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MATHEMATICAL ANALYSIS OF THE DYNAMICS OF A FRACTIONAL ECONOMIC CYCLE MODEL AND THE EXISTENCE OF SOLUTIONS BY MEANS OF FIXED POINT THEORY

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Abstract. This paper proposes a new fractional business cycle model with general investment and variable depreciation rate involving the generalized Hattaf fractional (GHF) derivative. The existence of the model solutions is rigorously established using fixed point theory. Additionally, the mathematical analysis of the dynamics of the proposed fractional model including the stability of economic equilibrium is fully investigated. Furthermore, numerical simulations are presented to illustrate our theoretical results.

Keywords: fixed point theory; economic cycle; depreciation rate; Hattaf fractional derivative; stability.

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1. Introduction

Economic cycles also named business cycles that are the recurring fluctuations in economic activity characterized by expansions, peaks, contractions and troughs. Such economic cycles

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reflect the fluctuations in real gross domestic product (GDP), employment rates, industrial production, and other economic indicators over time. Many factors contribute to economic cycles like the changes in consumer spending, investment levels, government policies, technological advancements and global economic conditions.

Various mathematical models have been proposed and developed to understand the dynamics of the economic cycle. In 1940, Kaldor [1] introduced the first model to investigate the business cycle using nonlinear dynamics which proved that the main reason for the fluctuations is an investment. The results shown in [1] proved by Chang and Smyth [2] in 1971, using mathematical theory and gave some additional conditions needed for the existence of limit cycles. In 2020, Mao and Liu [3] proposed a Kaldor business cycle model with variable depreciation rate of capital stock. In [4], the authors proposed a delayed business cycle model with variable depreciation rate of stock capital that extends the model presented in [3].

In economics, memory refers to the collective and historical knowledge, experiences, information that society has accumulated over time. This includes the knowledge of past economic events, policies and practices, as well as the cultural and social norms that shape economic behavior. Therefore, memory is an important factor in the economics, as it shapes decision-making, influences behavior, and can help to promote economic growth and development. However, all the above cited models neglected the memory effect by considering only integer-order derivative. Therefore, we propose a tool for describing business cycles with long-term memory using the new generalized Hattaf fractional (GHF) derivative which is a non-local operator and it has a non-singular kernel formulated by the Mittag-Leffler function with a parameter different to the order of the fractional derivative [5], which covers the most famous fractional derivatives with non-singular kernels. The GHF derivative was used by many authors to model the dynamics of many phenomena arising from various fields of science and engineering [6, 7, 8].

The rest of this paper is outlined as follows. The next section is devoted to the formulation of the model and some interesting preliminaries needed to the elaboration of this study. Section 3 presents the existence and uniqueness of the solution by means of fixed point theory. Section 4 focuses on the existence of economic equilibrium and stability analysis. Finally, some numerical simulations are given in Section 5.

2. Preliminary Results and Model Formulation

In this section, we first recall the definition of the GHF derivative and its proprieties necessary for elaboration of this study. After, we present our fractional economic cycle model.

Definition 2.1. [5] *Let $p \in [0,1)$, $q,\gamma > 0$, and $f \in H^1(a,b)$. The GHF derivative of order p in the Caputo sense of the function $f(t)$ with respect to the weight function $\omega(t)$ is defined as follows:*

$$D^{p,q,\gamma}_{a,t,\omega} f(t) = \frac{N(p)}{1-p} \frac{1}{\omega(t)} \int_a^t E_q[-\mu_p (t-\tau)^\gamma] \frac{d}{d\tau}(\omega f)(\tau) d\tau, \tag{1}$$

where $\omega \in C^1(a,b)$, $\omega > 0$ on $[a,b]$, $N(p)$ is a normalization function such that $N(0) = N(1) = 1$, $\mu_p = \frac{p}{1-p}$ and $E_q(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(qk+1)}$ is the Mittag-Leffler function of parameter q.

