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SOME FIXED POINT RESULTS ON E-METRIC SPACES USING CONTRACTION OPERATORS

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Abstract: In this paper, we prove the generalizations of the Bianchini contraction and the Cirić-Reich-Rus contraction mappings fixed point results using the concept of convergence criteria for semi-interior points in E-metric spaces with non-solid and non-normal set of positive elements E^+ of real normed space E (positive cone). Additionally, many examples are included to illustrate the existence of semi-interior points of E^+ with empty interior. In addition, we present some applications in the field of applied mathematics that support our main findings.

Keywords: fixed point; semi-interior points; B-contraction; Bianchini contraction.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Fixed point theory (FPT) in ordered normed spaces has played an important role in computer science, economics, optimization theory, astronomy, dynamical systems, decision theory, parameter estimation, and many other subjects over a period of several decades. Large scale problems requiring fixed point theory are highly esteemed for their lightning-fast solutions. As a result, in recent years, many scholars have focused on developing FPT approaches and have provided various useful techniques for discovering FP in complex issues. Poincare [20],

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a French mathematician, developed the concept of FPT by utilising operators in an abstract topological form while studying nonlinear equations in the early 1800's. Initially, Liouville [23] created the sequential approximation method in 1837, and Picard introduced its logical technique in 1890. The father of FPT, mathematician Brouwer [6], proposed FP theorems for continuous mappings on finite dimensional spaces. In 1922, Banach [4] established and confirmed the renowned Banach contraction principle. Later, several authors used the Banach contraction principle (BCP) in numerous ways and presented numerous FP results [see, [8], [26], [27], [29], [31], [32], [33] & [34]]. The author Kurepa [17], explained the notions of K-metric spaces by replacing the set of all real numbers $\mathbb{R}$ by a Banach space to define the vector valued metric space. This concept was further generalized in 1980 by Rzepecki [22] and a FP theorem of Maia's type was proved.

Originally, the concept of cone metric space, in which convergent and Cauchy sequences were defined in terms of interior points in the ordered Banach space, was studied by Huang and Zhang [13]. In particular, they proved FP theorems for the Banach type, the Kannan type, and the Chatterjea type contractions and their related consequences. Also, they redefined and proposed the concept of K-metric spaces and convergence in an ordered Banach space E with normal solid cone P. After that, Hamlbarani [21] extended these results and showed the theorems without the assumptions of normality on cone P of the Banach space E. Following that, a high proportion of FP outcomes in cone metric spaces was seen. Further, in [1], the authors define E-metric spaces and characterized the cone metric spaces in more general way by defining ordered normed spaces. The notions of tvs-valued cone Banach spaces were introduced in 2014 by Mehmood et al. [18]. In all above results, the Banach space E were considered with defined order with respect to the positive solid cone E^+ of E, which means, by considering that the interior of E^+ is non-empty. Only few results could be found in which the non-solid cones were considered [[15], [16]]. In 2019, Mehmood et al. [24] proved some FP results in the frame of E-metric spaces by inserting non-solid cones, (also, refer to [25]). Subsequently, in [16], the quasi-interior points of P were considered instead of interior points in the case of non-solid cones.

Recently in 2017, the concept of semi-interior points was defined by Polyrakis et al. in [5]. According to Proposition 3.2 of [5], any semi-interior point of E^+ is also an interior point of E^+ with respect to the norm $||.||$ of E which coincides with the initial norm $||.||$ of E in E^+. So E^+ is $||.||$-normal if and only if E^+ is $||.||$-normal. So all the main results of this paper, can followed automatically by using the new norm $||.||$ of E, but a systematic way to prove the fixed point results is defined in the main results. The class of cones with semi-interior point and empty interior is a class of cones larger than the one with nonempty interior as the examples of [5]. It is worth noting that fixed points results for ordered normed spaces can also proved for this larger class of cones with semi-interior points. For more relevant results, see also [2], [3], [7], [9], [10], [11], and [12].

On the other hand, the first researchers to investigate a generalization of the Banach fixed point theorem while simultaneously using a contraction condition of the rational type were Dass and Gupta [28]. Jaggi [30], used a contraction condition of the rational type to prove a fixed point theorems in complete metric spaces. Moreover, rational contraction conditions have been heavily employed in both the FP and common FP locations.

