Available online at http://scik.org
Adv. Fixed Point Theory, 2024, 14:7
https://doi.org/10.28919/afpt/8373
ISSN: 1927-6303

COMMON FIXED POINT THEOREMS FOR $(\phi, \mathfrak{F})$-CONTRACTION MAPPINGS IN CONE b-METRIC SPACES OVER BANACH ALGEBRA WITH APPLICATION

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Abstract. In this study, the notion of the common fixed point theorem on contraction mappings is established using the class functions. The primary result is a generalization of the common fixed point theorems for $(\phi, \mathfrak{F})$-contraction mappings on cone b-metric spaces over Banach algebra $\mathfrak{A}$. Investigated are the common fixed points criteria for existence and uniqueness. Suitable examples are provided an illustrate the validity of our results. At the end of this article the existence of common solution for a class of functional equations arising in dynamic programming are demonstrated with the help of our main results.

Keywords: cone b-metric space; common fixed point; Banach algebra.

2020 AMS Subject Classification: 47H10, 54H25, 54M20.

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Received December 01, 2023

1. Introduction

In 1981, the concept of quasimetric spaces (b-metric spaces) was introduced by Vulpe et all [27]. A brief overview of the early developments in fixed point theory on b-metric spaces is provided in [27], along with some significant related issues that Berinde and Pacurar are investigating [9]. Czerwik [11],[10] developed the idea of b-metric space, which expanded on conventional metric spaces. Huang and Zhang [16] generalized the concept of metric spaces and introduced cone metric space. They replaced the set of real numbers to real Banach space. Recently, many articles have discussed the results on cone metric spaces being identical to results on ordinary metric spaces. Finally, Liu and Xu [22] introduced the concept of cone metric spaces over Banach algebras and proved Banach contraction principle in the setting of cone metric spaces over Banach algebras. The authors presented some fixed point theorems of generalized Lipschitz mappings in the new setting without the assumption of normality, which are not equivalent to metric spaces in terms of the existence of the fixed points of the mappings. Khan et al. [19] used a new technique to prove fixed point theorems on metric space by altering distances between the points employing suitably equipped continuous control functions. Ansari [4] introduced the notion of C-class function as a major generalization of Banach contraction principle and obtained some fixed point results. Dhamodharan and Krishnakumar [12] generalized the results of common fixed point of four mappings with contractive modulus on cone Banach space and also presented cone c-class function with common fixed point theorems for cone b-metric space in 2017 [13]. In this direction several authors further established fixed point results in cone metric spaces (see [17, 2, 14]). In 2023, Maheshwaran and Jahir Hussain [23] developed fixed point theorem for $(\phi,\mathfrak{F})$-multi-valued mappings in cone b-metric spaces over Banach algebra and also worked on fixed point theorem for $(\phi,\mathfrak{F})$-expansive mappings in cone b-metric spaces over Banach algebra in 2023 [23], [24]. The purpose of this paper, we using the class functions and, we provide a common fixed point theorem for contraction mappings. A generalisation of the common fixed point theorem for $(\phi,\mathfrak{F})$- contraction mappings on cone b-metric spaces over Banach algebra is presented in the main theorem. Our results generalized and improve the results in [15].

Let $\mathfrak{A}$ be a real Banach algebra, i.e. $\mathfrak{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties:

For all $\eta,\zeta,\upsilon \in \mathfrak{A}, \mathfrak{d} \in \mathfrak{R}$

(i) $\eta(\zeta\varepsilon) = (\eta\zeta)\varepsilon$;

(ii) $\eta(\zeta+\varepsilon) = \eta\zeta+\eta\varepsilon$ *and* $(\eta+\zeta)\varepsilon = \eta\varepsilon+\zeta\varepsilon$;

(iii) $\mathfrak{d}(\eta\zeta) = (\mathfrak{d}\eta)\zeta = \eta(\mathfrak{d}\zeta)$;

(iv) $\| \eta\zeta \| \leq \| \eta \| \| \zeta \|$.

We shall assume that the Banach algebra $\mathfrak{A}$ has a unit, i.e., a multiplicative identity e such that $e\eta = \eta e = \eta$ for all $\eta \in \mathfrak{A}$. An element $\eta \in \mathfrak{A}$ is said to be invertible if there is an inverse element $\zeta \in \mathfrak{A}$ such that $\eta\zeta = \zeta\eta = e$. The inverse of η is denoted by η^{-1}.

Let $\mathfrak{A}$ be a real Banach algebra with a unit e and $\eta \in \mathfrak{A}$. If the spectral radius $\rho(\eta)$ of η is less than 1, that is

$$\rho(\eta) = \lim_{\mathfrak{n}\to+\infty} \| \eta^{\mathfrak{n}} \|^{\frac{1}{\mathfrak{n}}} = \inf_{n\succcurlyeq 1} \| \eta^{\mathfrak{n}} \|^{\frac{1}{\mathfrak{n}}} < 1$$

then $e-\eta$ is invertible. Actually,

$$(e-\eta)^{-1} = \sum_{i=0}^{+\infty} \eta_i.$$

A subset $\mathfrak{P}$ of $\mathfrak{A}$ is called a cone of $\mathfrak{A}$ if

i. $\{\theta, e\} \subset \mathfrak{P}$,

ii. $\mathfrak{P}^2 = \mathfrak{P}\mathfrak{P} \subset \mathfrak{P}$, $\mathfrak{P} \cap (-\mathfrak{P}) = \{\theta\}$,

iii. $\mathfrak{h}\mathfrak{P} + \beta\mathfrak{P} \subset \mathfrak{P}$, for all $\mathfrak{h},\beta \in \mathfrak{R}$.

For a given cone $\mathfrak{P} \subset \mathfrak{A}$, we define a partial ordering $\preccurlyeq$ with respect to $\mathfrak{P}$ by $\eta \preccurlyeq \zeta$ if and only if $\zeta - \eta \in \mathfrak{P}$. We shall write $\eta \prec \zeta$ to indicate that $\eta \preccurlyeq \zeta$ but $\eta \neq \zeta$, while $\eta \ll \zeta$ will indicate that $\zeta - \eta \in int\mathfrak{P}$, where $int\mathfrak{P}$ denotes the interior of $\mathfrak{P}$. If $int\mathfrak{P} \neq \varnothing$, then $\mathfrak{P}$ is called a solid cone. Write $\| \ . \|$ as the norm of $\mathfrak{A}$. A cone $\mathfrak{P}$ is called normal if there is a number $M > 0$ such that for all $\eta,\zeta \in \mathfrak{A}$, we have

$$\theta \preccurlyeq \eta \preccurlyeq \zeta \text{ implies } \| \eta \| \leq M \| \zeta \| .$$

The least positive number satisfying above is called the normal constant of $\mathfrak{P}$.

In the following we suppose that $\mathfrak{A}$ is a real Banach algebra with a unit $e, \mathfrak{P}$ is a solid cone in $\mathfrak{A}$, and $\preccurlyeq$ is a partial ordering with respect to $\mathfrak{P}$.

2. Preliminaries

Firstly, we recall several basic ideas in one b-metric spaces and Banach algebra that are necessary for the following sections.

Lemma 2.1. [22] *If $\mathfrak{E}$ is a real Banach space with a cone $\mathfrak{P}$ and if $\mathfrak{d} \leq \beta\mathfrak{d}$ with $\mathfrak{d} \in \mathfrak{P}$ and $0 \leq \beta < 1$, then $\mathfrak{d} = \theta$.*

Lemma 2.2. [22] *If $\mathfrak{E}$ is a real Banach space with a solid cone $\mathfrak{P}$ and if $\theta \leq \mathfrak{u} \ll \mathfrak{c}$ for each $\theta \ll \mathfrak{c}$, then $\mathfrak{u} = \theta$.*

Lemma 2.3. [22] *Let $\mathfrak{P}$ be a cone in a Banach algebra $\mathfrak{A}$ and $\mathfrak{K} \in \mathfrak{P}$ be a given vector. Let $\{\mathfrak{u}_\mathfrak{n}\}$ be a sequence in $\mathfrak{P}$. If for all $\mathfrak{c}_1 \gg \theta$, there exist(s) $\mathfrak{N}_1$ such that $\mathfrak{u}_\mathfrak{n} \ll \mathfrak{c}_1$ for all $\mathfrak{n} > \mathfrak{N}_1$, then for all $\mathfrak{c}_2 \gg \theta$, there exist(s) $\mathfrak{N}_2$ such that $\mathfrak{K}\,\mathfrak{u}_\mathfrak{n} \ll \mathfrak{c}_2$ for all $\mathfrak{n} > \mathfrak{N}_2$.*

Lemma 2.4. [22] *If $\mathfrak{E}$ is a real Banach space with a solid cone $\mathfrak{P}$ and $\{\eta_\mathfrak{n}\} \subset \mathfrak{P}$ is a sequence with $\| \eta_\mathfrak{n} \| \to 0(\mathfrak{n} \to +\infty)$, then for all $\theta \ll \mathfrak{c}$, there exist(s) $\mathfrak{N} \in \mathbb{N}$ such that $\mathfrak{n} > \mathfrak{N}$ we have, $\eta_\mathfrak{n} \ll \mathfrak{c}$ i.e., $\{\eta_\mathfrak{n}\}$ is a $\mathfrak{c}$-sequence.*

Lemma 2.5. [25] *Let $\mathfrak{A}$ be a Banach algebra with a unit $e, \mathfrak{i}, \mathfrak{j} \in \mathfrak{A}$. If $\mathfrak{i}$ commutes with $\mathfrak{j}$, then $\rho(\mathfrak{i}+\mathfrak{j}) \leq \rho(\mathfrak{i}) + \rho(\mathfrak{j})$, $\rho(\mathfrak{i}\mathfrak{j}) \leq \rho(\mathfrak{i})\rho(\mathfrak{j})$.*

Remark 2.1. [25] *If $\rho(\eta) < 1$, then $\| \eta_n \| \to 0$ as $\mathfrak{n} \to +\infty$.*

Definition 2.1. [18] *Let X be a non-empty set, $\Omega \geq 1$ be a constant and $\mathfrak{A}$ be a Banach algebra. A function $D_b : X \times X \to \mathfrak{A}$ is said to be a cone $\mathfrak{b}$-metric provide that, for all $\eta,\ \zeta,\ \varepsilon \in X$,*

(d1) $D_b(\eta,\ \zeta) = 0$ if and only if $\eta = \zeta$;

(d2) $D_b(\eta,\ \zeta) = D_b(\zeta,\ \eta)$;

(d3) $D_b(\eta,\ \varepsilon) \preccurlyeq \Omega[D_b(\eta,\ \zeta) + D_b(\zeta,\ \varepsilon)]$.

A pair $(X,\ D_b)$ is called a cone $\mathfrak{b}$-metric space over Banach algebra $\mathfrak{A}$.

Example 2.1. [22] *Let $\mathfrak{A}=C[\mathfrak{d},\mathfrak{b}]$ be the set of continuous functions on the interval $[\mathfrak{d},\mathfrak{b}]$ with the supermum norm. Define multiplication in the usual way. Then $\mathfrak{A}$ is a Banach algebra with a unit 1. Set $\mathfrak{P}=\{\eta\in\mathfrak{A}:\eta(t)\geq 0,\ \mathfrak{t}\in[\mathfrak{d},\mathfrak{b}]\}$ and $X=\mathfrak{R}$. Defined a mapping $D_b:X\times X\to\mathfrak{A}$ by $D_b(\eta,\zeta)(\mathfrak{t})=|\eta-\zeta|^{\mathfrak{p}}e^{\mathfrak{t}}$ for all $\eta,\zeta\in X$, where $\mathfrak{p}>1$ is a constant. This makes (X,D_b) into a cone $\mathfrak{b}$-metric space over Banach algebra $\mathfrak{A}$ with the coefficient $\mathfrak{b}=2^{\mathfrak{p}-1}$, but it is not a cone metric space over Banach algebra since the triangle inequality is not satisfied.*

Definition 2.2. [24] *Let (X,D_b) be a cone $\mathfrak{b}$-metric space over Banach algebra $\mathfrak{A},\eta\in X$, let $\{\eta_{\mathfrak{n}}\}$ be a sequence in X. Then*

i. $\{\eta_{\mathfrak{n}}\}$ converges to η whenever for every $\mathfrak{c}\in\mathfrak{A}$ with $\theta\ll\mathfrak{c}$ there is natural number $\mathfrak{n}_0$ such that $D_b(\eta_{\mathfrak{n}},\eta)\ll\mathfrak{c}$, for all $\mathfrak{n}\geq\mathfrak{n}_0$. We denote this by $\lim_{\mathfrak{n}\to+\infty}\eta_{\mathfrak{n}}=\eta$.

ii. $\{\eta_{\mathfrak{n}}\}$ is a Cauchy sequence whenever for every $\mathfrak{c}\in\mathfrak{A}$ with $\theta\ll\mathfrak{c}$ there is natural number $\mathfrak{n}_0$ such that $D_b(\eta_{\mathfrak{n}},\eta_{\mathfrak{m}})\ll\mathfrak{c}$, for all $\mathfrak{n},\mathfrak{m}\geq\mathfrak{n}_0$.

iii. $\{\eta,D_b\}$ is complete cone $\mathfrak{b}$-metric if every Cauchy sequence in X is convergent.