The GHF derivative introduced in the above definition generalizes and extends many special cases. In the fact, when $\omega(t) = 1$ and $q = \gamma = 1$, we get the Caputo-Fabrizio fractional derivative [9], which is given by

$$^{C}D^{p,1,1}_{a,t,1} f(t) = \frac{N(p)}{1-p} \int_a^t \exp[-\mu_p (t-\tau)] f'(\tau) d\tau.$$

We obtain the Atangana-Baleanu fractional derivative [10] when $\omega(t) = 1$ and $q = \gamma = p$, which is given by

$$^{C}D^{p,p,p}_{a,t,1} f(t) = \frac{N(p)}{1-p} \int_a^t E_p[-\mu_p (t-\tau)^p] f'(\tau) d\tau.$$

For $q = \gamma = p$, we get the weighted Atangana-Baleanu fractional derivative [11], which is given by

$$^{C}D^{p,p,p}_{a,t,\omega} f(t) = \frac{N(p)}{1-p} \frac{1}{\omega(t)} \int_a^t E_p[-\mu_p (t-\tau)^p] \frac{d}{d\tau}(\omega f)(\tau) d\tau.$$

For simplicity, we denote $^{C}D^{p,q,q}_{a,t,\omega}$ by $\mathscr{D}^{p,q}_{a,\omega}$. According to [5], the generalized Hattaf fractional integral operator associated to $\mathscr{D}^{p,q}_{a,\omega}$ is defined by

$$\mathscr{I}^{p,q}_{a,\omega} f(t) = \frac{1-p}{N(p)} f(t) + \frac{p}{N(p)} \, {}^{RL}\mathscr{I}^{q}_{a,\omega} f(t), \tag{2}$$

where ${}^{RL}\mathscr{I}^{q}_{a,\omega}$ is the standard weighted Riemann-Liouville fractional integral of order q defined by

$$ {}^{RL}\mathscr{I}^{q}_{a,\omega}f(t) = \frac{1}{\Gamma(q)}\frac{1}{\omega(t)}\int_a^t (t-\tau)^{q-1}\omega(\tau)f(\tau)d\tau. \tag{3} $$

Theorem 2.2. [5] *Let $p \in [0,1]$, $q > 0$ and $f \in H^1(a,b)$. Then we have the following property:*

$$ \mathscr{I}^{p,q}_{a,\omega}(\mathscr{D}^{p,q}_{a,\omega}f)(t) = f(t) - \frac{\omega(a)f(a)}{\omega(t)}. \tag{4} $$

Theorem 2.3. [5] *The Laplace transform of $\omega(t)\mathscr{D}^{p,q}_{0,\omega}$ is given by*

$$ \mathscr{L}\{\omega(t)\mathscr{D}^{p,q}_{0,\omega}f(t)\}(s) = \frac{N(p)}{1-p}\frac{s^q\mathscr{L}\{\omega(t)f(t)\}(s) - s^{q-1}\omega(0)f(0)}{s^q+\mu_p}. $$

Corollary 2.4. [12] *Let $q > 0$, $x(t)$, $u(t)$ be nonnegative functions and $v(t) = M \geq 0$ with $N(p) - (1-p)M > 0$. If*

$$ x(t) \leq u(t) + M\mathscr{I}^{p,q}_{0,\omega}x(t), $$

then

$$ x(t) \leq \frac{N(p)}{N(p)-(1-p)M}\left[u(t) + \int_0^t \sum_{n=1}^{+\infty}\frac{(pM)^n(t-\tau)^{nq-1}u(\tau)}{\Gamma(nq)\,[N(p)-(1-p)M]^n}d\tau\right]. $$

Furthermore, if in addition $u(t)$ is a nondecreasing function on $[0,T]$, we have

$$ x(t) \leq \frac{N(p)u(t)}{N(p)-(1-p)M}E_q\left(\frac{pMt^q}{N(p)-(1-p)M}\right). $$

Now, we propose the following business cycle involving GHF derivative:

$$ \begin{cases} \mathscr{D}^{p,q}_{0,\omega}Y(t) = \alpha[I(Y(t),K(t)) - \gamma Y(t)], \\ \mathscr{D}^{p,q}_{0,\omega}K(t) = I(Y(t),K(t)) - \delta(K(t))K(t), \end{cases} \tag{5} $$

subject to the given the initial condition:

$$ \begin{cases} Y(0) = Y_0, \\ K(0) = K_0. \end{cases} \tag{6} $$

Here, the state variables $Y(t)$ and $K(t)$ denote the gross product and capital stock at time t, respectively. Further, the coefficient $\gamma \in (0,1)$ is the saving constant. The investment function is represented by $I(Y,K)$ and it is assumed to be continuously differentiable in $\mathbb{R}^2$ with $\frac{\partial I}{\partial Y} > 0$ and $\frac{\partial I}{\partial K} < 0$. The adjustment coefficient in the goods market is denoted by α. Finally, $0 < \delta(K) < 1$ is the depreciation rate depending on K.