The remaining parts of this manuscript are displayed as follows: In Section 2, we recall the notations, basic notions, and essential definitions needed throughout the paper. In Section 3, we prove the main concept related to fixed point results using the B-contraction, the Bianchini contraction, and some rational type contraction mappings in the setting of E metric spaces with non-solid and possibly non-normal cones. In Section 4, we present some applications in the field of applied mathematics related to the main findings of this paper. Finally, in Section 5, we reach a conclusion.

2. PRELIMINARIES

In this section, some notations, basic notions, essential definitions and lemmas from earlier works are recalled. Throughout in this article, let E be an ordered normed space with norm $\|.\|$, which is ordered by its positive cone E^+, such that for all $p,q \in E, p \leq q$ iff $q-p \in E^+$. Let E^* be the dual space of E. The following are the basics of ordered normed spaces and E-metric spaces.

Definition 2.1. [24] *An ordered space E is a vector space over the real numbers, with a partial order relation $\preceq$ such that*

(N1) *for all p,q and $r \in E$, $p \preceq q$ implies $p+r \preceq q+r$;*

(N2) *for all $a \in \mathbb{R}^+$ and $p \in E$ with $p \succeq 0_E$, $ap \succeq 0_E$.*

Moreover if E is equipped with norm $\|.\|$, then E is called normed ordered space.

Definition 2.2. [24] *[7]The positive cone E^+ of a normed ordered space X is called;*

(D1) normal, *if there exist a constant $M \succ 0$ such that for all $p,q \in E$ where $0 \preceq p \preceq q$ implies $\|p\| \preceq M\|q\|$.*

(D2) solid, *if E^+ has non-empty interior,*

(D3) reflexive, *iff $E^+ \cap U$ is weakly compact, where U is the unit ball in X,*

(D4) strongly reflexive, *iff $E^+ \cap U$ is compact.*

Definition 2.3. [24] *Let T be a non-empty set and let E be an ordered space, over the real scalars. An ordered E-metric on T is an E-valued mapping $d : T \times T \to E$ such that for all $p,q,r \in T$, the following hold:*

(CM1) *$d(p,q) \succeq 0$ and $d(p,q) = 0$ if and only if $p = q$;*

(CM2) *$d(p,q) = d(q,p)$;*

(CM3) *$d(p,q) \preceq d(p,r) + d(r,q)$.*

Then the pair (T,d) is called E-metric space.

Let us assume that $int(E^+)$ is non-empty. Now, we recall the the definitions of convergent and Cauchy sequence in an E-metric space.

Definition 2.4. [24] *Let E be a normed ordered space and (T,d) be an E-metric space, then the sequence $\{t_n\}$ in T is called convergent to a point $t \in T$ if for all $c \in int(E^+)$, there exists a natural number N such that*

$$d(t_n,t) \ll c,$$

for all $n \geq N$ and we write

$$\lim_{n\to\infty} t_n = t,$$

or simply $t_n \to t$. The sequence $\{t_n\}$ is a Cauchy, if for all $c \in int(E^+)$, there exist a natural number N_1 such that

$$d(t_n,t_m) \ll c,$$

for all $n,m \leq N_1$.

Next, we present the the notion of semi-interior points of the positive cone E^+ of an ordered space E and we define new convergence criteria. Further, we provide non-trivial examples from literature to ensure the applications of our presented notions.

Remark 2.5. *Let E be an ordered normed space ordered by the positive cone E^+, we shall denote by 0_E the zero of E,*

$$U = \{p \in E : \|.\| \leq 1\}$$

be the closed unit ball of E, and by U_+ mean the positive part of unit ball defined by the set

$$U_+ = U \cap E^+.$$

Definition 2.6. [24] *The point $p_0 \in E^+$ is a semi-interior point of E^+ if there exists a real number $\mu > 0$ such that*

$$p_0 - \mu U_+ \subseteq E^+.$$

Clearly any interior point of E^+ is semi-interior point.

Note that the set of all semi-interior points of E^+ is denoted by $(E^+)^{\ominus}$, and for $p,q \in E^+$, $p \lll q$ if and only if $q - p \in (E^+)^{\ominus}$. Moreover, the following example from [5] insure the existence of semi-interior points of cones having empty interior, these spaces are useful for linear and nonlinear optimization problems and operator analysis.