Lemma 2.6. [18] *Let $\mathfrak{E}$ be a real Banach space with a solid cone $\mathfrak{P}$*

1) If $\mathfrak{d}_1,\mathfrak{d}_2,\mathfrak{d}_3\in\mathfrak{E}$ and $\mathfrak{d}_1\preccurlyeq\mathfrak{d}_2\ll\mathfrak{d}_3$, then $\mathfrak{d}_1\ll\mathfrak{d}_3$.

2) If $\mathfrak{d}_1\in\mathfrak{P}$ and $\mathfrak{d}_1\ll\mathfrak{d}_3$ for each $\mathfrak{d}_3\gg\theta$, then $\mathfrak{d}_1=\theta$.

Lemma 2.7. [18] *Let $\mathfrak{P}$ be a solid cone in a Banach algebra $\mathfrak{A}$. Suppose that $\mathfrak{h}\in\mathfrak{P}$ and $\{\eta_{\mathfrak{n}}\}\subset\mathfrak{P}$ is a $\mathfrak{c}$-sequence. Then $\{\mathfrak{h}\eta_{\mathfrak{n}}\}$ is a $\mathfrak{c}$-sequence.*

Lemma 2.8. [18] *Let $\mathfrak{A}$ be a Banach algebra with a unit $e,\xi\in\mathfrak{A}$, then $\lim_{\mathfrak{n}\to+\infty}\|\xi^{\mathfrak{n}}\|^{\frac{1}{\mathfrak{n}}}$ exists and the spectral radius $\rho(\xi)$ satisfies*

$$\rho(\xi)=\lim_{\mathfrak{n}\to+\infty}\|\xi^{\mathfrak{n}}\|^{\frac{1}{\mathfrak{n}}}=\inf\|\xi^{\mathfrak{n}}\|^{\frac{1}{\mathfrak{n}}}.$$

If, then $(\beta e-\xi)$ is invertible in $\mathfrak{A}$, moreover,

$$(\beta e-\xi)^{-1}=\sum_{i=0}^{+\infty}\frac{\xi^i}{\beta^{i+1}},$$

where β is a complex constant.

Definition 2.3. [1] *Let $\mathfrak{I},\mathfrak{L}:X\to X$ be a mappings on set X.*

1) *If* $\mathfrak{w} = \mathfrak{I}\eta = \mathfrak{L}\eta$ *for some* $\eta \in X$*, then* η *is called a coincidence point of* $\mathfrak{I}$ *and* $\mathfrak{L}$*, and* $\mathfrak{w}$ *is called a point of coincidence of* $\mathfrak{I}$ *and* $\mathfrak{L}$*.*
2) *The pair* $(\mathfrak{I}, \mathfrak{L})$ *is called weakly compatible if* $\mathfrak{I}$ *and* $\mathfrak{L}$ *commute at all of their coincidence points, that is,* $\mathfrak{I}\mathfrak{L}\eta = \mathfrak{L}\mathfrak{I}\eta$*, for all* $\eta \in \mathfrak{C}(\mathfrak{I}, \mathfrak{L}) = \{\eta \in X\ : \mathfrak{I}\eta = \mathfrak{L}\eta\}$*.*

Lemma 2.9. [1] *Let* $\mathfrak{I}$ *and* $\mathfrak{L}$ *be weakly compatible self-maps of a set* X*. If* $\mathfrak{I}$ *and* $\mathfrak{L}$ *have a unique point of coincidence* $\mathfrak{w} = \mathfrak{I}\eta = \mathfrak{L}\eta$*, then* $\mathfrak{w}$ *is the unique common fixed point of* $\mathfrak{I}$ *and* $\mathfrak{L}$*.*

Lemma 2.10. *Let* $\mathfrak{A}$ *be a Banach algebra with a unit* e *and* ξ *be a vector in* $\mathfrak{A}$*. If* β *is a complex constant and* $\rho(\xi) < |\beta|$*, then we have* $\rho\left((\beta e - \xi)^{-1}\right) \leq (|\beta| - \rho(\xi))^{-1}$*.*

Proof. Since $\rho(\xi) < |\beta|$, it follows by (2.8) that $(\beta e - \xi)$ is invertible and

$$(\beta e - \xi)^{-1} = \sum_{i=0}^{+\infty} \frac{\xi^i}{\beta^{i+1}}.$$

Set $\Omega = \sum_{i=0}^{+\infty} \frac{\xi^i}{\beta^{i+1}}$ *and* $\Omega_{\mathfrak{n}} = \sum_{i=0}^{\mathfrak{n}} \frac{\xi^i}{\beta^{i+1}}$, then $\Omega_{\mathfrak{n}} \to \Omega(\mathfrak{n} \to +\infty)$ and $\Omega_{\mathfrak{n}}$ commutes with Ω for all $\mathfrak{n}$. From lemma (2.5), it consequently follows that

$$\begin{aligned}
\rho(\Omega_{\mathfrak{n}}) &= \rho(\Omega_{\mathfrak{n}} - \Omega + \Omega) \\
&\leq \rho(\Omega - \Omega_{\mathfrak{n}}) + \rho(\Omega) \\
&\Rightarrow \rho\left(\textstyle\sum_{i=0}^{n} \frac{\xi^i}{\beta^{i+1}} - \sum_{i=0}^{+\infty} \frac{\xi^i}{\beta^{i+1}}\right) \\
&= \rho\left(\textstyle\sum_{i=0}^{n} \frac{\xi^i}{\beta^{i+1}}\right) - \rho\left(\textstyle\sum_{i=0}^{+\infty} \frac{\xi^i}{\beta^{i+1}}\right) \\
&\leq \rho(\Omega - \Omega_{\mathfrak{n}}), \\
\rho(\Omega) &= \rho(\Omega - \Omega_{\mathfrak{n}} + \Omega) \\
&\leq \rho(\Omega - \Omega_{\mathfrak{n}}) + \rho(\Omega_{\mathfrak{n}}) \\
&\Rightarrow \rho\left(\textstyle\sum_{i=0}^{+\infty} \frac{\xi^i}{\beta^{i+1}} - \sum_{i=0}^{n} \frac{\xi^i}{\beta^{i+1}}\right) \\
&= \rho\left(\textstyle\sum_{i=0}^{+\infty} \frac{\xi^i}{\beta^{i+1}}\right) - \rho\left(\textstyle\sum_{i=0}^{n} \frac{\xi^i}{\beta^{i+1}}\right) \\
&\leq \rho(\Omega - \Omega_{\mathfrak{n}})
\end{aligned}$$

may suggest

$$\begin{aligned}
|\rho(\Omega_{\mathfrak{n}}) - \rho(\Omega)| &\leq \rho(\Omega - \Omega_n) \\
&\leq \| \Omega - \Omega_{\mathfrak{n}} \| \\
&\Rightarrow \rho\left(\textstyle\sum_{i=0}^{n} \frac{\xi^i}{\beta^{i+1}}\right) \to \rho\left(\textstyle\sum_{i=0}^{+\infty} \frac{\xi^i}{\beta^{i+1}}\right)(\mathfrak{n} \to +\infty).
\end{aligned}$$

Consequently, by (2.5),

$$\begin{aligned}\rho\left((\beta e-\xi)^{-1}\right) &= \rho\left(\sum_{i=0}^{+\infty}\frac{\xi^i}{\beta^{i+1}}\right)=\lim_{\mathfrak{n}\to+\infty}\rho(\Omega_{\mathfrak{n}})\\ &= \lim_{\mathfrak{n}\to+\infty}\rho\left(\sum_{i=0}^{\mathfrak{n}}\frac{\xi^i}{\beta^{i+1}}\right)\\ &\le \lim_{\mathfrak{n}\to+\infty}\sum_{i=0}^{\mathfrak{n}}\frac{[\rho(\mathfrak{h})]^i}{\beta^{i+1}}\\ &= \lim_{\mathfrak{n}\to+\infty}\sum_{i=0}^{+\infty}\frac{[\rho(\mathfrak{h})]^i}{\beta^{i+1}}=\frac{1}{|\beta|-\rho(\xi)}\\ &= (|\beta|-\rho(\xi))^{-1}\end{aligned}$$

□

Lemma 2.11. [15] *Let $\mathfrak{A}$ be a Banach algebra with a unit e and $\mathfrak{P}$ be a solid cone in $\mathfrak{A}$. Let $\gamma\in\mathfrak{A}$ and $\eta_{\mathfrak{n}}=\gamma^{\mathfrak{n}}$. If $\rho(\gamma)<1$, then $\{\eta_{\mathfrak{n}}\}$ is a $\mathfrak{c}$-sequence.*

Proof. Since $\rho(\gamma)=\lim_{\mathfrak{n}\to+\infty}\|\gamma^{\mathfrak{n}}\|^{\frac{1}{\mathfrak{n}}}<1$, then there exist(s) $\tau>0$ such that

$$\lim_{\mathfrak{n}\to+\infty}\|\gamma^{\mathfrak{n}}\|^{\frac{1}{\mathfrak{n}}}<\tau<1$$

. Letting $\mathfrak{n}$ be big enough, we obtain $\|\gamma^{\mathfrak{n}}\|^{\frac{1}{\mathfrak{n}}}\le\tau$, which implies that $\|\gamma^{\mathfrak{n}}\|^{\frac{1}{\mathfrak{n}}}\le\tau^{\mathfrak{n}}\to 0(\mathfrak{n}\to+\infty)$. So $\|\gamma^{\mathfrak{n}}\|\to 0$, i.e., $\|\eta_{\mathfrak{n}}\|\to 0(\mathfrak{n}\to+\infty)$. Note that for all $\mathfrak{c}\gg\theta$, there is $\beta>0$ such that

$$U(\mathfrak{c},\beta)=\{\eta\in E\ :\|\eta-\mathfrak{c}\|<\beta\}\subset\mathfrak{P}.$$

In view of $\|\eta_n\|\to 0(\mathfrak{n}\to+\infty)$, there exist(s) $\mathfrak{N}$ such that $\|\eta_{\mathfrak{n}}\|<\beta$ for all $\mathfrak{n}>\mathfrak{N}$. Consequently, $\|(\mathfrak{c}-\eta_n)-\mathfrak{c}\|=\|\eta_n\|<\beta$, this loads to $\mathfrak{c}-\eta_n\in U(\mathfrak{c},\beta)\subset\mathfrak{P}$, that is, $\mathfrak{c}-\eta_n\in int\mathfrak{P}$, thus $\eta_n\ll\mathfrak{c}$ for all $\mathfrak{n}>\mathfrak{N}$. □

Definition 2.4. [19] *Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{P}=\mathfrak{R}_0^+$ be a cone in $\mathfrak{A}$. A mapping $\mathfrak{F}:\mathfrak{P}\to\mathfrak{P}$ such that*

1) $\mathfrak{F}$ is non-decreasing and continuous;

2) $\mathfrak{F}(\mathfrak{t})<\mathfrak{t}$ for each $\mathfrak{t}>0$;

3) $\mathfrak{F}(\theta)=\theta$;

4) $\mathfrak{F}(\mathfrak{a}+\mathfrak{b})\le\mathfrak{F}(\mathfrak{a})+\mathfrak{F}(\mathfrak{b})$ for all, $\mathfrak{a},\mathfrak{b}\in[0,+\infty)$.