3. The Existence and Uniqueness of the Solution

In this section, we investigate the existence and uniqueness of solutions of system (5) by means of fixed point theory.

Let $\mathscr{C} = C([0,b],\mathbb{R}^2)$ be the Banach space of continuous functions g from $[0,b]$ into $\mathbb{R}^2$ equipped with the sup-norm

$$\|g\|_{\mathscr{C}} = \sup_{t\in[0,b]} ||g(t)||.$$

The system (5) can be written as follows:

$$\begin{cases} \mathscr{D}^{p,q}_{0,\omega} Z(t) = F(t,Z(t)), \\ Z(0) = Z_0, \end{cases} \tag{7}$$

where $Z(t) = (Y(t),K(t))^T$, $Z_0 = (Y(0),K(0))^T$ and the vector function F is given by $F = \begin{pmatrix} F_1 \\ \\ F_2 \end{pmatrix} = \begin{pmatrix} \alpha[I(Y,K)-\gamma Y] \\ \\ I(Y,K)-\delta(K)K \end{pmatrix}.$

Lemma 3.1. *$Z(t)$ is a solution of (7) if and only if $Z(t)$ satisfies the following integral equation:*

$$Z(t) = \frac{\omega(0)Z_0}{\omega(t)} + \mathscr{I}^{p,q}_{0,\omega} F(t,Z(t)). \tag{8}$$

Proof. Let $Z(t)$ be solution of (7). Applying the Hattaf fractional integral to both sides of (5), we get

$$Z(t) = \frac{\omega(0)Z(0)}{\omega(t)} + \mathscr{I}^{p,q}_{0,\omega} F(t,Z(t)). \tag{9}$$

As $Z(0) = Z_0$, we have (8).

Now, we assume that $Z(t)$ satisfies (8). Then

$$Z(0) = \frac{\omega(0)Z_0}{\omega(0)} + \mathscr{I}^{p,q}_{0,\omega} F(0,Z(0)) = Z_0 + \mathscr{I}^{p,q}_{0,\omega} F(0,Z(0)).$$

For $F(0,Z(0)) = 0$, we obtain $Z(0) = Z_0$. Since Z(t) satisfies (8), we have

$$\mathscr{D}^{p,q}_{0,\omega} Z(t) = \mathscr{D}^{p,q}_{0,\omega}\left(\frac{\omega(0)Z_0}{\omega(t)}\right) + F(t,Z(t)) = 0 + F(t,Z(t)).$$

Hence, $Z(t)$ satisfies (8). $\square$

To prove that F is Lipschitz in its second variable, we assume that the depreciation rate $\delta(K)$ satisfies the following hypothesis

(H_1): There exists a constant $m_1 > 0$ such that

$|K_1\delta(K_1) - K_2\delta(K_2)| \leq m_1|K_1 - K_2|, \; \forall K_1, K_2 \in \mathbb{R}.$

Then we have the following lemma.

Lemma 3.2. *Assume that (H_1) holds. Then the vector F is Lipschtiz in its second variable.*

Proof. We have

$$
\begin{aligned}
&||F(t,Z_1(t)) - F(t,Z_2(t))|| \\
&= |F_1(t,Z_1(t)) - F_1(t,Z_2(t))| + |F_2(Z_1(t)) - F_2(Z_2(t))| \\
&= \alpha|I(Y_1(t),K_1(t)) - \gamma Y_1(t) - I(Y_2(t),K_2(t)) + \gamma Y_2(t)| \\
&+ |I(Y_1(t),K_1(t)) - \delta(K_1(t))K_1(t) - I(Y_2(t),K_2(t)) + \delta(K_2(t))K_2(t)| \\
&\leq (\alpha+1)|I(Y_1(t),K_1(t)) - I(Y_2(t),K_2(t))| + \gamma\alpha|Y_1(t) - Y_2(t)| \\
&+ |\delta(K_1(t))K_1(t) - \delta(K_2(t))K_2(t)| \\
&\leq (\alpha+1)(L_1|Y_1(t) - Y_2(t)| + L_2|K_1(t) - K_2(t)|) + \gamma\alpha|Y_1(t) - Y_2(t)| \\
&+ |\delta(K_1(t))K_1(t) - \delta(K_2(t))K_2(t)|,
\end{aligned}
$$

