let \(T: X \to X\) be a self-mapping. If there exist \(\alpha, \beta \in [0, 1)\) with \(\alpha + \beta < 1\) such that

\[
d(Tx, Ty) \leq \alpha d(x, y) + \beta \frac{1 + d(x, Tx) + d(y, Ty)}{1 + d(x, y)}
\]

for all \(x, y \in X\), then \(T\) has a unique fixed point \(x^* \in X\).

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A genuine generalization of the Dass-Gupta fixed point theorem within the framework of dualistic partial metric spaces was demonstrated by Nazam et al. [27]. As generalizations of metric spaces, Czerwik [1] presented a new class of generalized metric spaces known as b-metric spaces.

**Definition 1 (see [1])** Let $X$ be a nonempty set and $s \geq 1$. A function $d_b: X \times X \rightarrow [0, \infty)$ is said to be a b-metric if for all $x, y, z \in X$,

(b1). $d_b(x, y) = 0$ iff $x = y$

(b2). $d_b(x, y) = d_b(y, x)$ for all $x, y \in X$

(b3). $d_b(x, z) \leq s \left[d_b(x, y) + d_b(y, z)\right]$

Then, we refer to the pair $(X, d_b)$ as a b-metric space. Many fixed-point findings on such spaces were subsequently obtained (see to [2–7]).

Extended b-metric spaces are a concept first introduced by Kamran et al. [8].

**Definition 2 (see [8])** Let $X$ be a nonempty set and $p: X \times X \rightarrow [1, \infty)$ be a function. A function $d_e: X \times X \rightarrow [0, \infty)$ is called an extended b-metric if for all $x, y, z \in X$,

(e1). $d_e(x, y) = 0$ iff $x = y$

(e2). $d_e(x, y) = d_e(y, x)$ for all $x, y \in X$

(e3). $d_e(x, z) \leq p(x, z)[d_e(x, y) + d_e(y, z)]$

The pair $(X, d_e)$ is called an extended b-metric space.

Mlaiki et al. have presented a novel type of generalized b-metric space [9].

**Definition 3 (see [9])** Let $X$ be a nonempty set and $p: X \times X \rightarrow [1, \infty)$ be a function. A function $d_c: X \times X \rightarrow [0, \infty)$ is called a controlled metric if for all $x, y, z \in X$,

(c1). $d_c(x, y) = 0$ iff $x = y$

(c2). $d_c(x, y) = d_c(y, x)$ for all $x, y \in X$

(c3). $d_c(x, z) \leq p(x, y)d_c(x, y) + p(y, z)d_c(y, z)$

The pair $(X, d_c)$ is called a controlled metric space (see also [10]).

**Definition 4 (see [9])** Let $(X, d_c)$ be a controlled metric space and $\{x_n\}_{n \geq 0}$ be a sequence in $X$. Then,
1. The sequence \( \{x_n\} \) converges to some \( x \) in \( X \) if for every \( \varepsilon > 0 \), there exists \( N = N(\varepsilon) \in \mathbb{N} \) such that \( d_c(x_n, x) < \varepsilon \) for all \( n \geq N \). In this case, we write \( \lim_{n \to \infty} x_n = x \).

2. The sequence \( \{x_n\} \) is Cauchy if for every \( \varepsilon > 0 \), there exists \( N = N(\varepsilon) \in \mathbb{N} \) such that \( d_c(x_n, x_m) < \varepsilon \) for all \( n, m \geq N \).

3. The controlled metric space \((X, d_c)\) is called complete if every Cauchy sequence is convergent.

**Definition 5 (see [9])** Let \((X, d_c)\) be a controlled metric space. Let \( x \in X \) and \( \varepsilon > 0 \).

1. The open ball \( B(x, \varepsilon) \) is
   \[
   B(x, \varepsilon) = \{ y \in X : d_c(y, x) < \varepsilon \}.
   \]

2. The mapping \( F: X \to X \) is said to be continuous at \( x \in X \) if for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( F(B(x, \varepsilon)) \subseteq B(Fx, \varepsilon) \).