Definition 2.5. [19] *Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{P}=\mathfrak{R}_0^+$ be a cone in $\mathfrak{A}$. A mapping $\phi:\ \mathfrak{P}\to\mathfrak{P}$ such that:*

1) ϕ *is monotone non-decreasing and continuous;*

2) $\{\phi^{\mathfrak{n}}(\mathfrak{t})\}$ $(\mathfrak{t}>0)$ *is a* $\mathfrak{c}$*-sequence in* $\mathfrak{P}$*;*

3) If $\{\mathfrak{u}_n\}$ *is a* $\mathfrak{c}$*-sequence in* $\mathfrak{P}$*, then* $\{\phi(\mathfrak{u}_\mathfrak{n})\}$ *is also a* $\mathfrak{c}$*-sequence in* $\mathfrak{P}$*;*

4) $\phi(\mathfrak{t})=\mathfrak{K}\mathfrak{t}$*, for some* $(\mathfrak{K}\in\mathfrak{P})$*,* $\mathfrak{K}>0.$*;*

3. Main Results

We prove a unique common fixed point for generalized $(\phi,\mathfrak{F})$*- contraction mappings via the class functions* Φ *and* Ψ*. Furthermore, we give examples to support our main results.*

Definition 3.1. *Let* (X,D_b) *be a cone b-metric space over Banach algebra* $\mathfrak{A}$ *and* $\mathfrak{P}$ *be a solid cone in* $\mathfrak{A}$ *with the coefficient* $\Omega\succcurlyeq 1$*. Let mappings* $\mathfrak{I},\mathfrak{L}$ *be a two self-mappings on X,* ϕ *and* $\mathfrak{F}$ *be a self mapping on* $\mathfrak{R}_0^+$*, then* $\phi\in\Phi$ *and* $\mathfrak{F}\in\Psi$*. The generalized* $(\phi,\mathfrak{F})$*-contraction mappings, satisfies that*

$$\phi\left(D_b\left(\mathfrak{I}\eta,\mathfrak{L}\zeta\right)\right)\preccurlyeq\mathfrak{F}(\phi\left(\urcorner(\eta,\zeta)\right)) \tag{3.1}$$

where, $$\urcorner(\eta,\zeta)=\begin{pmatrix}\xi_1\frac{D_b(\eta,\mathfrak{I}\eta)}{1+D_b(\eta,\mathfrak{I}\eta)}+\xi_2\frac{D_b(\zeta,\mathfrak{L}\zeta)}{1+D_b(\zeta,\mathfrak{L}\zeta)}\\ +\xi_3\frac{D_b(\eta,\mathfrak{L}\zeta)}{1+D_b(\eta,\mathfrak{L}\zeta)}+\xi_4\frac{D_b(\zeta,\mathfrak{I}\eta)}{1+D_b(\zeta,\mathfrak{I}\eta)}+\xi_5 D_b(\eta,\zeta)\end{pmatrix}$$

such that for all $\eta,\zeta\in X$*, where* $\xi_i\in\mathfrak{P}$ $(i\ =1,\ 2,\ \ldots 5,\ )$ *be a generalized Lipschitz constant with* $2\Omega\rho\left(\xi_5\right)+(\Omega+1)\rho\left(\xi_1+\xi_2+\Omega\ \xi_3+\Omega\xi_4\ \right)\prec 2$*. Moreover, if* $\mathfrak{I}$ *and* $\mathfrak{L}$ *are weakly compatible, then* $\mathfrak{I}$ *and* $\mathfrak{L}$ *have a unique common fixed point.*

Theorem 3.1. *Let* (X,D_b) *be a cone b-metric space over Banach algebra* $\mathfrak{A}$ *and* $\mathfrak{P}$ *be a solid cone in* $\mathfrak{A}$ *with the coefficient* $\Omega\geq 1$*.* $\xi_i\in\mathfrak{P}$ $(i\ =1,\ 2,\ \ldots 5,\ )$ *be a generalized Lipschitz constant with* $2\Omega\rho\left(\xi_5\right)+(\Omega+1)\rho\left(\xi_1+\xi_2+\Omega\ \xi_3+\Omega\xi_4\ \right)<2$*. Suppose that* ξ_5 *commutes with* $\xi_1+\xi_2+\Omega\ \xi_3+\Omega\xi_4$ *and the mappings* $\mathfrak{I},\mathfrak{L}\ :X\to X$ *be generalized* $(\phi,\mathfrak{F})$*-contraction mapping, satisfies (3.1). Where,* $\mathfrak{F}\in\Psi$*,* $\phi\in\Phi$ *such that for all* $\eta,\zeta\in X$*. Moreover, if* $\mathfrak{I}$ *and* $\mathfrak{L}$ *are weakly compatible, then* $\mathfrak{I}$ *and* $\mathfrak{L}$ *have a unique common fixed point.*

Proof. Fix any $\eta\in X$. Define $\eta_0=\eta$ and let $\eta_1\in\mathfrak{I}\eta_0,\ \ \eta_2\in\mathfrak{L}\eta_1$ such that $\eta_{2n+1}=\mathfrak{I}\eta_{2n}$, $\eta_{2n+2}=\mathfrak{L}\eta_{2n+1}$, by lemma 2.6, we may choose $\eta_2\in\mathfrak{L}\eta_1$ such that

$$\begin{aligned}
\phi\left(D_b\left(\eta_1,\ \eta_2\right)\right) &= \phi\left(D_b\left(\mathfrak{I}\eta_0,\mathfrak{L}\eta_1\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1\frac{D_b(\eta_0,\mathfrak{I}\eta_0)}{1+D_b(\eta_0,\mathfrak{I}\eta_0)}+\xi_2\frac{D_b(\eta_1,\mathfrak{L}\eta_1)}{1+D_b(\eta_1,\mathfrak{L}\eta_1)}+\xi_3\frac{D_b(\eta_0,\mathfrak{L}\eta_1)}{1+D_b(\eta_0,\mathfrak{L}\eta_1)}\\ +\xi_4\frac{D_b(\eta_1,\mathfrak{I}\eta_0)}{1+D_b(\eta_1,\mathfrak{I}\eta_0)}+\xi_5 D_b\left(\eta_0,\eta_1\right)\end{pmatrix}\right)\right)\\
&= \left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1\frac{D_b(\eta_0,\eta_1)}{1+D_b(\eta_0,\eta_1)}+\xi_2\frac{D_b(\eta_1,\eta_2)}{1+D_b(\eta_1,\eta_2)}+\xi_3\frac{D_b(\eta_0,\eta_2)}{1+D_b(\eta_0,\eta_2)}\\ +\xi_4\frac{D_b(\eta_1,\eta_1)}{1+D_b(\eta_1,\eta_1)}+\xi_5 D_b\left(\eta_0,\eta_1\right)\end{pmatrix}\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1 D_b\left(\eta_0,\eta_1\right)+\xi_2 D_b\left(\eta_1,\eta_2\right)+\xi_3 D_b\left(\eta_0,\eta_2\right)\\ +\xi_4 D_b\left(\eta_1,\eta_1\right)+\xi_5 D_b\left(\eta_0,\eta_1\right)\end{pmatrix}\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1 D_b\left(\eta_0,\eta_1\right)+\xi_2 D_b\left(\eta_1,\eta_2\right)\\ +\xi_3\Omega\ \left[D_b\left(\eta_0,\eta_1\right)+D_b\left(\eta_1,\eta_2\right)\right]\\ +\xi_5 D_b\left(\eta_0,\eta_1\right)\end{pmatrix}\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix}\left(\xi_1+\Omega\xi_3+\xi_5\right) D_b\left(\eta_0,\eta_1\right)\\ +\left(\xi_2+\Omega\xi_3\right) D_b\left(\eta_1,\eta_2\right)\end{pmatrix}\right)\right)
\end{aligned}$$

which implies that

$$(e-\xi_2-\Omega\xi_3)\phi\left(D_b\left(\eta_1,\eta_2\right)\right)\preccurlyeq\left(\xi_1+\xi_5+\Omega\ \xi_3\right)\mathfrak{F}\left(\phi\left(D_b\left(\eta_0,\eta_1\right)\right)\right) \tag{3.2}$$

Then,

$$\begin{aligned}
\phi(D_b\left(\eta_2,\eta_1\right)) &= \phi\left(\ D_b\left(\mathfrak{L}\eta_1,\mathfrak{I}\eta_0\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1\frac{D_b(\eta_1,\mathfrak{L}\eta_1)}{1+D_b(\eta_1,\mathfrak{L}\eta_1)}+\xi_2\frac{D_b(\eta_0,\mathfrak{I}\eta_0)}{1+D_b(\eta_0,\mathfrak{I}\eta_0)}+\xi_3\frac{D_b(\eta_1,\mathfrak{I}\eta_0)}{1+D_b(\eta_1,\mathfrak{I}\eta_0)}\\ +\xi_4\frac{D_b(\eta_0,\mathfrak{L}\eta_1)}{1+D_b(\eta_0,\mathfrak{L}\eta_1)}+\xi_5 D_b\left(\eta_1,\eta_0\right)\end{pmatrix}\right)\right)\\
&= \left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1\frac{D_b(\eta_1,\eta_2)}{1+D_b(\eta_1,\eta_2)}+\xi_2\frac{D_b(\eta_0,\eta_1)}{1+D_b(\eta_0,\eta_1)}+\xi_3\frac{D_b(\eta_1,\eta_1)}{1+D_b(\eta_1,\eta_1)}\\ +\xi_4\frac{D_b(\eta_0,\eta_2)}{1+D_b(\eta_0,\eta_2)}+\xi_5 D_b\left(\eta_0,\eta_1\right)\end{pmatrix}\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1 D_b\left(\eta_1,\eta_2\right)+\xi_2 D_b\left(\eta_0,\eta_1\right)+\xi_3 D_b\left(\eta_1,\eta_1\right)\\ +\xi_4 D_b\left(\eta_0,\eta_2\right)+\xi_5 D_b\left(\eta_0,\eta_1\right)\end{pmatrix}\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1 D_b\left(\eta_1,\eta_2\right)+\xi_2 D_b\left(\eta_0,\eta_1\right)\\ +\xi_4\Omega\left[D_b\left(\eta_0,\eta_1\right)+D_b\left(\eta_1,\eta_2\right)\right]\\ +\xi_5 D_b\left(\eta_0,\eta_1\right)\end{pmatrix}\right)\right)\\
&= \left(\mathfrak{F}\left(\phi\begin{pmatrix}\left(\xi_2+\xi_5+\Omega\xi_4\right) D_b\left(\eta_0,\eta_1\right)\\ +\left(\xi_1+\Omega\xi_4\right) D_b\left(\eta_2,\eta_1\right)\end{pmatrix}\right)\right)
\end{aligned}$$

which implies that

$$(e-\xi_1-\Omega\xi_4)\,\phi\left(D_b\left(\eta_2,\eta_1\right)\right)\preccurlyeq\left(\xi_2+\xi_5+\Omega\xi_4\right)\mathfrak{F}\left(\phi\left(D_b\left(\eta_0,\eta_1\right)\right)\right) \tag{3.3}$$

Adding inequalities (3.2) and (3.3), we obtain $\phi(D_b(\eta_1,\eta_2))$ where,

$$(2e-\xi_1-\xi_2-\Omega\xi_3-\Omega\xi_4)\,\phi\left(D_b\left(\eta_1,\eta_2\right)\right) \preccurlyeq \begin{pmatrix} 2\xi_5+\xi_1+\xi_2 \\ +\Omega\ \xi_3+\Omega\xi_4 \end{pmatrix} \mathfrak{F}\left(\phi\left(D_b\left(\eta_0,\eta_1\right)\right)\right) \tag{3.4}$$

Denote $\xi_1+\xi_2+\Omega\ \xi_3+\Omega\xi_4=\xi$, then (3.4) yields that

$$(2e-\xi)\,\phi\left(D_b\left(\eta_1,\eta_2\right)\right) \preccurlyeq (2\xi_5+\xi)\,\mathfrak{F}\left(\phi\left(D_b\left(\eta_0,\eta_1\right)\right)\right) \tag{3.5}$$