with $L_1 = \sup\limits_{t\in[0,b]} |\frac{\partial I(Y(t),K_2(t))}{\partial Y}|$ and $L_2 = \sup\limits_{t\in[0,b]} |\frac{\partial I(Y_1(t),K(t))}{\partial K}|$.
Hence, the Lipschitz condition holds and F satisfies

$$||F(t,Z_1(t)) - F(t,Z_2(t))|| \leq L||Z_1(t) - Z_2(t)||, \tag{10}$$

where $L = \max\{(\alpha+1)L_1 + \alpha\gamma, (\alpha+1)L_2 + m_1|\}$. □

Theorem 3.3. *Assume that $L < \frac{N(p)}{1-p}$. If Z and X are two solutions of (7), then $Z = X$. This implies the uniqueness of solution.*

Proof. Let X and Z are two solutions of (7). Based on Lemma 3.1, we get

$$Z(t) - X(t) = \mathscr{I}^{p,q}_{0,\omega}\left(F(t,Z(t) - F(t,X(t))\right).$$

According to Lemma 3.2, we deduce that

$$|Z(t)-X(t)| \leq L\mathscr{I}_{0,\omega}^{p,q}|Z(t)-X(t)|.$$

Using Corollary 2.4, we have

$$|Z(t)-X(t)| \leq \frac{N(p)\times 0}{N(p)-(1-p)L}E_q\left(\frac{pLt^q}{N(p)-(1-p)L}\right).$$

This implies that $Z(t)=X(t)$ for all $t\in[0,b]$. □

Theorem 3.4. *If $L(\frac{1-p}{N(p)}+\frac{b^q}{\Gamma(q+1)})<1$, then system (7) has a unique solution for any initial condition.*

Proof. We consider the operator $\Upsilon:\mathscr{C}\to\mathscr{C}$ as follows

$$(\Upsilon Z)(t)=\frac{\omega(0)Z(0)}{\omega(t)}+\mathscr{I}_{0,\omega}^{p,q}F(t,Z(t)),\ t\in[0,b]$$

According to Lemma 3.1, it suffices to prove that the operator Υ has a unique fixed point. We first prove that Υ is well defined. We have

$$\begin{aligned}|(\Upsilon Z)(t)| &= |\frac{\omega(0)Z_0}{w(t)}+\mathscr{I}_{0,\omega}^{p,q}F(t,Z(t))| \\ &\leq |Z_0|\frac{\omega(0)}{\omega(t)}+\mathscr{I}_{0,\omega}^{p,q}|F(t,Z(t))|.\end{aligned}$$

As $\omega(0)<\omega(t)$ for all $t\geq 0$, F is Lipschitz continuous and $t\leq b$, we deduce that$|F(Z(t))|$ is bounded by constant ξ and

$$\begin{aligned}|(\Upsilon Z)(t)| &\leq |Z_0|+\xi\mathscr{I}_{0,w}^{p,q}(1) \\ &\leq |Z_0|+\xi\left(\frac{1-p}{N(p)}+\frac{pb^q}{N(p)\Gamma(q+1)}\right),\end{aligned}$$

which implies that the operator is well defined. Therefore, for all $Z_1,Z_2\in\mathscr{C}$ and $t\in[0,b]$, we have

$$\begin{aligned}|(\Upsilon Z_1)(t)-(\Upsilon Z_2)(t)| &= |\mathscr{I}_{0,w}^{p,q}F(t,Z_1(t))-F(t,Z_2(t))| \\ &\leq |\frac{1-p}{N(p)}(F(t,Z_1(t))-F(t,Z_2(t)))+\frac{p}{N(p)} \\ &\quad {}^{RL}\mathscr{I}_{0,w}^{q}(F(t,Z_1(t))-F(t,Z_2(t)))|\end{aligned}$$

$$\leq \frac{1-p}{N(p)}L|Z_1 - Z_2| + \frac{p}{N(p)}L||Z_1 - Z_2||_{\mathscr{C}} \frac{t^q}{\Gamma(q+1)}.$$

As a result

$$||\Upsilon Z_1 - \Upsilon Z_2||_{\mathscr{C}} \leq L\left(\frac{1-p}{N(p)} + \frac{pb^q}{N(p)\Gamma(q+1)}\right)||Z_1 - Z_2||_{\mathscr{C}}.$$