Example 2.7. [24] *This example clearly shown that a strong reflexive cone E^+ of $L_1([0,1])$ exist which generate a dense subspace T of $L_1([0,1])$ i.e.,*

$$L_1([0,1]) = E^+ - E^-$$

and

$$L_1([0,1]) = \bar{T}.$$

Let $V = co((B^+(0,1)) \cup (-B^+(0,1)))$ *and* E_1^+ *be a set of positive elements of* $L_1([0,1])$ *generated by the set* $\omega = 3q + V$*, where* $q = \sum_{k=1}^{\infty} \beta^{k-1} e_k$ *for* $\beta \in (0,1)$*, where* $\{e_i\}$ *is the set of standard normalized basis. It has been shown in [5], that the cone* E^+ *has the empty interior but has semi-interior points.*

For more examples related to the positive cone with semi-interior points but empty interior point, we refer the readers to [5]. Also, it is clear that, if $int(E^+)$ is nonempty then every interior point of the cone E^+ is the semi-interior point. The following lemma is the converse in complete, ordered normed spaces.

Lemma 2.8. [24] *If* E *is the complete ordered normed space with generating and closed cone* E^+*, then any semi-interior point of* E^+ *is an interior point of* E^+*.*

Now we define the e-convergence and e-Cauchy convergence criteria in the ordered normed space E, with non-solid cone E^+.

Definition 2.9. [24] *Let* E *be a ordered normed space with assumption that* $(E^+)^{\ominus}$ *is nonempty and* (T,d) *be an* E*-metric space. Let* $\{p_n\}$ *be a sequence in* T *and* $p \in T$*. Then;*
(i) *A sequence* $\{p_n\}$ *is e-convergence to some* p *whenever for every* $e \ggg 0$*, there exists a natural number* k *such that* $d(p_n, p) \lll e$ *for all* $n \geq k$*. We denote this by* $\lim_{n\to\infty} p_n = p$ *or* $p_n \to p$*.*
(ii) *A sequence* $\{p_n\}$ *is e-Cauchy, if for every* $e \in E$ *with* $0 \lll e$ *there is natural number* N *such that for all* $m, n > N, d(p_n, p_m) \lll e$*.*
(iii) (T,d) *is said to be e-complete if every e-Cauchy sequence is e-convergent.*

3. Main Results

In this section, we prove some FP by using various contraction mappings such as the Bianchini contraction and the Ćirić-Reich-Rus contraction and their related consequences on E-metric spaces. For that, assume (K,d) be a e-complete e-metric space. Firstly, we consider the Bianchini contraction mapping to prove our main FP theorem.

Theorem 3.1. *Let* (K,d) *be an e-complete e-metric space with closed positive cone* E^+ *such that* $(E^+)^{\ominus} \neq \phi$*. Let* $T : K \to K$ *be a mapping satisfying the Bianchini contraction condition,* *i.e.,*

$$d(Tp,Tq) \leq ed(p,q),$$

where $d(p,q) = max\{d(p,Tp), d(q,Tq)\}$*, for all* $p,q \in K$ *and* $e \in [0,1)$. *Then* T *has a unique* FP *in* K *and for each* $p \in K$*, the iterative sequence* $\{T^n p\}_{n\geq 0}$ *converges to the* FP p *of* T.

Proof. For any $p_0 \in K$, let us take the iterative sequence $p_{n+1} = Tp_n = T^n p_0$ with $p_n \neq p_{n+1}$ for some $n \in \mathbb{N}$.

Case 1. *Consider* $d(p,q) = d(p,Tp)$. *Then, we have*

$$\begin{aligned} d(p_n, p_{n+1}) &= d(Tp_{n-1}, Tp_n) \\ &\preceq ed(p_{n-1}, Tp_{n-1}) \\ &\preceq ed(p_{n-1}, p_n) \\ &\preceq ed(Tp_{n-2}, Tp_{n-1}) \\ &\preceq e^2 d(p_{n-2}, Tp_{n-2}) \\ &\preceq e^2 d(p_{n-2}, p_{n-1}) \\ &\preceq \dots \\ &\preceq e^n d(p_0, p_1) \end{aligned}$$