Similarly, it can be shown that, there exists $\eta_2\in\mathfrak{I}\eta_1,\ \eta_3\in\mathfrak{L}\eta_2$ such that

$$\begin{aligned}
\phi\left(D_b\left(\eta_2,\eta_3\right)\right) &= \phi\left(D_b\left(\mathfrak{I}\eta_1,\mathfrak{L}\eta_2\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix} \xi_1\frac{D_b(\eta_1,\mathfrak{I}\eta_1)}{1+D_b(\eta_1,\mathfrak{I}\eta_1)}+\xi_2\frac{D_b(\eta_2,\mathfrak{L}\eta_2)}{1+D_b(\eta_2,\mathfrak{L}\eta_2)}+\xi_3\frac{D_b(\eta_1,\mathfrak{L}\eta_2)}{1+D_b(\eta_1,\mathfrak{L}\eta_2)} \\ +\xi_4\frac{D_b(\eta_2,\mathfrak{I}\eta_1)}{1+D_b(\eta_2,\mathfrak{I}\eta_1)}+\xi_5 D_b\left(\eta_1,\eta_2\right)\end{pmatrix}\right)\right)\\
&= \left(\mathfrak{F}\left(\phi\begin{pmatrix} \xi_1\frac{D_b(\eta_1,\eta_2)}{1+D_b(\eta_1,\eta_2)}+\xi_2\frac{D_b(\eta_2,\eta_3)}{1+D_b(\eta_2,\eta_3)}+\xi_3\frac{D_b(\eta_1,\eta_3)}{1+D_b(\eta_1,\eta_3)} \\ +\xi_4\frac{D_b(\eta_2,\eta_2)}{1+D_b(\eta_2,\eta_2)}+\xi_5 D_b\left(\eta_1,\eta_2\right)\end{pmatrix}\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix} \xi_1 D_b\left(\eta_1,\eta_2\right)+\xi_2 D_b\left(\eta_2,\eta_3\right)+\xi_3 D_b\left(\eta_1,\eta_3\right) \\ +\xi_4 D_b\left(\eta_2,\eta_2\right)+\xi_5 D_b\left(\eta_1,\eta_2\right)\end{pmatrix}\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix} \xi_1 D_b\left(\eta_1,\eta_2\right)+\xi_2 D_b\left(\eta_2,\eta_3\right) \\ +\xi_3\Omega\ \left[D_b\left(\eta_1,\eta_2\right)+D_b\left(\eta_2,\eta_3\right)\right] \\ +\xi_5 D_b\left(\eta_1,\eta_2\right)\end{pmatrix}\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix} (\xi_1+\Omega\xi_3+\xi_5)\,D_b\left(\eta_1,\eta_2\right) \\ +\left(\xi_2+\Omega\xi_3\right)D_b\left(\eta_2,\eta_3\right)\end{pmatrix}\right)\right)
\end{aligned}$$

which implies that

$$(e-\xi_2-\Omega\xi_3)\,\phi\left(D_b\left(\eta_1,\eta_2\right)\right) \preccurlyeq (\xi_1+\xi_5+\Omega\ \xi_3)\,\mathfrak{F}\left(\phi\left(D_b\left(\eta_0,\eta_1\right)\right)\right) \tag{3.6}$$

Then,

$$\begin{aligned}
\phi\left(D_b\left(\eta_3,\eta_2\right)\right) &= \phi\left(\ D_b\left(\mathfrak{L}\eta_2,\mathfrak{I}\eta_1\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix} \xi_1\frac{D_b(\eta_2,\mathfrak{L}\eta_2)}{1+D_b(\eta_2,\mathfrak{L}\eta_2)}+\xi_2\frac{D_b(\eta_1,\mathfrak{I}\eta_1)}{1+D_b(\eta_1,\mathfrak{I}\eta_1)}+\xi_3\frac{D_b(\eta_2,\mathfrak{I}\eta_1)}{1+D_b(\eta_2,\mathfrak{I}\eta_1)} \\ +\xi_4\frac{D_b(\eta_1,\mathfrak{L}\eta_2)}{1+D_b(\eta_1,\mathfrak{L}\eta_2)}+\xi_5 D_b\left(\eta_2,\eta_1\right)\end{pmatrix}\right)\right)\\
&= \left(\mathfrak{F}\left(\phi\begin{pmatrix} \xi_1\frac{D_b(\eta_2,\eta_3)}{1+D_b(\eta_2,\eta_3)}+\xi_2\frac{D_b(\eta_1,\eta_2)}{1+D_b(\eta_1,\eta_2)}+\xi_3\frac{D_b(\eta_2,\eta_2)}{1+D_b(\eta_2,\eta_2)} \\ +\xi_4\frac{D_b(\eta_1,\eta_3)}{1+D_b(\eta_1,\eta_3)}+\xi_5 D_b\left(\eta_2,\eta_1\right)\end{pmatrix}\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\begin{pmatrix} \xi_1 D_b\left(\eta_2,\eta_3\right)+\xi_2 D_b\left(\eta_1,\eta_2\right)+\xi_3 D_b\left(\eta_2,\eta_2\right) \\ +\xi_4 D_b\left(\eta_1,\eta_3\right)+\xi_5 D_b\left(\eta_2,\eta_1\right)\end{pmatrix}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\preccurlyeq \left(\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1 D_b(\eta_2,\eta_3)+\xi_2 D_b(\eta_1,\eta_2)\\ +\xi_4\Omega\left[D_b(\eta_1,\eta_2)+D_b(\eta_2,\eta_3)\right]\\ +\xi_5 D_b(\eta_2,\eta_1)\end{array}\right)\right)\right)\\
&=\left(\mathfrak{F}\left(\phi\left(\begin{array}{c}(\xi_2+\xi_5+\Omega\xi_4)D_b(\eta_1,\eta_2)\\ +(\xi_1+\Omega\xi_4)D_b(\eta_2,\eta_3)\end{array}\right)\right)\right)
\end{aligned}$$

which implies that

$$(e-\xi_1-\Omega\xi_4)\,\phi(D_b(\eta_3,\eta_2)) \preccurlyeq (\xi_2+\xi_5+\Omega\xi_4)\,\mathfrak{F}(\phi(D_b(\eta_0,\eta_1))) \tag{3.7}$$

Adding inequalities (3.6) and (3.7), we obtain $\phi(D_b(\eta_1,\eta_2))$ where,

$$\left(\begin{array}{c}2e-\xi_1-\xi_2\\ -\Omega\xi_3-\Omega\xi_4\end{array}\right)\phi(D_b(\eta_2,\eta_3)) \preccurlyeq \left(\begin{array}{c}2\xi_5+\xi_1+\xi_2\\ +\Omega\,\xi_3+\Omega\xi_4\end{array}\right)\mathfrak{F}(\phi(D_b(\eta_0,\eta_1))) \tag{3.8}$$

Denote $\xi_1+\xi_2+\Omega\,\xi_3+\Omega\xi_4=\xi$, then (3.8) yields that

$$(2e-\xi)\,\phi(D_b(\eta_1,\eta_2)) \preccurlyeq (2\xi_5+\xi)\,\mathfrak{F}(\phi(D_b(\eta_0,\eta_1))) \tag{3.9}$$

Continuing this process, we obtain by induction a sequence $\{\eta_n\}$ such that $\eta_{2n+1}\in\mathfrak{I}\eta_{2n}$, $\eta_{2n+2}\in\mathfrak{L}\eta_{2n+1}$ such that

$$\begin{aligned}
&\phi(D_b(\eta_{2n+1},\eta_{2n+2}))=\phi(D_b(\mathfrak{I}\eta_{2n},\mathfrak{L}\eta_{2n+1}))\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1\frac{D_b(\eta_{2n},\mathfrak{I}\eta_{2n})}{1+D_b(\eta_{2n},\mathfrak{I}\eta_{2n})}+\xi_2\frac{D_b(\eta_{2n+1},\mathfrak{L}\eta_{2n+1})}{1+D_b(\eta_{2n+1},\mathfrak{L}\eta_{2n+1})}+\xi_3\frac{D_b(\eta_{2n},\mathfrak{L}\eta_{2n+1})}{1+D_b(\eta_{2n},\mathfrak{L}\eta_{2n+1})}\\ +\xi_4\frac{D_b(\eta_{2n+1},\mathfrak{I}\eta_{2n})}{1+D_b(\eta_{2n+1},\mathfrak{I}\eta_{2n})}+\xi_5 D_b(\eta_{2n},\eta_{2n+1})\end{array}\right)\right)\right)\\
&=\left(\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1\frac{D_b(\eta_{2n},\eta_{2n+1})}{1+D_b(\eta_{2n},\eta_{2n+1})}+\xi_2\frac{D_b(\eta_{2n+1},\eta_{2n+2})}{1+D_b(\eta_{2n+1},\eta_{2n+2})}+\xi_3\frac{D_b(\eta_{2n},\eta_{2n+2})}{1+D_b(\eta_{2n},\eta_{2n+2})}\\ +\xi_4\frac{D_b(\eta_{2n+1},\eta_{2n+1})}{1+D_b(\eta_{2n+1},\eta_{2n+1})}+\xi_5 D_b(\eta_{2n},\eta_{2n+1})\end{array}\right)\right)\right)\\
&\preccurlyeq \left(\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1 D_b(\eta_{2n},\eta_{2n+1})+\xi_2 D_b(\eta_{2n+1},\eta_{2n+2})\\ +\xi_3\Omega\left[D_b(\eta_{2n},\eta_{2n+1})+D_b(\eta_{2n+1},\eta_{2n+2})\right]\\ +\xi_5 D_b(\eta_{2n},\eta_{2n+1})\end{array}\right)\right)\right)\\
&=\left(\mathfrak{F}\left(\phi\left(\begin{array}{c}(\xi_1+\xi_5+\Omega\xi_3)D_b(\eta_{2n},\eta_{2n+1})\\ +(\xi_2+\Omega\xi_3)D_b(\eta_{2n+1},\eta_{2n+2})\end{array}\right)\right)\right)
\end{aligned}$$

which implies that

$$(e-\xi_2-\Omega\xi_3)\,\phi(D_b(\eta_{2n+1},\eta_{2n+2})) \preccurlyeq (\xi_1+\xi_5+\Omega\xi_3)\,\mathfrak{F}(\phi(D_b(\eta_{2n},\eta_{2n+1}))) \tag{3.10}$$

Also,

$$
\begin{aligned}
&\phi\left(D_b\left(\eta_{2n+2},\eta_{2n+1}\right)\right)=\phi\left(\,D_b\left(\mathfrak{L}\eta_{2n+1},\mathfrak{I}\eta_{2n}\right)\right)\\
&\preccurlyeq\left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1\frac{D_b(\eta_{2n+1},\mathfrak{L}\eta_{2n+1})}{1+D_b(\eta_{2n+1},\mathfrak{L}\eta_{2n+1})}+\xi_2\frac{D_b(\eta_{2n},\mathfrak{I}\eta_{2n})}{1+D_b(\eta_{2n},\mathfrak{I}\eta_{2n})}+\xi_3\frac{D_b(\eta_{2n+1},\mathfrak{I}\eta_{2n})}{1+D_b(\eta_{2n+1},\mathfrak{I}\eta_{2n})}\\+\xi_4\frac{D_b(\eta_{2n},\mathfrak{L}\eta_{2n+1})}{1+D_b(\eta_{2n},\mathfrak{L}\eta_{2n+1})}+\xi_5 D_b\left(\eta_{2n+1},\eta_{2n}\right)\end{pmatrix}\right)\right)\\
&=\left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1\frac{D_b(\eta_{2n+1},\eta_{2n+2})}{1+D_b(\eta_{2n+1},\eta_{2n+2})}+\xi_2\frac{D_b(\eta_{2n},\eta_{2n+1})}{1+D_b(\eta_{2n},\eta_{2n+1})}+\xi_3\frac{D_b(\eta_{2n+1},\eta_{2n+1})}{1+D_b(\eta_{2n+1},\eta_{2n+1})}\\+\xi_4\frac{D_b(\eta_{2n},\eta_{2n+2})}{1+D_b(\eta_{2n},\eta_{2n+2})}+\xi_5 D_b\left(\eta_{2n+1},\eta_{2n}\right)\end{pmatrix}\right)\right)\\
&\preccurlyeq\left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1 D_b\left(\eta_{2n+1},\eta_{2n+2}\right)+\xi_2 D_b\left(\eta_{2n},\eta_{2n+1}\right)\\+\xi_4\Omega\left[D_b\left(\eta_{2n},\eta_{2n+1}\right)+D_b\left(\eta_{2n+1},\eta_{2n+2}\right)\right]\\+\xi_5 D_b\left(\eta_{2n+1},\eta_{2n}\right)\end{pmatrix}\right)\right)\\
&=\left(\mathfrak{F}\left(\phi\begin{pmatrix}\left(\xi_2+\xi_5+\Omega\xi_4\right)D_b\left(\eta_{2n},\eta_{2n+1}\right)\\+\left(\xi_1+\Omega\xi_4\right)D_b\left(\eta_{2n+1},\eta_{2n+2}\right)\end{pmatrix}\right)\right)
\end{aligned}
$$

which implies that

$$
(e-\xi_1+\Omega\xi_4)\,\phi\left(D_b\left(\eta_{2n+2},\eta_{2n+1}\right)\right)\preccurlyeq\left(\xi_2+\xi_5+\Omega\xi_4\right)\mathfrak{F}\left(\phi\left(D_b\left(\eta_{2n},\eta_{2n+1}\right)\right)\right)\tag{3.11}
$$