Since $L\left(\frac{1-p}{N(p)} + \frac{pb^q}{N(p)\Gamma(q+1)}\right) < 1$, we deduce that Υ is a contraction mapping. Hence, by applying the Banach contraction mapping principle, system (7) has a unique solution □

4. The Economic Equilibrium and Its Stability

In the order to investigate the existence of equilibria of (5), we consider the following hypotheses:

(H_2): There exists two constants $A > 0$ and $\bar{q} \geq 0$ such that $|I(Y,K) + \bar{q}K| \leq A$ for all $Y, K \in \mathbb{R}$.

(H_3): There exists a $\delta_1 > 0$ such that $\delta(K) \geq \delta_1$ for all $K \in \mathbb{R}$.

(H_4): $I(0,0) > 0$.

(H_5): $\frac{\delta'(K)K+\delta(K)}{\gamma}\frac{\partial I}{\partial Y} - \frac{\gamma\delta'(K)K+\gamma\delta(K)}{\gamma} + \frac{\partial I}{\partial K} < 0$ for all $Y,\ K \in \mathbb{R}$.

Theorem 4.1. *If $(H_2)-(H_5)$ hold, then system (5) has a unique economic equilibrium of the form $E^*(\frac{\delta(K^*)K^*}{\gamma}, K^*)$, where K^* is the unique solution of the equation $I(\frac{\delta(K)K}{\gamma}, K) - \delta(K)K = 0$.*

Proof. Economic equilibrium is the solution of the following system:

$$\left\{\begin{array}{l} \alpha[I(Y,K) - \gamma Y] = 0, \\ I(Y,K) - \delta(K)K = 0. \end{array}\right. \tag{11}$$

Then

$$Y = \frac{\delta(K)K}{\gamma}. \tag{12}$$

Replacing (12) in (11), we find

$$I\left(\frac{\delta(K)K}{\gamma}, K\right) - \delta(K)K = 0.$$

Let V be the function defined on the interval $[0, +\infty)$ by

$$V(K) = I\left(\frac{\delta(K)K}{\gamma}, K\right) - \delta(K)K.$$

Using assumptions $(H_1)-(H_4)$, we have $V(0)=I(0,0)>0$,

$\lim_{K\to+\infty} V(K)=-\infty$ and

$$V'(K)=\frac{\delta'(K)K+\delta(K)}{\gamma}\frac{\partial I}{\partial Y}-\frac{\gamma\delta'(K)K+\gamma\delta(K)}{\gamma}+\frac{\partial I}{\partial K}<0.$$

Therefore, there is a unique economic equilibrium $E^*(Y^*,K^*)$, where K* is the solution of the equation $V(K)=0$ and $Y^*=\frac{\delta(K^*)K^*}{\gamma}$. □

Next, we establish stability analysis of the economic equilibrium. Let $y=Y-Y^*$ and $k=K-K^*$. By substituting y and k into system (5) and linearizing, we get the following system

$$\left\{\begin{array}{l} D^{p,q}_{0,\omega}y(t)=\alpha[ay(t)+\beta k(t)-\gamma y(t)],\\ D^{p,q}_{0,\omega}k(t)=ay(t)+\beta k(t)-\bar{\delta}k(t),\end{array}\right. \tag{13}$$

where $a=\frac{\partial I}{\partial Y}(Y^*,K^*)$, $\beta=\frac{\partial I}{\partial K}(Y^*,K^*)$ and $\bar{\delta}=K^*\delta'(K^*)+\delta(K^*)$.

By applying the Laplace transform to system (13), we obtain

$$\Delta(s).\begin{pmatrix}\tilde{Y}(s)\\ \tilde{K}(s)\end{pmatrix}=\begin{pmatrix}b_1(s)\\ b_2(s)\end{pmatrix},$$

where $\tilde{Y}(s)=\mathscr{L}\{\omega(t)y(t)\}$, $\tilde{K}(s)=\mathscr{L}\{\omega(t)k(t)\}$,

$$\left\{\begin{array}{l} b_1(s)=s^{q-1}N(p)\omega(0)y(0),\\ b_2(s)=s^{q-1}N(p)\omega(0)k(0),\end{array}\right.$$

and

$$\Delta(s)=\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix},$$

with

$$\begin{aligned}
x_1 &= s^q\left[N(p)-\alpha(a-\gamma)(1-p)\right]-\alpha\mu_p(a-\gamma)(1-p),\\
x_2 &= -\left[\alpha\beta s^q(1-p)+\mu_p\alpha\beta(1-p)\right],\\
x_3 &= -\left[as^q(1-p)+a\mu_p(1-p)\right],\\
x_4 &= s^q\left[N(p)-(\beta-\bar{\delta})(1-p)\right]-\mu_p(\beta-\bar{\delta})(1-p).
\end{aligned}$$