This study aims to introduce a fixed-point theorem for \((\lambda, \alpha)\)-interpolative Kannan contraction, \((\lambda, \alpha, \beta)\)-interpolative Kannan contraction and \((\lambda, \alpha, \beta, \gamma)\)-interpolative Reich contraction in the context of complete controlled metric spaces. These theorems also extend and apply to the controlled metric environment several interesting results from metric fixed-point theory. Our result generalizes and extends some well-known results in the literature.

## 2. Main Result

We begin by defining the terms below.

**Definition 2.1** Let \((X, d_c)\) be a controlled metric space. Let \( F: X \to X \) be a self-map. We shall call \( F \) a \((\lambda, \alpha)\)-interpolative Kannan contraction, if there exist \( \lambda \in [0, 1) \) and \( \alpha \in (0, 1) \) such that
\[
    d_c(Fx, Fy) \leq \lambda d_c(x, Fx) + \alpha (d_c(y, Fy))^{1-\alpha}
\]
for all \( x, y \in X \), with \( x \neq y \).

**Definition 2.2** Let \((X, d_c)\) be a controlled metric space. Let \( F: X \to X \) be a self-map. We shall call \( F \) a \((\lambda, \alpha, \beta)\)-interpolative Kannan contraction, if there exist \( \lambda \in [0, 1) \), \( \alpha, \beta \in (0, 1) \) such that
\[
    d_c(Fx, Fy) \leq \lambda d_c(x, Fx) + \alpha (d_c(y, Fy)) + \beta (d_c(x, y))^{1-\alpha}
\]
for all \( x, y \in X \), with \( x \neq y \).
\[ d_c(Fx, Fy) \leq \lambda (d_c(x, Fx))^\alpha (d_c(y, Fy))^\beta \]  \hspace{1cm} (2.2)

for all \( x, y \in X \), with \( x \neq y \).

**Definition 2.3** Let \((X, d_c)\) be a controlled metric space. Let \(F: X \rightarrow X\) be a self-map. We shall call \(F\) a \((\lambda, \alpha, \beta, \gamma)\)-interpolative Reich contraction, if there exist \(\lambda \in [0, 1), \alpha, \beta, \gamma \in (0, 1), \alpha + \beta + \gamma < 1\) such that

\[ d_c(Fx, Fy) \leq \lambda (d_c(x, y))^\alpha (d_c(x, Fx))^\beta (d_c(y, Fy))^\gamma \]  \hspace{1cm} (2.3)

for all \( x, y \in X \), with \( x \neq y \).

Our first main result as follows.

**Theorem 2.4** Let \((X, d_c)\) be a complete controlled metric space. Let \(F: X \rightarrow X\) be a \((\lambda, \alpha)\)-interpolative Kannan contraction. For \(x_0 \in X\), take \(x_n = F^n x_0\). Assume that

\[ \sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{p(x_{i+1} - x_{i+2}) p(x_{i+2} - x_m)}{p(x_{i+1})} < \frac{1}{\lambda} \]  \hspace{1cm} (2.4)

Then \(F\) has a unique fixed point.

**Proof.** Let \(x_0 \in X\) be initial point. Define a sequence \(\{x_n\}\) as \(x_{n+1} = Fx_n, \forall n \in \mathbb{N}\). Obviously, if \( \exists n_0 \in \mathbb{N} \) for which \(x_{n_0+1} = x_{n_0}\), then \(F x_{n_0} = x_{n_0}\), and the proof is finished. Thus, we suppose that \(x_{n+1} \neq x_n\) for each \(n \in \mathbb{N}\). Thus, by (2.1), we have

\[ d_c(x_n, x_{n+1}) = d_c(Fx_{n-1}, Fx_n) \]

\[ \leq \lambda (d_c(x_{n-1}, Fx_{n-1}))^\alpha (d_c(x_n, Fx_n))^{1-\alpha} \]

\[ = \lambda (d_c(x_{n-1}, x_n))^\alpha (d_c(x_n, x_{n+1}))^{1-\alpha} \]