Add up (3.10) and (3.11) yields that

$$
\begin{pmatrix}2e-\xi_1-\xi_2\\-\Omega\xi_3-\Omega\xi_4\end{pmatrix}\phi\left(D_b\left(\eta_{2n+1},\ \eta_{2n+2}\right)\right)\preccurlyeq\begin{pmatrix}2\xi_5+\xi_1+\xi_2\\+\Omega\xi_3+\Omega\xi_4\end{pmatrix}\mathfrak{F}\left(\phi\left(D_b\left(\eta_{2n},\ \eta_{2n+1}\right)\right)\right)\tag{3.12}
$$

Denote $\xi_1+\xi_2+\Omega\ \xi_3+\Omega\xi_4=\xi$, then (3.12) yields that

$$
(2e-\xi)\,\phi\left(D_b\left(\eta_{2n+1},\ \eta_{2n+2}\right)\right)\preccurlyeq\left(2\xi_5+\xi\right)\mathfrak{F}\left(\phi\left(D_b\left(\eta_{2n},\ \eta_{2n+1}\right)\right)\right)\tag{3.13}
$$

Therefore,

$$
\phi(D_b\left(\eta_n\ ,\eta_{n+1}\right))=\phi\left(D_b\left(\mathfrak{I}\eta_{n-1},\mathfrak{L}\eta_n\right)\right)
$$

$$
\begin{pmatrix}2e-\xi_1-\xi_2\\-\Omega\xi_3-\Omega\xi_4\end{pmatrix}\phi(D_b(\eta_n,\eta_{n+1}))\preccurlyeq\begin{pmatrix}2\xi_5+\xi_1+\xi_2\\+\Omega\xi_3+\Omega\xi_4\end{pmatrix}\mathfrak{F}\left(\phi\left(D_b\left(\eta_{n-1},\eta_n\right)\right)\right)\tag{3.14}
$$

Denote $\xi_1+\xi_2+\Omega\ \xi_3+\Omega\xi_4=\xi$, then (3.14) yields that

$$
(2e-\xi)\,\phi\left(D_b\left(\eta_n,\eta_{n+1}\right)\right)\preccurlyeq\left(2\xi_5+\xi\right)\mathfrak{F}\left(\phi\left(D_b\left(\eta_{n-1},\eta_n\right)\right)\right)\tag{3.15}
$$

Note that

$$
2\rho\left(\xi\right)\leq(\Omega+1)\rho\left(\xi\right)\leq 2\Omega\rho\left(\xi_5\right)+(\Omega+1)\rho\left(\xi\right)<2
$$

$\rho(\xi) < 1 < 2$, then by Lemma (2.8) it follows that $(2e-\xi)$ is invertible. Furthermore,

$$(2e-\xi)^{-1} = \sum_{i=0}^{+\infty} \frac{\xi^i}{2^{i+1}}$$

By multiplying in both sides of (3.15) by $(2e-\xi)^{-1}$, we get

$$\phi(D_b(\eta_n,\eta_{n+1})) \preccurlyeq (2e-\xi)^{-1}(2\xi_5+\xi)\,\mathfrak{F}(\phi(D_b(\eta_{n-1},\eta_n))) \tag{3.16}$$

Let $\gamma = (2e-\xi)^{-1}(2\xi_5+\xi)$, then by (3.16) we get

$$\begin{aligned}
\phi(D_b(\eta_n,\eta_{n+1})) &\preccurlyeq \mathfrak{F}(\gamma\phi(D_b(\eta_{n-1},\eta_n)))\\
&\preccurlyeq \gamma\mathfrak{F}(\gamma\phi(D_b(\eta_{n-2},\eta_{n-1})))\\
&= \mathfrak{F}\left(\gamma^2\phi(D_b(\eta_{n-2},\eta_{n-1}))\right)\\
&\vdots\\
&\preccurlyeq \gamma^n\mathfrak{F}(\phi(D_b(\eta_0,\eta_1)))
\end{aligned}$$

Since ξ_5 commutes with ξ, it follows that

$$\begin{aligned}
(2e-\xi)^{-1}(2\xi_5+\xi) &= \left(\sum_{i=0}^{+\infty}\tfrac{\xi^i}{2^{i+1}}\right)(2\xi_5+\xi)\\
&= 2\left(\sum_{i=0}^{+\infty}\tfrac{\xi^i}{2^{i+1}}\right)\xi_5 + \left(\sum_{i=0}^{+\infty}\tfrac{\xi^{i+1}}{2^{i+1}}\right)\\
&= 2\xi\left(\sum_{i=0}^{+\infty}\tfrac{\xi^i}{2^{i+1}}\right) + \xi\left(\sum_{i=0}^{+\infty}\tfrac{\xi^i}{2^{i+1}}\right)\\
&= (2\xi_5+\xi)\left(\sum_{i=0}^{+\infty}\tfrac{\xi^i}{2^{i+1}}\right) = (2\xi_5+\xi)(2e-\xi)^{-1}
\end{aligned}$$

then, $(2e-\xi)^{-1}$ commutes with $(2\xi_5+\xi)$. Note by Lemma (2.5) and Lemma (2.10) that

$$\begin{aligned}
\rho(\gamma) &= \rho\left((2\xi_5+\xi)(2e-\xi)^{-1}\right)\\
&\leq \rho\left((2e-\xi)^{-1}\right)\,\rho((2\xi_5+\xi))\\
&\leq \tfrac{1}{2-\rho(\xi)}\left[2\rho(\xi_5)+\rho(\xi)\right]\\
&= \tfrac{1}{2-\rho(\xi_1+\xi_2+\Omega\,\xi_3+\Omega\xi_4)}\left[2\rho(\xi_5)+\rho(\xi_1+\xi_2+\Omega\,\xi_3+\Omega\xi_4)\right]\\
&< \tfrac{1}{\Omega}, \quad [\textit{since } 2\Omega\rho(\xi_5)+(\Omega+1)(\rho(\xi_1)+\rho(\xi_2)+\Omega\,\rho(\xi_3)+\Omega\,\rho(\xi_4)) < 2\,]
\end{aligned}$$

Which establishes that $(e-\Omega\gamma)$ is invertible and $\parallel\gamma^m\parallel\to 0(m\to+\infty)$. Hence, for any $m\geq 1$, $\mathfrak{p}\geq 1$ and $\gamma\in\mathfrak{P}$ with $\rho(\gamma)<\frac{1}{\Omega}$, we have that

$$\begin{aligned}
\phi\left(D_b\left(\eta_m,\eta_{m+\mathfrak{p}}\right)\right) &\preccurlyeq \mathfrak{F}\left(\phi\Omega\left[D_b\left(\eta_m,\eta_{m+1}\right)+D_b\left(\eta_{m+1},\eta_{m+\mathfrak{p}}\right)\right]\right)\\
&\preccurlyeq \Omega\mathfrak{F}\left(\phi D_b\left(\eta_m,\eta_{m+1}\right)\right)+\Omega^2\mathfrak{F}\left(\phi\begin{bmatrix} D_b\left(\eta_{m+1},\eta_{m+2}\right)\\ +D_b\left(\eta_{m+2},\eta_{m+\mathfrak{p}}\right)\end{bmatrix}\right)\\
&\preccurlyeq \begin{pmatrix} \Omega\mathfrak{F}\left(\phi D_b\left(\eta_m,\eta_{m+1}\right)\right)+\Omega^2\mathfrak{F}\left(\phi D_b\left(\eta_{m+1},\eta_{m+2}\right)\right)\\ +\Omega^3\mathfrak{F}\left(\phi D_b\left(\eta_{m+2},\eta_{m+3}\right)\right)+\dots\\ +\Omega^{\mathfrak{p}-1}\mathfrak{F}\left(\phi D_b\left(\eta_{m+\mathfrak{p}-2},\eta_{m+\mathfrak{p}-1}\right)\right)\\ +\Omega^{\mathfrak{p}-1}\mathfrak{F}\left(\phi D_b\left(\eta_{m+\mathfrak{p}-1},\eta_{m+\mathfrak{p}}\right)\right)\end{pmatrix}\\
&\preccurlyeq \begin{pmatrix} \Omega\gamma^m\mathfrak{F}\left(\phi D_b\left(\eta_0,\eta_1\right)\right)+\Omega^2\gamma^{m+1}\mathfrak{F}\left(\phi D_b\left(\eta_0,\eta_1\right)\right)\\ +\Omega^3\gamma^{m+2}\mathfrak{F}\left(\phi D_b\left(\eta_0,\eta_1\right)\right)+\dots\\ +\Omega^{\mathfrak{p}-1}\gamma^{m+\mathfrak{p}-2}\mathfrak{F}\left(\phi D_b\left(\eta_0,\eta_1\right)\right)\\ +\Omega^{\mathfrak{p}}\gamma^{m+\mathfrak{p}-1}\mathfrak{F}\left(\phi D_b\left(\eta_0,\eta_1\right)\right)\end{pmatrix}\\
&= \Omega\gamma^m\left[e+\Omega\gamma+\Omega^2\gamma^2+\dots+(\Omega\gamma)^{\mathfrak{p}-1}\right]\mathfrak{F}\left(\phi D_b\left(\eta_0,\eta_1\right)\right)\\
&\preccurlyeq \Omega\gamma^m(e-\Omega\gamma)^{-1}\mathfrak{F}\left(\phi D_b\left(\eta_0,\eta_1\right)\right).
\end{aligned}$$

In view, $\parallel \Omega\gamma^m\mathfrak{F}\left(\phi\left(D_b\left(\eta_0,\eta_1\right)\right)\right)\parallel\leq\parallel\Omega\gamma^m\parallel\parallel\mathfrak{F}\left(\phi\left(D_b\left(\eta_0,\eta_1\right)\right)\right)\parallel\to 0$ $(m\to+\infty)$, it follows that, for any $\mathfrak{c}\ll\mathfrak{A}$, with $\theta\ll\mathfrak{c}$, there exists $\mathfrak{N}\in\mathbb{N}$ such that for any $\mathfrak{n}>\mathfrak{m}>\mathfrak{N}$, we have

$$\phi(D_b(\eta_n,\eta_{m+\mathfrak{p}}))\preccurlyeq\Omega\gamma^m(e-\Omega\gamma)^{-1}\mathfrak{F}\left(\phi D_b\left(\eta_0,\eta_1\right)\right)\ll\mathfrak{c}.$$

Next by using Lemma (2.11) and Lemma (2.7), we conclude that $\{\eta_n\}$ is a Cauchy sequence. Since $(X,\ D_b)$ is complete, there exists $\mathfrak{s}\in X$ such that $\eta_n\to\mathfrak{s}$ $(n\to+\infty)$. We shall prove that $\mathfrak{s}$ is a common fixed point of $\mathfrak{I}$ and $\mathfrak{L}$.

$$\begin{aligned}
&\phi\left(D_b\left(\eta_{2n+1},\ \mathfrak{L}\mathfrak{s}\right)\right)=\phi\left(D_b\left(\mathfrak{I}\eta_{2n},\mathfrak{L}\mathfrak{s}\right)\right)\\
&\preccurlyeq\left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1\frac{D_b(\eta_{2n},\mathfrak{I}\eta_{2n})}{1+D_b(\eta_{2n},\mathfrak{I}\eta_{2n})}+\xi_2\frac{D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}{1+D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}+\xi_3\frac{D_b(\eta_{2n},\mathfrak{L}\mathfrak{s})}{1+D_b(\eta_{2n},\mathfrak{L}\mathfrak{s})}\\ +\xi_4\frac{D_b(\mathfrak{s},\mathfrak{I}\eta_{2n})}{1+D_b(\mathfrak{s},\mathfrak{I}\eta_{2n})}+\xi_5 D_b\left(\eta_{2n},\mathfrak{s}\right)\end{pmatrix}\right)\right)\\
&=\left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1\frac{D_b(\eta_{2n},\eta_{2n+1})}{1+D_b(\eta_{2n},\eta_{2n+1})}+\xi_2\frac{D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}{1+D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}+\xi_3\frac{D_b(\eta_{2n},\mathfrak{L}\mathfrak{s})}{1+D_b(\eta_{2n},\mathfrak{L}\mathfrak{s})}\\ +\xi_4\frac{D_b(\mathfrak{s},\eta_{2n+1})}{1+D_b(\mathfrak{s},\eta_{2n+1})}+\xi_5 D_b\left(\eta_{2n},\mathfrak{s}\right)\end{pmatrix}\right)\right)\\
&\preccurlyeq\left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1 D_b\left(\eta_{2n},\eta_{2n+1}\right)+\xi_2 D_b\left(\mathfrak{s},\mathfrak{L}\mathfrak{s}\right)+\xi_3 D_b\left(\eta_{2n},\mathfrak{L}\mathfrak{s}\right)\\ +\xi_4 D_b\left(\mathfrak{s},\eta_{2n+1}\right)+\xi_5 D_b\left(\eta_{2n},\mathfrak{s}\right)\end{pmatrix}\right)\right)
\end{aligned}$$