Thus, the characteristic equation about E^* is given by

$$a_0 s^{2p} + a_1 s^p + a_2 = 0, \tag{14}$$

where

$$\begin{aligned}
a_0 &= [N(p) - \alpha(a-\gamma)(1-p)][N(p) - (\beta - \bar{\delta})(1-p)] - \alpha\beta a(1-p)^2,\\
a_1 &= -\mu_p(1-p)\{N(p)[(\beta-\bar{\delta}) + \alpha(a-\gamma)] - 2\alpha\},\\
a_2 &= -a\mu_\alpha^2 \bar{\delta}(1-\alpha)^2(I_Y - \gamma).
\end{aligned}$$

Theorem 4.2. *The economic equilibrium E^* is locally asymptotically stable if and only if*

$$a - \gamma < \min\left\{ \frac{N(p)}{\alpha(1-p)} - \frac{\beta a(1-p)}{N(p) - (\beta-\bar{\delta})(1-p)}, \frac{2\gamma(1-p)(\gamma\bar{\delta} - a\bar{\delta} - \gamma\beta)}{N(p)} + \frac{\bar{\delta}-\beta}{\alpha}, \frac{-a\beta}{\bar{\delta}-\beta} \right\}. \tag{15}$$

Proof. Let $s^\alpha = \lambda$ and substitute it into (14), we have

$$a_0\lambda^2 + a_1\lambda + a_2 = 0. \tag{16}$$

Based on Routh-Hurwitz criterion, all the roots of equation (16) have negative real parts if and only if

$$a_0 > 0, \;\; a_1 > 0 \;\; and \;\; a_2 > 0.$$

Therefore, the economic equilibrium E^* is locally asymptotically stable. □

Remark 4.3. *If $p \to 1^-$, then we obtain the same result presented in* [4] *for the case without delays.*

5. Numerical Simulation

In this section, we present some numerical simulations to illustrate our theoretical results. Let $t_n = n\Delta t$, with $n \in \mathbb{N}$. Using the numerical method proposed in [13] to discretizate fractional business cycle model (5). Hence, we get the following discrete model

$$\begin{cases}
Y(t_{n+1}) &= \frac{Y_0\omega(0)}{\omega(t_n)} + \frac{1-p}{N(p)}F_1(t_n, Z(t_n)) + \frac{p(\Delta t)^q}{N(p)\Gamma(q+2)\omega(t_n)} \sum\limits_{k=0}^{n} (\omega(t_k)F_1(t_k, Z(t_k))\mathscr{A}_{n,k,q}\\
&\quad +\omega(t_{k-1})F_1(t_{k-1}, Z(t_{k-1}))\mathscr{B}_{n,k,q}),\\
K(t_{n+1}) &= \frac{K_0\omega(0)}{\omega(t_n)} + \frac{1-p}{N(p)}F_2(t_n, Z(t_n)) + \frac{p(\Delta t)^q}{N(p)\Gamma(q+2)\omega(t_n)} \sum\limits_{k=0}^{n} (\omega(t_k)F_2(t_k, Z(t_k))\mathscr{A}_{n,k,q}\\
&\quad +\omega(t_{k-1})F_2(t_{k-1}, Z(t_{k-1}))\mathscr{B}_{n,k,q}),
\end{cases} \tag{17}$$