Case 2. *Consider* $d(p,q) = d(q,Tq)$. *Then, we have*

$$\begin{aligned} d(p_n, p_{n+1}) &= d(Tp_{n-1}, Tp_n) \\ &\preceq ed(p_n, Tp_n) \\ &\preceq ed(p_{n-1}, p_n) \\ &\preceq ed(p_n, p_{n+1}) \end{aligned}$$

which is impossible because $e \in [0,1)$. Therefore, **Case 2** does not exist. Now, by **Case 1** and $n \succ m$, we get

$$
\begin{aligned}
d(p_m,p_n) &\preceq d(p_m,p_{m+1})+d(p_{m+1},p_{m+2})+...+d(p_{n-1},p_n)\\
&\preceq (\mu^m+\mu^{m+1}+...+\mu^{n-1})d(p_0,p_1)\\
&=\mu^m(1+\mu+\mu^2+....+\mu^{n-m-1})d(p_0,p_1)\\
&=\mu^m\left(\frac{1-\mu^{n-m}}{1-\mu}\right)d(p_0,p_1)
\end{aligned}
$$

Let $e \ggg 0$ be given, choose $\nu > 0$ such that $e-\nu U_+ \subseteq E^+$ and a natural number k_1 such that $\mu^m\left(\frac{1-\mu^{n-m}}{1-\mu}\right)d(p_0,p_1) \in \frac{\mu}{2}U_+$ for any $m,n \geq k_1$, therefore $e-\frac{\mu^m}{1-\mu}d(p_0,p_1)-\frac{\mu}{2} \subseteq e-\nu U_+ \subseteq E^+$, hence $d(p_m,p_n) \leq \mu^m\left(\frac{1-\mu^{n-m}}{1-\mu}\right)d(p_0,p_1) \lll e$, for all $m,n \geq k_1$ which implies $\{p_n\}$ is an e-cauchy sequence, since K is e-complete so there exists some $p \in K$ such that $p_n \to p$. For a given $e \ggg 0_E$, choose $k_2 \in \mathbb{N}$, such that $d(p,p_n) \lll \frac{e}{2}$, for all $n \geq k_2$. Consider for all $n \geq k_2$,

$$
\begin{aligned}
d(p,Tp) &\leq d(p,p_n)+d(p_n,Tp)\\
&\leq d(p,p_n)+\nu d(p,p_{n-1})\\
&\lll e
\end{aligned}
$$

Since $d(p,Tp) \lll \frac{e}{m}$ for any $\frac{e}{m} \ggg 0_E$ and $m \in \mathbb{N}$, therefore $\frac{e}{m}-d(p,Tp) \in E^+$, for all $m \in \mathbb{N}$, which implies $-d(p,Tp) \in E^+$, but $d(p,Tp) \in E^+$, therefore $d(p,Tp) = 0_E$. Hence $p = Tp$. Let $q \in k$ be such that $p \neq q = Tq$, then consider

$$
\begin{aligned}
d(p,q) &= d(Tp,Tq)\\
&\leq \mu d(p,q)
\end{aligned}
$$

which implies $d(p,q) = 0_E$. This proves the theorem. □

Corollary 3.2. *Let (K,d) be an e-complete e-metric space with closed positive cone E^+ such that $(E^+)^{\ominus} \neq \phi$. Let $T : K \to K$ be a mapping satisfying the contraction condition, i.e.,*

$$d(Tp,Tq) \leq ed(p,Tp),$$

for all $p,q \in K$ and $e \in (0,1)$. Then T has a unique FP in K and for each $p \in K$, the iterative sequence $\{T^n p\}_{n\geq 0}$ converges to the FP p of T.

Proof. Substituting $d(p,q) = d(p,Tp)$ in Theorem 3.1 completes this corollary. □

Secondly, we consider another contraction mapping known as the Cirić-Reich-Rus contraction to prove our next main FP theorem.

Theorem 3.3. *Let (K,d) be an e-complete e-metric space with closed positive cone E^+ such that $(E^+)^\ominus \neq \phi$. Let $T : K \to K$ be a mapping satisfying the condition*

$$d(Tp,Tq) \leq e_1 d(p,q) + e_2(d(p,Tp) + d(q,Tq)),$$

for all $p,q \in K$ and some $e_1, e_2 \in [0,1)$ with $e_1 + 2e_2 < 1$. Then T has a unique FP in K and for each $p \in K$, the iterative sequence $\{T^n p\}_{n\geq 0}$ converges to the FP p of T.