$$\preccurlyeq \left(\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1\Omega[D_b(\eta_{2n},\mathfrak{s})+D_b(\mathfrak{s},\eta_{2n+1})]\\ +\xi_2\Omega[D_b(\mathfrak{s},\eta_{2n+1})+D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})]\\ +\xi_3\Omega[D_b(\eta_{2n},\eta_{2n+1})+D_b(\eta_{2n+1},\eta_{2n})]\\ +\xi_4 D_b(\mathfrak{s},\eta_{2n+1})+\xi_5 D_b(\eta_{2n},\mathfrak{s})\end{array}\right)\right)\right)$$
$$\preccurlyeq \left(\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1\Omega D_b(\eta_{2n},\mathfrak{s})+\xi_1\Omega D_b(\mathfrak{s},\eta_{2n+1})\\ +\xi_2\Omega D_b(\mathfrak{s},\eta_{2n+1})+\xi_2\Omega D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})\\ +\xi_3\Omega^2[D_b(\eta_{2n},\mathfrak{s})+D_b(\mathfrak{s},\eta_{2n+1})]\\ +\xi_3\Omega D_b(\eta_{2n+1},\eta_{2n})+\xi_4 D_b(\mathfrak{s},\eta_{2n+1})+\xi_5 D_b(\eta_{2n},\mathfrak{s})\end{array}\right)\right)\right)$$
$$\preccurlyeq \left(\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1\Omega D_b(\eta_{2n},\mathfrak{s})+\xi_1\Omega D_b(\mathfrak{s},\eta_{2n+1})\\ +\xi_2\Omega D_b(\mathfrak{s},\eta_{2n+1})+\xi_2\Omega D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})\\ +\xi_3\Omega^2 D_b(\eta_{2n},\mathfrak{s})+\xi_3\Omega^2 D_b(\mathfrak{s},\eta_{2n+1})\\ +\xi_3\Omega D_b(\eta_{2n+1},\eta_{2n})+\xi_4 D_b(\mathfrak{s},\eta_{2n+1})+\xi_5 D_b(\eta_{2n},\mathfrak{s})\end{array}\right)\right)\right)$$
$$\preccurlyeq \left(\mathfrak{F}\left(\phi\left(\begin{array}{c}(\xi_1\Omega+\xi_3\Omega^2+\xi_5)D_b(\eta_{2n},\mathfrak{s})\\ +(\xi_1\Omega+\xi_2\Omega+\xi_3\Omega^2+\xi_4)D_b(\mathfrak{s},\eta_{2n+1})\\ +(\xi_2\Omega+\xi_3\Omega)D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})\end{array}\right)\right)\right)$$

which implies that

$$\left(\begin{array}{c}(e-\xi_2\Omega-\xi_3\Omega)\\ \phi(D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s}))\end{array}\right)\preccurlyeq\left(\mathfrak{F}\left(\phi\left(\begin{array}{c}(\xi_1\Omega+\xi_3\Omega^2+\xi_5)D_b(\eta_{2n},\mathfrak{s})\\ +(\xi_1\Omega+\xi_2\Omega+\xi_3\Omega^2+\xi_4)D_b(\mathfrak{s},\eta_{2n+1})\end{array}\right)\right)\right) \tag{3.17}$$

For another thing,

$$\begin{aligned}\phi(D_b(\mathfrak{L}\mathfrak{s},\eta_{2n+1})) &= \phi(D_b(\mathfrak{L}\mathfrak{s},\mathfrak{I}\eta_{2n}))\\ &\preccurlyeq \left(\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1\frac{D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}{1+D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}+\xi_2\frac{D_b(\eta_{2n},\mathfrak{I}\eta_{2n})}{1+D_b(\eta_{2n},\mathfrak{I}\eta_{2n})}+\xi_3\frac{D_b(\mathfrak{s},\mathfrak{I}\eta_{2n})}{1+D_b(\mathfrak{s},\mathfrak{I}\eta_{2n})}\\ +\xi_4\frac{D_b(\eta_{2n},\mathfrak{L}\mathfrak{s})}{1+D_b(\eta_{2n},\mathfrak{L}\mathfrak{s})}+\xi_5 D_b(\mathfrak{s},\eta_{2n})\end{array}\right)\right)\right)\\ &= \left(\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1\frac{D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}{1+D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}+\xi_2\frac{D_b(\eta_{2n},\eta_{2n+1})}{1+D_b(\eta_{2n},\eta_{2n+1})}+\xi_3\frac{D_b(\mathfrak{s},\eta_{2n+1})}{1+D_b(\mathfrak{s},\eta_{2n+1})}\\ +\xi_4\frac{D_b(\eta_{2n},\mathfrak{L}\mathfrak{s})}{1+D_b(\eta_{2n},\mathfrak{L}\mathfrak{s})}+\xi_5 D_b(\mathfrak{s},\eta_{2n})\end{array}\right)\right)\right)\\ &\preccurlyeq \left(\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1 D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})+\xi_2 D_b(\eta_{2n},\eta_{2n+1})+\xi_3 D_b(\mathfrak{s},\eta_{2n+1})\\ +\xi_4 D_b(\eta_{2n},\mathfrak{L}\mathfrak{s})+\xi_5 D_b(\mathfrak{s},\eta_{2n})\end{array}\right)\right)\right)\end{aligned}$$

$$\preceq \left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1\Omega[D_b(\mathfrak{s},\eta_{2n+1})+D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})]\\+\xi_2\Omega[D_b(\eta_{2n},\mathfrak{s})+D_b(\mathfrak{s},\eta_{2n+1})]\\+\xi_3D_b(\mathfrak{s},\eta_{2n+1})\\+\xi_4\Omega[D_b(\eta_{2n},\eta_{2n+1})+D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})]\\+\xi_5D_b(\mathfrak{s},\eta_{2n})\end{pmatrix}\right)\right)$$

$$\preceq \left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1\Omega D_b(\mathfrak{s},\eta_{2n+1})+\xi_1\Omega D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})\\+\xi_2\Omega D_b(\eta_{2n},\mathfrak{s})+\xi_2\Omega D_b(\mathfrak{s},\eta_{2n+1})\\+\xi_3D_b(\mathfrak{s},\eta_{2n+1})++\xi_4\Omega D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})\\+\xi_4\Omega^2[D_b(\eta_{2n},\mathfrak{s})+D_b(\mathfrak{s},\eta_{2n+1})]\\+\xi_5D_b(\mathfrak{s},\eta_{2n})\end{pmatrix}\right)\right)$$

$$\preceq \left(\mathfrak{F}\left(\phi\begin{pmatrix}\xi_1\Omega D_b(\mathfrak{s},\eta_{2n+1})+\xi_1\Omega D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})\\+\xi_2\Omega D_b(\eta_{2n},\mathfrak{s})+\xi_2\Omega D_b(\mathfrak{s},\eta_{2n+1})\\+\xi_3D_b(\mathfrak{s},\eta_{2n+1})++\xi_4\Omega D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})\\+\xi_4\Omega^2D_b(\eta_{2n},\mathfrak{s})+\xi_4\Omega^2D_b(\mathfrak{s},\eta_{2n+1})\\+\xi_5D_b(\mathfrak{s},\eta_{2n})\end{pmatrix}\right)\right)$$

$$\preceq \left(\mathfrak{F}\left(\phi\begin{pmatrix}(\xi_2\Omega+\xi_4\Omega^2+\xi_5)D_b(\mathfrak{s},\eta_{2n})\\+(\xi_1\Omega+\xi_2\Omega+\xi_3+\xi_4\Omega^2)D_b(\mathfrak{s},\eta_{2n+1})\\+(\xi_1\Omega+\xi_4\Omega)D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})\end{pmatrix}\right)\right)$$

which implies that

(3.18)

$$(e-\xi_1\Omega-\Omega\xi_4)\,\phi\left(D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})\right)\preceq\left(\mathfrak{F}\left(\phi\begin{pmatrix}(\xi_2\Omega+\xi_4\Omega^2+\xi_5)D_b(\mathfrak{s},\eta_{2n})\\+(\xi_1\Omega+\xi_2\Omega+\xi_3+\xi_4\Omega^2)D_b(\mathfrak{s},\eta_{2n+1})\end{pmatrix}\right)\right)$$

Adding inequalities (3.17) and (3.18), it follows that,

(3.19)

$$(2e-\Omega\xi)\,\phi\left(D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})\right)\preceq\begin{pmatrix}2e-\Omega\xi_1-\Omega\xi_2\\-\Omega\xi_3-\Omega\xi_4\end{pmatrix}\phi\left(D_b(\eta_{2n+1},\mathfrak{L}\mathfrak{s})\right)$$

$$\preceq\left(\mathfrak{F}\left(\phi\begin{pmatrix}(2\xi_5+\Omega\xi)D_b(\mathfrak{s},\eta_{2n})\\+(\xi_1\Omega+\xi_2\Omega+\xi_3+\xi_4+\Omega\xi)D_b(\mathfrak{s},\eta_{2n+1})\end{pmatrix}\right)\right)$$

Note that

$$\rho(\Omega\xi)=\Omega\rho(\xi)\leq(\Omega+1)\rho(\xi)\leq 2\Omega\rho(\xi_5)+(\Omega+1)\rho(\xi)<2$$

then by Lemma (2.8) it follows that $(2e-\Omega\xi)$ is invertible. As a result, it follows immediately from (3.19) that

$$\phi\left(D_b\left(\eta_{2n+1},\mathfrak{L}\mathfrak{s}\right)\right)\preccurlyeq(2e-\Omega\xi)^{-1}\left(\mathfrak{F}\left(\phi\left(\begin{array}{c}(2\xi_5+\Omega\xi)D_b\left(\mathfrak{s},\eta_{2n}\right)\\+(\xi_1\Omega+\xi_2\Omega+\xi_3+\xi_4+\Omega\xi)D_b\left(\mathfrak{s},\eta_{2n+1}\right)\end{array}\right)\right)\right)$$

Since $\{D_b(\mathfrak{s},\eta_{2n})\}$ and $\{D_b(\mathfrak{s},\eta_{2n+1})\}$ are $\mathfrak{c}$-sequence, then by Lemma (2.7), we deduce that$\{D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})\}$ is a $\mathfrak{c}$-sequence, thus $\eta_{2n+1}\to\mathfrak{s}\ (n\to+\infty)$. Hence $\mathfrak{I}\mathfrak{s}=\mathfrak{L}\mathfrak{s}=\mathfrak{w}$. In the following we shall prove $\mathfrak{L}$ and $\mathfrak{I}$ have a unique point of coincidence. Such that $\mathfrak{s}\neq\mathfrak{s}^*$ then from (3.1) we have

$$\begin{aligned}
\phi\left(D_b\left(\mathfrak{s},\mathfrak{s}^*\right)\right)&=\phi(\mathfrak{H}\left(\mathfrak{I}\mathfrak{s},\mathfrak{L}\mathfrak{s}^*\right))\\
&\preccurlyeq\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1\frac{D_b(\mathfrak{s},\mathfrak{I}\mathfrak{s})}{1+D_b(\mathfrak{s},\mathfrak{I}\mathfrak{s})}+\xi_2\frac{D_b(\mathfrak{s}^*,\mathfrak{L}\mathfrak{s}^*)}{1+D_b(\mathfrak{s}^*,\mathfrak{L}\mathfrak{s}^*)}+\xi_3\frac{D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s}^*)}{1+D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s}^*)}\\+\xi_4\frac{D_b(\mathfrak{s}^*,\mathfrak{I}\mathfrak{s})}{1+D_b(\mathfrak{s}^*,\mathfrak{I}\mathfrak{s})}+\xi_5D_b\left(\mathfrak{s},\mathfrak{s}^*\right)\end{array}\right)\right)\\
&\preccurlyeq\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1D_b\left(\mathfrak{s},\mathfrak{I}\mathfrak{s}\right)+\xi_2D_b\left(\mathfrak{s}^*,\mathfrak{L}\mathfrak{s}^*\right)+\xi_3D_b\left(\mathfrak{s},\mathfrak{L}\mathfrak{s}^*\right)\\+\xi_4D_b\left(\mathfrak{s}^*,\mathfrak{I}\mathfrak{s}\right)+\xi_5D_b\left(\mathfrak{s},\mathfrak{s}^*\right)\end{array}\right)\right)\\
&\preccurlyeq\mathfrak{F}\left(\phi\left(\xi_3D_b\left(\mathfrak{s},\mathfrak{s}^*\right)+\xi_4D_b\left(\mathfrak{s},\mathfrak{s}^*\right)+\xi_5D_b\left(\mathfrak{s},\mathfrak{s}^*\right)\right)\right)\\
&=\left(\xi_3+\xi_4+\xi_5\right)\mathfrak{F}\left(\phi D_b\left(\mathfrak{s},\mathfrak{s}^*\right)\right)
\end{aligned}$$