The last inequality gives

\[ d_c(x_n, x_{n+1})^\alpha \leq \lambda d_c(x_{n-1}, x_n)^\alpha \]  \hspace{1cm} (2.5)

Since \(\alpha < 1\), we have

\[ d_c(x_n, x_{n+1}) \leq \frac{1}{\lambda^\alpha} d_c(x_{n-1}, x_n) \leq \lambda d_c(x_{n-1}, x_n) \]

and then

\[ d_c(x_n, x_{n+1}) \leq \lambda d_c(x_{n-1}, x_n) \leq \lambda^2 d_c(x_{n-2}, x_n) \leq \cdots \leq \lambda^n d_c(x_0, x_1) \]  \hspace{1cm} (2.6)
For all \( n, m \in \mathbb{N} \) and \( n < m \), we have

\[
d_c(x_n, x_m) \leq p(x_n, x_{n+1})d_c(x_n, x_{n+1}) + p(x_{n+1}, x_m)d_c(x_{n+1}, x_m)
\]

\[
\leq p(x_n, x_{n+1})d_c(x_n, x_{n+1}) + p(x_{n+1}, x_m)p(x_{n+1}, x_{n+2})d_c(x_{n+1}, x_{n+2})
\]

\[
+ p(x_{n+1}, x_m)p(x_{n+2}, x_m)d_c(x_{n+2}, x_m)
\]

\[
\leq p(x_n, x_{n+1})d_c(x_n, x_{n+1}) + p(x_{n+1}, x_m)p(x_{n+1}, x_{n+2})d_c(x_{n+1}, x_{n+2})
\]

\[
+ p(x_{n+1}, x_m)p(x_{n+2}, x_m)p(x_{n+2}, x_{n+3})d_c(x_{n+2}, x_{n+3})
\]

\[
+ p(x_{n+1}, x_m)p(x_{n+2}, x_m)p(x_{n+3}, x_m)d_c(x_{n+3}, x_m)
\]

\[
\leq p(x_n, x_{n+1})d_c(x_n, x_{n+1})
\]

\[
+ \sum_{i=n+1}^{m-2} (\prod_{j=n+1}^{i} p(x_j, x_m)) p(x_i, x_{i+1})d_c(x_i, x_{i+1})
\]

\[
+ \prod_{i=n+1}^{m-1} p(x_j, x_m) d_c(x_{m-1}, x_m)
\]

(2.7)

This implies that

\[
d_c(x_n, x_m) \leq p(x_n, x_{n+1})d_c(x_n, x_{n+1})
\]

\[
+ \sum_{i=n+1}^{m-2} (\prod_{j=n+1}^{i} p(x_j, x_m)) p(x_i, x_{i+1})d_c(x_i, x_{i+1})
\]

\[
+ \prod_{i=n+1}^{m-1} p(x_j, x_m) d_c(x_{m-1}, x_m)
\]

\[
\leq p(x_n, x_{n+1})d_c(x_0, x_1)
\]

\[
+ \sum_{i=n+1}^{m-2} (\prod_{j=n+1}^{i} p(x_j, x_m)) p(x_i, x_{i+1})\lambda^i d_c(x_0, x_1)
\]

\[
+ \prod_{i=n+1}^{m-1} p(x_j, x_m) \lambda^{m-1} d_c(x_0, x_1)
\]

\[
\leq p(x_n, x_{n+1})\lambda^n d_c(x_0, x_1)
\]

\[
+ \sum_{i=n+1}^{m-1} (\prod_{j=n+1}^{i} p(x_j, x_m)) p(x_i, x_{i+1})\lambda^i d_c(x_0, x_1)
\]

(2.8)

Let

\[
\eta_r = \sum_{i=0}^{r} (\prod_{j=0}^{i} p(x_j, x_m)) p(x_i, x_{i+1})\lambda^i d_c(x_0, x_1)
\]

(2.9)