Proof. For any $p_0 \in K$, let us take the iterative sequence $p_{n+1} = Tp_n = T^n p_0$ with $p_n \neq p_{n+1}$ for some $n \in \mathbb{N}$. Consider,

$$\begin{aligned}
d(p_n, p_{n+1}) &= d(Tp_{n-1}, Tp_n) \\
&\preceq e_1 d(p_{n-1}, p_n) + e_2(d(p_{n-1}, Tp_{n-1}) + d(p_n, Tp_n)) \\
&\preceq e_1 d(p_{n-1}, p_n) + e_2 d(p_{n-1}, p_n) + e_2 d(p_n, p_{n+1}) \\
&= \mu d(p_{n-1}, p_n), \text{ where } \mu = \frac{e_1 + e_2}{1 - e_2} \\
&\preceq \mu^2 d(p_{n-2}, p_{n-1}) \\
&\dots \\
&\preceq \mu^n d(p_0, p_1)
\end{aligned}$$

Now for $n \succ m$, consider,

$$\begin{aligned}
d(p_m, p_n) &\preceq d(p_m, p_{m+1}) + d(p_{m+1}, p_{m+2}) + \dots + d(p_{n-1}, p_n) \\
&\preceq (\mu^m + \mu^{m+1} + \dots + \mu^{n-1}) d(p_0, p_1)
\end{aligned}$$

$$= \mu^m(1+\mu+\mu^2+....+\mu^{n-m-1})d(p_0,p_1)$$
$$= \mu^m\left(\frac{1-\mu^{n-m}}{1-\mu}\right)d(p_0,p_1)$$

Let $e \ggg 0$ be given, choose $\nu > 0$ such that $e - \nu U_+ \subseteq E^+$ and a natural number k_1 such that $\mu^m\left(\frac{1-\mu^n}{1-\mu}\right)d(p_0,p_1) \in \frac{\mu}{2}U_+$ for any $m,n \geq k_1$, therefore $e - \frac{\mu^m}{1-\mu}d(p_0,p_1) - \frac{\mu}{2} \subseteq e - \nu U_+ \subseteq E^+$, hence $d(p_m,p_n) \leq \mu^m\left(\frac{1-\mu^n}{1-\mu}\right)d(p_0,p_1) \lll e$, for all $m,n \geq k_1$ which implies $\{p_n\}$ is an e-cauchy sequence, since K is e-complete so there exists some $p \in K$ such that $p_n \to p$. For a given $e \ggg 0_E$, choose $k_2 \in \mathbb{N}$, such that $d(p,p_n) \lll \frac{e}{2}$, for all $n \geq k_2$. Consider for all $n \geq k_2$,

$$d(p,Tp) \leq d(p,p_n) + d(p_n,Tp)$$
$$\leq d(p,p_n) + \nu d(p,p_{n-1})$$
$$\lll e$$

Since $d(p,Tp) \lll \frac{e}{m}$ for any $\frac{e}{m} \ggg 0_E$ and $m \in \mathbb{N}$, therefore $\frac{e}{m} - d(p,Tp) \in E^+$, for all $m \in \mathbb{N}$, which implies $-d(p,Tp) \in E^+$, but $d(p,Tp) \in E^+$, therefore $d(p,Tp) = 0_E$. Hence $p = Tp$. Let $q \in k$ be such that $p \neq q = Tq$, then consider

$$d(p,q) = d(Tp,Tq)$$
$$\leq \mu d(p,q)$$

which implies $d(p,q) = 0_E$. This proves the theorem. □

Corollary 3.4. *Let (K,d) be an e-complete e-metric space with closed positive cone E^+ such that $(E^+)^\ominus \neq \phi$. Let $T : K \to K$ be a mapping satisfying the condition*

$$d(Tp,Tq) \leq e_1 d(p,q),$$

for all $p,q \in K$ and some $e_1 \in (0,1)$. Then T has a unique FP in K and for each $p \in K$, the iterative sequence $\{T^n p\}_{n\geq 0}$ converges to the FP p of T.

Proof. Substituting $e_2 = 0$ in Theorem 3.2 completes this corollary. □

Corollary 3.5. *Let (K,d) be an e-complete e-metric space with closed positive cone E^+ such that $(E^+)^\ominus \neq \phi$. Let $T : K \to K$ be a mapping satisfying the condition*

$$d(Tp,Tq) \leq e_2(d(p,Tp)+d(q,Tq)),$$

for all $p,q \in K$ and some $e_2 \in (0,1)$. Then T has a unique FP in K and for each $p \in K$, the iterative sequence $\{T^n p\}_{n\geq 0}$ converges to the FP p of T.