Set $(\xi_3+\xi_4+\xi_5)=\tau$, then it follows that

$$\phi\left(D_b\left(\mathfrak{s},\mathfrak{s}^*\right)\right)\preccurlyeq\tau\mathfrak{F}\left(\phi\left(D_b\left(\mathfrak{s},\mathfrak{s}^*\right)\right)\right)\preccurlyeq\ldots\preccurlyeq\tau^n\mathfrak{F}\left(\phi\left(D_b\left(\mathfrak{s},\mathfrak{s}^*\right)\right)\right)\tag{3.20}$$

Because of

$$2\rho(\xi_5)+2\rho(\xi)\leq 2\Omega\rho(\xi_5)+(\Omega+1)\rho(\xi)<2,$$

It follows that $\rho(\xi_5)+\rho(\xi)<1$. Since ξ_5 commutes with ξ, then by Lemma (2.5),

$$\rho(\xi_5+\xi)\leq\rho(\xi_5)+\rho(\xi)<1.$$

Accordingly, by Lemma (2.11), we speculate that $\{(\xi_5+\xi)^n\}$ is a $\mathfrak{c}$-sequence. Noticing that $\tau\leq\xi_5+\xi$ leads to $\tau^n\leq(\xi_5+\xi)^n$, we claim that $\{\tau^n\}$ is a $\mathfrak{c}$-sequence. Consequently, in view (3.20), it easy to see $\mathfrak{F}(\phi D_b(\mathfrak{s},\mathfrak{s}^*))=0$, that is $\mathfrak{s}=\mathfrak{s}^*$.

Finally, if $(\mathfrak{I},\mathfrak{L})$ is weakly compatible, then Lemma (2.9), we claim that $\mathfrak{L}$ and $\mathfrak{I}$ have a unique common fixed point. □

Corollary 3.1. *Let (X, D_b) be a cone b-metric space over Banach algebra $\mathfrak{A}$ and $\mathfrak{P}$ be a solid cone in $\mathfrak{A}$ with the coefficient $\Omega \geq 1$. $\xi_i \in \mathfrak{P}$ $(i = 1, 2, \ldots 4,)$ be a generalized Lipschitz constant with $2\Omega\rho(\xi_4) + (\Omega+1)\rho(\xi_1 + \Omega\, \xi_2 + \Omega\xi_3) < 2$. Suppose that ξ_4 commutes with $\xi_1 + \Omega\, \xi_2 + \Omega\xi_3$ and the mappings $\mathfrak{I}, \mathfrak{L} : X \to X$ be generalized $(\phi, \mathfrak{F})$-contraction mapping, satisfies that*

$$\phi\left(\xi\left(\mathfrak{I}\eta, \mathfrak{L}\Omega\right)\right) \preccurlyeq \mathfrak{F}(\phi(\urcorner(\eta, \zeta))$$

where,

$$\urcorner(\eta, \zeta) = \xi_1 \frac{D_b(\eta, \mathfrak{I}\eta)}{1 + D_b(\eta, \mathfrak{I}\eta)} + \xi_2 \frac{D_b(\eta, \mathfrak{L}\zeta)}{1 + D_b(\eta, \mathfrak{L}\zeta)} + \xi_3 \frac{D_b(\zeta, \mathfrak{I}\eta)}{1 + D_b(\zeta, \mathfrak{I}\eta)} + \xi_4 D_b(\eta, \zeta)$$

where, $\mathfrak{F} \in \Psi$, $\phi \in \Phi$ such that for all $\eta, \zeta \in X$. Moreover, if $\mathfrak{I}$ and $\mathfrak{L}$ are weakly compatible, then $\mathfrak{I}$ and $\mathfrak{L}$ have a unique common fixed point.

Proof. Choose $\xi_1 = \xi_3 = \xi_4 = \xi_5 = \xi$ and $\xi_2 = 0$ in theorem 3.1, the proof is valid. □

Corollary 3.2. *Let (X, D_b) be a cone b-metric space over Banach algebra $\mathfrak{A}$ and $\mathfrak{P}$ be a solid cone in $\mathfrak{A}$ with the coefficient $\Omega \geq 1$. $\xi_i \in \mathfrak{P}$ $(i = 1, 2, \ldots 4,)$ be a generalized nonnegative real constant with $2\Omega(\xi_4) + (\Omega+1)(\xi_1 + \Omega\, \xi_2 + \Omega\xi_3) < 2$. Let mappings $\mathfrak{I}, \mathfrak{L} : X \to X$ be generalized $(\phi, \mathfrak{F})$-contraction mapping, satisfies that*

$$\phi\left(\xi\left(\mathfrak{I}\eta, \mathfrak{L}\zeta\right)\right) \preccurlyeq \mathfrak{F}(\phi(\xi_1 D_b(\eta, \mathfrak{I}\eta) + \xi_2 D_b(\eta, \mathfrak{L}\zeta) + \xi_3 D_b(\zeta, \mathfrak{I}\eta) + \xi_4 D_b(\eta, \zeta))$$

where, $\mathfrak{F} \in \Psi$, $\phi \in \Phi$ such that for all $\eta, \zeta \in X$. Moreover, if $\mathfrak{I}$ and $\mathfrak{L}$ are weakly compatible, then $\mathfrak{I}$ and $\mathfrak{L}$ have a unique common fixed point.

Proof. Taking $\xi_1, \xi_3, \xi_4, \xi_5 \in \mathfrak{R}^+$ in theorem 3.1, we obtain the desired result. □

Corollary 3.3. *Let (X, D_b) be a cone b-metric space over Banach algebra $\mathfrak{A}$ and $\mathfrak{P}$ be a solid cone in $\mathfrak{A}$ with the coefficient $\Omega \leq 1$. $\xi_i \in \mathfrak{P}$ be a generalized Lipschitz constant with $\rho(\xi) < \frac{1}{\Omega^2+\Omega}$. Suppose that the mappings $\mathfrak{I}, \mathfrak{L} : X \to X$ be generalized $(\phi, \mathfrak{F})$-contraction mapping, satisfies that*

$$\phi\left(\xi\left(\mathfrak{I}\eta, \mathfrak{L}\zeta\right)\right) \preccurlyeq \mathfrak{F}(\phi(\xi\left(D_b(\eta, \mathfrak{L}\zeta) + D_b(\zeta, \mathfrak{I}\eta)\right))$$

where, $\mathfrak{F} \in \Psi$, $\phi \in \Phi$ such that for all $\eta, \zeta \in X$. Moreover, if $\mathfrak{I}$ and $\mathfrak{L}$ are weakly compatible, then $\mathfrak{I}$ and $\mathfrak{L}$ have a unique common fixed point.

Proof. Putting $\xi_1=\xi_2=\xi_5=0$ and $\xi_3=\xi_4=\xi$ in theorem 3.1, we complete the proof. $\square$

Example 3.1. *Let $\mathfrak{P}=\Re^2,\mathfrak{P}=\{(\eta,\zeta)\in\mathfrak{A}|\eta,\zeta\geq 0\}$ and let $X=[0,1]$. Define a function $D_b:X\times X\to\mathfrak{A}$ by $D_b(\eta,\zeta)=|\eta-\zeta|$. Clearly, (X,D_b) is a complete cone b-metric space over Banach algebra $\mathfrak{A}$ with coefficient $\Omega=2$. Now define $\mathfrak{F}:\mathfrak{P}\to\mathfrak{P}$ by $\mathfrak{F}(\mathfrak{t})=\mathfrak{t}$ for all $\mathfrak{t}>0$. Then $\mathfrak{F}\in\Psi$. Also define $\phi:\mathfrak{P}\to\mathfrak{P}$ by $\phi(\mathfrak{t})=\mathscr{K}\mathfrak{t}$, for all $\mathfrak{t}>0$. Then $\mathfrak{F}$ is a continuous comparison function.*

Define $\mathfrak{I},\mathfrak{L}:X\to X)$ by $\mathfrak{I}(\eta)=\frac{\eta}{16},\ \mathfrak{L}(\zeta)=\frac{\zeta}{4}$, for all $\eta,\zeta\in X$. Then,

$$\phi\left(D_b\left(\mathfrak{I}\eta,\mathfrak{L}\zeta\right)\right)\leq\mathfrak{F}(\phi(\urcorner(\eta,\zeta)))$$

where,

$$\urcorner(\eta,\zeta)=\xi_1\frac{D_b(\eta,\mathfrak{I}\eta)}{1+D_b(\eta,\mathfrak{I}\eta)}+\xi_2\frac{D_b(\zeta,\mathfrak{L}\zeta)}{1+D_b(\zeta,\mathfrak{L}\zeta)}+\xi_3\frac{D_b(\eta,\mathfrak{L}\zeta)}{1+D_b(\eta,\mathfrak{L}\zeta)}+\xi_4\frac{D_b(\zeta,\mathfrak{I}\eta)}{1+D_b(\zeta,\mathfrak{I}\eta)}+\xi_5 D_b(\eta,\zeta)$$

$$\begin{aligned}
\phi\left(D_b\left(\mathfrak{I}\eta,\mathfrak{L}\zeta\right)\right) &\leq\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1\frac{|\eta-\mathfrak{I}\eta|}{1+|\eta-\mathfrak{I}\eta|}+\xi_2\frac{|\zeta-\mathfrak{L}\zeta|}{1+|\zeta-\mathfrak{L}\zeta|}+\xi_3\frac{|\eta-\mathfrak{L}\zeta|}{1+|\eta-\mathfrak{L}\zeta|}\\+\xi_4\frac{|\zeta-\mathfrak{I}\eta|}{1+|\zeta-\mathfrak{I}\eta|}+\xi_5|\eta-\zeta|\end{array}\right)\right)\\
&\leq\mathfrak{F}\left(\phi\left(\begin{array}{c}\xi_1\frac{|\eta-\frac{\eta}{16}|}{1+|\eta-\frac{\eta}{16}|}+\xi_2\frac{\left|\zeta-\frac{\zeta}{4}\right|}{1+\left|\zeta-\frac{\zeta}{4}\right|}+\xi_3\frac{\left|\eta-\frac{\zeta}{4}\right|}{1+\left|\eta-\frac{\zeta}{4}\right|}\\+\xi_4\frac{|\zeta-\frac{\eta}{16}|}{1+|\zeta-\frac{\eta}{16}|}+\xi_5|\eta-\zeta|\end{array}\right)\right)\\
&\leq\mathfrak{F}\left(\phi\left(\tfrac{1}{6}\left|\eta-\zeta\right|\right)\right),\\
&\leq\mathfrak{F}\left(\tfrac{1}{6}\left(\ \phi|\eta-\zeta|\right)\right),\\
&\leq\mathfrak{F}\left(\tfrac{1}{6}\left|\eta-\zeta\right|\right)\\
&\leq\tfrac{\alpha}{6}\left|\eta-\zeta\right|, \qquad\qquad for\ \ 0<\alpha<1\\
&\leq\tfrac{\alpha}{6}(M(\eta,\zeta))\\
\phi\left(D_b\left(\mathfrak{I}\eta,\mathfrak{L}\zeta\right)\right) &\leq\mathfrak{F}(\phi(\urcorner(\eta,\zeta))).
\end{aligned}$$

Choose $\xi_1=\xi_2=\xi_3=\xi_4=0,\ \xi_5=\frac{1}{6}$. Note that $\mathfrak{I}$ and $\mathfrak{L}$ commute at the coincidence point $\eta=0$. The pair $(\mathfrak{I},\mathfrak{L})$ is weakly compatible, it is easy to see that all the conditions of theorem 3.1 holds trivially good and 0 is the unique common fixed point of $\mathfrak{I}$ and $\mathfrak{L}$.