Consider

\[
\mu_i = \sum_{i=0}^{r} (\prod_{j=0}^{i} p(x_j, x_m)) p(x_i, x_{i+1})\lambda^i d_c(x_0, x_1)
\]

(2.10)
In view of condition (2.4) and the ratio test, we ensure that the series $\sum \mu_i$ converges. Thus,
$$\lim_{n \to \infty} \eta_n$$
exists. Hence, the real sequence $\{\eta_n\}$ is Cauchy. Now, using (2.6), we get
$$d_c(x_n, x_m) \leq d_c(x_0, x_1)[\lambda^n p(x_n, x_{n+1}) + (\eta_{m-1} - \eta_n)]$$
(2.11)
Above, we used $p(x, y) \geq 1$. Letting $n, m \to \infty$ in (2.11), we obtain
$$\lim_{n, m \to \infty} d_c(x_n, x_m) = 0$$
(2.12)
Thus, the sequence $\{x_n\}$ is Cauchy in the complete controlled metric space $(X, d_c)$. So, there is some $x^* \in X$. So that
$$\lim_{n \to \infty} d_c(x_n, x^*) = 0;$$
(2.13)
that is, $x_n \to x^*$ as $n \to \infty$. Now, we will prove that $x^*$ is a fixed point of $F$. By (2.1) and condition (c3), we get
$$d_c(x^*, Fx^*) \leq p(x^*, x_{n+1})d_c(x^*, x_{n+1}) + p(x_{n+1},Fx^*)d_c(x_{n+1},Fx^*)$$
$$= p(x^*, x_{n+1})d_c(x^*, x_{n+1}) + p(x_{n+1},Fx^*)d_c(Fx_n,Fx^*)$$
$$\leq p(x^*, x_{n+1})d_c(x^*, x_{n+1})$$
$$+p(x_{n+1},Fx^*) \left[\lambda(d_c(x_n, Fx_n))^\alpha (d_c(x^*,Fx^*))^{1-\alpha}\right]$$
$$\leq p(x^*, x_{n+1})d_c(x^*, x_{n+1})$$
$$+p(x_{n+1},Fx^*) \left[\lambda(d_c(x_n, x_{n+1}))^\alpha (d_c(x^*,Fx^*))^{1-\alpha}\right]$$
(2.14)
Taking the limit as $n \to \infty$ and using (2.10), (2.11) we obtain that
$$d_c(x^*, Fx^*) = 0$$
(2.15)
This yields that $x^* = Fx^*$. Now, we prove the uniqueness of $x^*$. Let $y^*$ be another fixed point of $F$ in $X$, then $Fy^* = y^*$. Now, by (2.1), we have
$$d_c(x^*, y^*) = d_c(Fx^*,Fy^*)$$
$$\leq \lambda(d_c(x^*,x^*))^\alpha (d_c(y^*,y^*))^{1-\alpha} = 0$$
(2.16)
This yields that $x^* = y^*$. It completes the proof.
**Theorem 2.5** Let \((X, d_c)\) be a complete controlled metric space. Let \(F: X \rightarrow X\) be a \((\lambda, \alpha, \beta)\)-interpolative Kannan contraction with (2.4) and for \(x_0 \in X\), \(x_n = F^n x_0\). Then \(F\) has a unique fixed point.

**Proof.** Following the steps of proof of Theorem 2.4, we construct the sequence \(\{x_n\}\) by iterating

\[
x_{n+1} = Fx_n, \quad \forall \ n \in \mathbb{N},
\]

where \(x_0 \in X\) is arbitrary starting point. Then, by (2.2), we have

\[
d_c(x_n, x_{n+1}) = d_c(Fx_{n-1}, Fx_n) \\
\leq \lambda (d_c(x_{n-1}, Fx_{n-1}))^\alpha (d_c(x_n, Fx_n))^\beta \\
= \lambda (d_c(x_{n-1}, x_n))^\alpha (d_c(x_n, x_{n+1}))^\beta
\]