Proof. Substituting $e_1 = 0$ in Theorem 3.2 completes this corollary. □

Next, we prove the same kind of result by using rational-type contraction mapping to find the existence and uniqueness of the FP.

Theorem 3.6. *Let (K,d) be an e-complete e-metric space with closed positive cone E^+ such that $(E^+)^\ominus \neq \phi$. Let $T : K \to K$ be a mapping satisfying the condition*

$$d(Tp,Tq) \leq \frac{e_1 d(q,Tq)[1+d(p,Tp)]}{1+d(p,q)} + e_2 d(p,q),$$

for all $p,q \in K$ and some $e_1,e_2 \in [0,1)$ with $2e_1+e_2 < 1$. Then T has a unique FP in K and for each $p \in K$, the iterative sequence $\{T^n p\}_{n\geq 0}$ converges to the FP p of T.

Proof. For any $p_0 \in K$, let us take the iterative sequence $p_{n+1} = Tp_n = T^n p_0$ with $p_n \neq p_{n+1}$ for some $n \in \mathbb{N}$. Consider,

$$\begin{aligned}
d(p_n,p_{n+1}) &= d(Tp_{n-1},Tp_n) \\
&\preceq \frac{e_1 d(p_n,Tp_n)[1+d(p_{n-1},Tp_{n-1})]}{1+d(p_{n-1},Tp_{n-1})} + e_2 d(p_{n-1},p_n) \\
&\preceq e_1 d(p_n,p_{n+1}) + e_2 d(p_{n-1},p_n) \\
&= \mu d(p_{n-1},p_n), \text{ where } \mu = \frac{e_2}{1-e_1} \\
&\preceq \mu^2 d(p_{n-2},p_{n-1}) \\
&\dots \\
&\preceq \mu^n d(p_0,p_1)
\end{aligned}$$

Now for $n \succ m$, consider,

$$\begin{aligned} d(p_m,p_n) &\preceq d(p_m,p_{m+1})+d(p_{m+1},p_{m+2})+...+d(p_{n-1},p_n) \\ &\preceq (\mu^m+\mu^{m+1}+...+\mu^{n-1})d(p_0,p_1) \\ &= \mu^m(1+\mu+\mu^2+....+\mu^{n-m-1})d(p_0,p_1) \\ &= \mu^m\left(\frac{1-\mu^{n-m}}{1-\mu}\right)d(p_0,p_1) \end{aligned}$$

Let $e \ggg 0$ be given, choose $\nu > 0$ such that $e-\nu U_+ \subseteq E^+$ and a natural number k_1 such that $\mu^m\left(\frac{1-\mu^{n-m}}{1-\mu}\right)d(p_0,p_1) \in \frac{\mu}{2}U_+$ for any $m,n \geq k_1$, therefore $e-\frac{\mu^m}{1-\mu}d(p_0,p_1)-\frac{\mu}{2} \subseteq e-\nu U_+ \subseteq E^+$, hence $d(p_m,p_n) \leq \mu^m\left(\frac{1-\mu^{n-m}}{1-\mu}\right)d(p_0,p_1) \lll e$, for all $m,n \geq k_1$ which implies $\{p_n\}$ is an e-cauchy sequence, since K is e-complete so there exists some $p \in K$ such that $p_n \to p$. For a given $e \ggg 0_E$, choose $k_2 \in \mathbb{N}$, such that $d(p,p_n) \lll \frac{e}{2}$, for all $n \geq k_2$. Consider for all $n \geq k_2$,

$$\begin{aligned} d(p,Tp) &\leq d(p,p_n)+d(p_n,Tp) \\ &\leq d(p,p_n)+d(Tp_{n-1},Tp) \\ &\leq d(p,p_n)+\frac{e_1d(p,Tp)[1+d(p_{n-1},Tp_{n-1})]}{1+d(p_{n-1},p)}+e_2d(p_{n-1},p) \\ &\leq d(p,p_n)+\frac{e_1d(p,Tp)[1+d(p_{n-1},p_n)]}{1+d(p_{n-1},p)}+e_2d(p_{n-1},p) \\ &\lll e \end{aligned}$$