where

$\mathscr{A}_{n,k,q} = (n-k+1)^q(n-k+2+q)-(n-k)^q(n-k+2+2q)$,

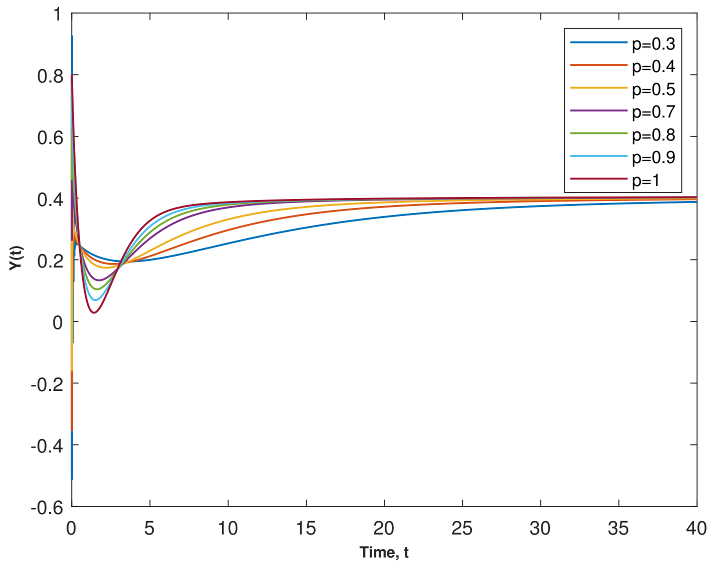
$\mathscr{B}_{n,k,q} = (n-k)^q(n-k+1+q)-(n-k+1)^{q+1}$.

For the simulation, we choose $N(p) = 1-p+\frac{p}{\Gamma(p)}$ and we consider $I(Y,K) = \frac{e^Y}{1+e^Y}+\frac{cK}{\sqrt{1+\varepsilon K^2}}$, where $c<0$, $\varepsilon \geq 0$. When $\varepsilon = 0$, $I(Y,K)$ is the Kaldor-type investment function. The depreciation rate function is chosen as $\delta(K) = \delta_1 + \frac{\delta_0-\delta_1}{1+K}$, where $0<\delta_1<\delta_0$. Let $\alpha = 3$, $c=-0.5$, $\varepsilon = 0.01$, $\delta_1 = 0.2$ and $\delta_0 = 0.3$.

By simple calculation, we get $a = 0.2399$, $\beta = 0.8914$ and $a-\gamma = -0.2601$. By applying Theorem 4.2, the stability of the economic equilibrium $E^*(0.4085, 0.7992)$ depend on the memory effect. When we change values of p. For instance,

- $p = 0.3$, for this choice the economic equilibrium $E^*(0.4085, 0.7992)$ is stable if and only if $a-\gamma < \min\{-0.2562, -0.1786, 0.3237\}$.
- $p = 0.5$, for this choice the economic equilibrium $E^*(0.4085, 0.7992)$ is stable if and only if $a-\gamma < \min\{-0.1063, -0.215, 0.3237\}$.
- $p = 0.8$, for this choice the economic equilibrium $E^*(0.4085, 0.7992)$ is stable if and only if $a-\gamma < \min\{0.0070, -0.2202, 0.3237\}$.

Figure 1 shows the impact of memory effect on the dynamical behaviors of model (5) for different values of the parameter p.



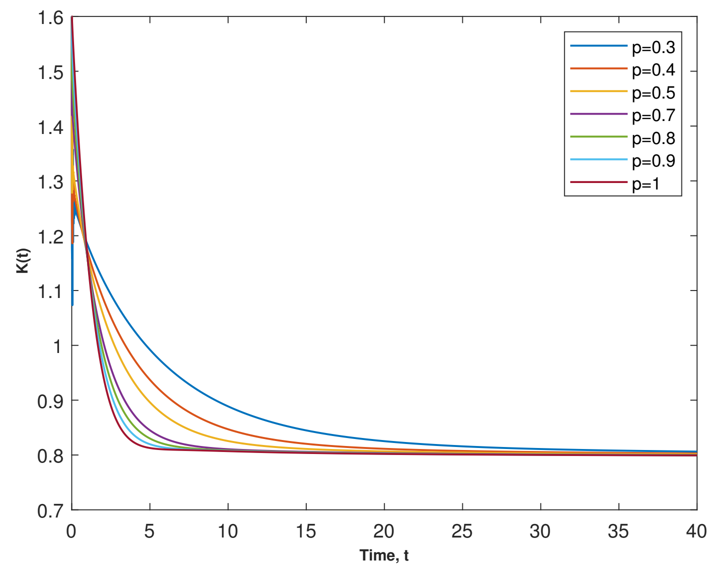


FIGURE 1. The impact of memory effect on the dynamics of model (5).

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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