Since \(\alpha < 1 - \beta\), the last inequality gives

\[
d_c(x_n, x_{n+1})^{1-\beta} \leq \lambda d_c(x_{n-1}, x_n)^\alpha \leq \lambda d_c(x_{n-1}, x_n)^{1-\beta} \tag{2.17}
\]

Hence

\[
d_c(x_n, x_{n+1}) \leq \lambda^{1-\beta} d_c(x_{n-1}, x_n) \leq \lambda d_c(x_{n-1}, x_n)
\]

and then

\[
d_c(x_n, x_{n+1}) \leq \lambda d_c(x_{n-1}, x_n) \leq \lambda^2 d_c(x_{n-2}, x_{n-1}) \leq \cdots \leq \lambda^n d_c(x_0, x_1) \tag{2.18}
\]

As already elaborated in the proof of Theorem 2.4, the classical procedure leads to the existence of a fixed-point \(x^* \in X\). Now, we prove the uniqueness of \(x^*\). Let \(y^*\) be another fixed point of \(F\) in \(X\), then \(Fy^* = y^*\). Now, by (2.2), we have

\[
d_c(x^*, y^*) = d_c(Fx^*, Fy^*) \\
\leq \lambda (d_c(x^*, x^*))^\alpha (d_c(y^*, y^*))^\beta = 0 \tag{2.19}
\]

This yields that \(x^* = y^*\). This completes the proof.

**Theorem 2.6** Let \((X, d_c)\) be a complete controlled metric space. Let \(F: X \rightarrow X\) be a \((\lambda, \alpha, \beta, \gamma)\)-interpolative Reich contraction and assume that (2.4) hold for \(x_0 \in X\) and \(x_n = F^n x_0\). Then \(F\) has a unique fixed point.

**Proof.** Following the steps of proof of Theorem 2.4, we construct the sequence \(\{x_n\}\) by iterating
\[ x_{n+1} = Fx_n, \forall \ n \in \mathbb{N}, \]

where \( x_0 \in X \) is arbitrary starting point. Then, by (2.2), we have
\[
d_c(x_n, x_{n+1}) = d_c(Fx_n, Fx_n) \\
\leq \lambda (d_c(x_{n-1}, x_n)^\alpha (d_c(x_{n-1}, Fx_{n-1}))^\beta (d_c(x_n, Fx_n))^\gamma \\
= \lambda (d_c(x_{n-1}, x_n)^{\alpha+\beta} (d_c(x_n, x_{n+1}))^\gamma \\
\]

Since \( \alpha + \beta < 1 - \gamma \), the last inequality gives
\[
d_c(x_n, x_{n+1})^{1-\gamma} \leq \lambda d_c(x_{n-1}, x_n)^{\alpha+\beta} \leq \lambda d_c(x_{n-1}, x_n)^{1-\gamma} \tag{2.20}
\]

Hence
\[
d_c(x_n, x_{n+1}) \leq \lambda^{1-\gamma} d_c(x_{n-1}, x_n) \leq \lambda d_c(x_{n-1}, x_n)
\]

and then
\[
d_c(x_n, x_{n+1}) \leq \lambda d_c(x_{n-1}, x_n) \leq \lambda^2 d_c(x_{n-2}, x_{n-1}) \leq \cdots \leq \lambda^n d_c(x_0, x_1) \tag{2.21}
\]

As already elaborated in the proof of Theorem 2.4, the classical procedure leads to the existence of a fixed-point \( x^* \in X \). Now, we prove the uniqueness of \( x^* \). Let \( y^* \) be another fixed point of \( F \) in \( X \), then \( Fy^* = y^* \). Now, by (2.3), we have
\[
d_c(x^*, y^*) = d_c(Fx^*, Fy^*) \\
\leq \lambda (d_c(x^*, y^*))^\alpha (d_c(x^*, x^*))^\beta (d_c(y^*, y^*))^\gamma = 0 \tag{2.22}
\]

This yields that \( x^* = y^* \). This completes the proof.

**Authors’ Contributions**

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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**Conflict of Interests**

The authors declare that there is no conflict of interests.
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