Since $d(p,Tp) \lll \frac{e}{m}$ for any $\frac{e}{m} \ggg 0_E$ and $m \in \mathbb{N}$, therefore $\frac{e}{m}-d(p,Tp) \in E^+$, for all $m \in \mathbb{N}$, which implies $-d(p,Tp) \in E^+$, but $d(p,Tp) \in E^+$, therefore $d(p,Tp) = 0_E$. Hence $p = Tp$. Let $q \in k$ be such that $p \neq q = Tq$, then consider

$$\begin{aligned} d(p,q) &= d(Tp,Tq) \\ &\leq \mu^n d(p,q) \end{aligned}$$

which implies $d(p,q) = 0_E$. This proves the theorem. □

4. Applications

The FP covers a wide range of applications in the field of mathematics, particularly differential geometry, numerical analysis, and so on. By reading [35] and the references therein, one can find a variety of applications involving FP results in the field of applied mathematics. The examples below demonstrate how to apply FP findings in differential equations.

Example 4.1. *Let $T = C([0,1],\mathbb{R})$ and T is e-complete e-metric space defined by $d(p,q) = \sup_{t\in[0,1]} |p-q|^2$. Also, consider $y''(t) = 3y^2(t)/2$, $0 \le t \le 1$ and the initial conditions $y(0) = 4$, $y(1) = 1$. Here, the exact solution is $y(t) = 4/(1+t)^2$. We have, $y_0(t) = c_1 t + c_2$. By using the initial conditions, we get $y_0(t) = 4-3t$. Now, define the integral operator,*

$$A(y) = y + \int_0^1 G(t,s)[y'' - f(s,y,y')]ds \tag{4.1}$$

where

$$G(t,s) = \begin{cases} s(1-t) & 0 \le s \le t \\ t(1-s) & t \le s \le 1 \end{cases}$$

Then, the equation (4.1) becomes

$$\begin{aligned} A(y) &= y(t) + \int_0^1 G(t,s)y''(s)ds - \int_0^1 G(t,s)f(s,y,y')ds \\ &= (4-3t) - \int_0^1 G(t,s)[-3/2y^2(s)]ds \\ &= 4 - 3t + \frac{3}{2}\left\{\int_0^1 G(t,s)y^2(s)ds\right\} \end{aligned}$$

Consider,

$$\begin{aligned} d(Ap,Aq) &= \sup_{t\in[0,1]} |Ap - Aq|^2 \\ &= \sup_{t\in[0,1]} \left|\frac{3}{2}\int_0^1 G(t,s)p^2(s)ds - \frac{3}{2}\int_0^1 G(t,s)q^2(s)ds\right|^2 \\ &\le \frac{9}{4}\left(\int_0^1 |G(t,s)|^2 ds\right)\left(\int_0^1 |p^2(s) - q^2(s)|^2 ds\right) \\ &\le \frac{3}{4}\frac{t^2(1-t)^2}{3}\int_0^1 \left|p^2(s) - q^2(s)\right|^2 ds \end{aligned}$$

$$\leq \frac{3}{4}(\frac{1}{4})(\frac{1}{4})\int_0^1 \left|p^2(s) - q^2(s)\right|^2 ds$$
$$\leq \frac{3}{64}\sup_{t\in[0,1]} |p(s) - q(s)|^2$$
$$\leq \frac{3}{64}d(p,q)$$

Then, we have

$$d(Tp,Tq) \leq e_1 d(p,q) + e_2(d(p,Tp) + d(q,Tq)).$$

Thus, $e_1 = 3/64$ *and* $e_2 = 0$ *satisfies all the conditions of Theorem3.3. Also, by Theorem3.3, A has* FP *in* $T = C([0,1],\mathbb{R})$. *Therefore, the given bounded value problem has* FP *in* T.

5. Conclusion

This paper has introduced some new FP theorems that are applicable to both contraction and rational contraction operators on E-metric spaces. In particular, going in the same direction as [24], we provide the results in the setting of contraction mappings, namely the Bianchini contraction type, Ciric-Reich-Rus contraction type, and their consequences. Additionally, we provide the FP theorems by using the rational contraction mapping, which were discussed mostly in [28] and [30]. In order to confirm the presence of the FP theorems, alternative discoveries presented in the later can be demonstrated in a lower environment.

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Author Contributions

All authors contributed equally, read and approved the final manuscript.

Conflict of Interests

The authors declare that there is no conflict of interests.

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