Example 3.2. *Let $\mathfrak{P}=\Re^2,\ \mathfrak{P}=\{(\eta,\zeta)\in\mathfrak{A}|\eta,\zeta\succcurlyeq 0\}$ and let $X=\left\{0,\frac{1}{2},\frac{3}{2}\ \right\}$, be a cone b-metric space over Banach algebra $\mathfrak{A}$ with coefficient $\Omega=\frac{4}{3}>1$. let $D_b:X\times X\to\mathfrak{A}$. and*

$\mathfrak{F}:\mathfrak{P}\to\mathfrak{P}$ by $\mathfrak{F}(\mathfrak{t})<\mathfrak{t}$ for all $\mathfrak{t}>0$. Then $\mathfrak{F}\in\mathfrak{F}$. Also define $\phi:\mathfrak{P}\to\mathfrak{P}$ by $\phi(\mathfrak{t})=\mathfrak{K}\mathfrak{t}$ for some $\mathfrak{K}>0$. Then ϕ is a continuous comparison function. Now define the mapping $\mathfrak{I},\mathfrak{L}:X\to X$ by

1) $D_b(\eta,\zeta)=0$, *where* $\zeta=\eta$, *for all* $\zeta,\eta\in X$.
2) $D_b\left(0,\frac{1}{2}\right)=D_b\left(\frac{1}{2},0\right)=\frac{1}{4}$, $D_b\left(0,\frac{3}{2}\right)=D_b\left(\frac{3}{2},0\right)=\frac{1}{8}$, $D_b\left(\frac{1}{2},\frac{3}{2}\right)=D_b\left(\frac{3}{2},\frac{1}{2}\right)=\frac{1}{2}$.

Define by

$$\mathfrak{I}(0)=0,\mathfrak{I}\left(\frac{1}{2}\right)=0,\mathfrak{I}\left(\frac{3}{2}\right)=0.$$

$$\mathfrak{L}(0)=0,\mathfrak{L}\left(\frac{1}{2}\right)=\frac{3}{2},\mathfrak{L}\left(\frac{3}{2}\right)=0.$$

Let $\xi_1=\xi_2=\xi_3=\xi_4=\xi_5=\frac{1}{6}$, clearly, $\xi_1+\xi_2+\xi_3+\xi_4+\xi_5<1$. Next we will verify the condition (3.1). It have the following cases to be considered.

Case (i): *$\phi\left(D_b\left(\mathfrak{I}\eta,\mathfrak{L}\zeta\right)\right)=0$, the inequality (3.1) holds.*

Case (ii): *$\phi\left(D_b\left(\mathfrak{I}\eta,\mathfrak{L}\zeta,\right)\right)\neq 0$, we have following cases to be considered.*

Case (ii-a): *$\eta=0,\ \zeta=\frac{1}{2}$, we can get $\phi\left(D_b\left(\mathfrak{I}\eta,\mathfrak{L}\zeta\right)\right)=\frac{1}{8}$, then*

$$\phi\left(D_b\left(\mathfrak{I}\eta,\mathfrak{L}\zeta\right)\right)\preccurlyeq\mathfrak{F}(\phi\left(\urcorner(\eta,\zeta)\right)$$

where,

$$\urcorner(\eta,\zeta)=\xi_1\frac{D_b\left(\eta,\mathfrak{I}\eta\right)}{1+D_b\left(\eta,\mathfrak{I}\eta\right)}+\xi_2\frac{D_b\left(\zeta,\mathfrak{L}\zeta\right)}{1+D_b\left(\zeta,\mathfrak{L}\zeta\right)}+\xi_3\frac{D_b\left(\eta,\mathfrak{L}\zeta\right)}{1+D_b\left(\eta,\mathfrak{L}\zeta\right)}+\xi_4\frac{D_b\left(\zeta,\mathfrak{I}\eta\right)}{1+D_b\left(\zeta,\mathfrak{I}\eta\right)}+\xi_5 D_b(\eta,\zeta)$$

$$\begin{aligned}
\frac{1}{8} &\leq \mathfrak{F}\left(\phi\left(\begin{array}{c}\frac{1}{6}\frac{D_b(0,\mathfrak{I}(0))}{1+D_b(0,\mathfrak{I}(0))}+\frac{1}{6}\frac{D_b\left(\frac{1}{2},\mathfrak{L}\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\mathfrak{L}\left(\frac{1}{2}\right)\right)}+\frac{1}{6}\frac{D_b\left(0,\mathfrak{L}\left(\frac{1}{2}\right)\right)}{1+D_b\left(0,\mathfrak{L}\left(\frac{1}{2}\right)\right)}\\ +\frac{1}{6}\frac{D_b\left(\frac{1}{2},\mathfrak{I}(0)\right)}{1+D_b\left(\frac{1}{2},\mathfrak{I}(0)\right)}+\frac{1}{6}D_b\left(0,\frac{1}{2}\right)\end{array}\right)\right)\\
&\leq \frac{1}{6}\left(\begin{array}{c}\frac{D_b(0,0)}{1+D_b(0,0)}+\frac{D_b\left(\frac{1}{2},\frac{3}{2}\right)}{1+D_b\left(\frac{1}{2},\frac{3}{2}\right)}+\frac{D_b\left(0,\frac{3}{2}\right)}{1+D_b\left(0,\frac{3}{2}\right)}\\ +\frac{D_b\left(\frac{1}{2},0\right)}{1+D_b\left(\frac{1}{2},0\right)}+D_b\left(0,\frac{1}{2}\right)\end{array}\right)\\
&\leq \frac{1}{6}\left(0+\frac{\frac{1}{2}}{1+\frac{1}{2}}+\frac{\frac{1}{8}}{1+\frac{1}{8}}+\frac{\frac{1}{4}}{1+\frac{1}{4}}+\frac{1}{4}\right)\\
&\leq \frac{1}{6}\left(\frac{1}{2}\times\frac{2}{3}+\frac{1}{8}\times\frac{8}{9}+\frac{1}{4}\times\frac{4}{5}+\frac{1}{4}\right)\\
&\leq \frac{1}{6}\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{5}+\frac{1}{4}\right)\\
\frac{1}{8} &\leq \frac{1}{6}\left(\frac{161}{180}\right)
\end{aligned}$$

$$\phi\left(D_b\left(\mathfrak{I}\zeta,\mathfrak{L}\eta\right)\right)\preccurlyeq\mathfrak{F}\left(\phi\left(\urcorner(\zeta,\eta)\right)\right)$$

Therefore, the inequality (3.1) holds.

Case (ii-b): $\zeta=\frac{1}{2},\ \eta=\frac{1}{2}$, *we can get* $\phi\left(D_b\left(\Im\zeta,\mathfrak{L}\eta\right)\right)=\frac{1}{8}$, *then*

$$\begin{aligned}
\tfrac{1}{8} &\leq \mathfrak{F}\left(\phi\left(\begin{array}{c}\frac{1}{6}\frac{D_b\left(\frac{1}{2},\Im\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\Im\left(\frac{1}{2}\right)\right)}+\frac{1}{6}\frac{D_b\left(\frac{1}{2},\mathfrak{L}\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\mathfrak{L}\left(\frac{1}{2}\right)\right)}+\frac{1}{6}\frac{D_b\left(\frac{1}{2},\mathfrak{L}\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\mathfrak{L}\left(\frac{1}{2}\right)\right)}\\ +\frac{1}{6}\frac{D_b\left(\frac{1}{2},\Im\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\Im\left(\frac{1}{2}\right)\right)}+\frac{1}{6}D_b\left(\frac{1}{2},\frac{1}{2}\right)\end{array}\right)\right)\\
&\leq \frac{1}{6}\left(\begin{array}{c}\frac{D_b\left(\frac{1}{2},0\right)}{1+D_b\left(\frac{1}{2},0\right)}+\frac{D_b\left(\frac{1}{2},\frac{3}{2}\right)}{1+D_b\left(\frac{1}{2},\frac{3}{2}\right)}+\frac{D_b\left(\frac{1}{2},\frac{3}{2}\right)}{1+D_b\left(\frac{1}{2},\frac{3}{2}\right)}\\ +\frac{D_b\left(\frac{1}{2},0\right)}{1+D_b\left(\frac{1}{2},0\right)}+D_b\left(\frac{1}{2},\frac{1}{2}\right)\end{array}\right)\\
&\leq \frac{1}{6}\left(\frac{\frac{1}{4}}{1+\frac{1}{4}}+\frac{\frac{1}{2}}{1+\frac{1}{2}}+\frac{\frac{1}{2}}{1+\frac{1}{2}}+\frac{\frac{1}{4}}{1+\frac{1}{4}}+0\right)\\
&\leq \frac{1}{6}\left(\frac{1}{2}\times\frac{5}{4}+\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{5}{4}\right)\\
&\leq \frac{1}{6}\left(\frac{5}{8}+\frac{1}{3}+\frac{1}{3}+\frac{5}{8}\right)\\
\tfrac{1}{8} &\leq \frac{1}{6}\left(\frac{46}{24}\right)
\end{aligned}$$

$$\phi\left(D_b\left(\Im\zeta,\mathfrak{L}\eta\right)\right)\preccurlyeq\mathfrak{F}\left(\phi\left(\urcorner(\zeta,\eta)\right)\right)$$

Therefore, the inequality (3.1) holds.

Case (ii-c): $\zeta=\frac{3}{2},\ \eta=\frac{1}{2}$, *we can get* $\phi\left(D_b\left(\Im\zeta,\mathfrak{L}\eta\right)\right)=\frac{1}{8}$, *then*

$$\begin{aligned}
\tfrac{1}{8} &\leq \mathfrak{F}\left(\phi\left(\begin{array}{c}\frac{1}{6}\frac{D_b\left(\frac{3}{2},\Im\left(\frac{3}{2}\right)\right)}{1+D_b\left(\frac{3}{2},\Im\left(\frac{3}{2}\right)\right)}+\frac{1}{6}\frac{D_b\left(\frac{1}{2},\mathfrak{L}\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\mathfrak{L}\left(\frac{1}{2}\right)\right)}+\frac{1}{6}\frac{D_b\left(\frac{3}{2},\mathfrak{L}\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{3}{2},\mathfrak{L}\left(\frac{1}{2}\right)\right)}\\ +\frac{1}{6}\frac{D_b\left(\frac{1}{2},\Im\left(\frac{3}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\Im\left(\frac{3}{2}\right)\right)}+\frac{1}{6}D_b\left(\frac{3}{2},\frac{1}{2}\right)\end{array}\right)\right)\\
&\leq \frac{1}{6}\left(\begin{array}{c}\frac{D_b\left(\frac{3}{2},0\right)}{1+D_b\left(\frac{3}{2},0\right)}+\frac{D_b\left(\frac{1}{2},\frac{3}{2}\right)}{1+D_b\left(\frac{1}{2},\frac{3}{2}\right)}+\frac{D_b\left(\frac{3}{2},\frac{3}{2}\right)}{1+D_b\left(\frac{3}{2},\frac{3}{2}\right)}\\ +\frac{D_b\left(\frac{1}{2},0\right)}{1+D_b\left(\frac{1}{2},0\right)}+D_b\left(\frac{3}{2},\frac{1}{2}\right)\end{array}\right)\\
&\leq \frac{1}{6}\left(\frac{\frac{1}{8}}{1+\frac{1}{8}}+\frac{\frac{1}{2}}{1+\frac{1}{2}}+0+\frac{\frac{1}{4}}{1+\frac{1}{4}}+\frac{1}{2}\right)\\
&\leq \frac{1}{6}\left(\frac{1}{8}\times\frac{8}{9}+\frac{1}{2}\times\frac{2}{3}+\frac{1}{4}\times\frac{4}{5}+\frac{1}{2}\right)\\
&\leq \frac{1}{6}\left(\frac{1}{9}+\frac{1}{3}+\frac{1}{5}+\frac{1}{2}\right)\\
\tfrac{1}{8} &\leq \frac{1}{6}\left(\frac{103}{90}\right)
\end{aligned}$$

$$\phi\left(D_b\left(\Im\zeta,\mathfrak{L}\eta\right)\right)\preccurlyeq\mathfrak{F}\left(\phi\left(\urcorner(\zeta,\eta)\right)\right)$$

Therefore, the inequality (3.1) holds. We showed that the condition (3.1) is satisfied in all cases. Finally, if $(\Im,\mathfrak{L})$ *is weakly compatible, then Lemma (2.9), we claim that* $\mathfrak{L}$ *and* $\Im$ *have a unique common fixed point* $\eta=0$.

4. Application

We find the existence and uniqueness of common solutions for a system of functional equations arising in dynamic programming, which was initiated by Bellman and Lee [8]

Let $(P, \|.\|)$ and $(Q, \|.\|)$ are Banach spaces, $M \subseteq P$ and $D \subseteq Q$. Taking M and D signify the state and decision spaces, respectively. Let $B(M)$ denotes the set of all real-valued functions on M. It is easy to verify that $B(M)$ is a linear space over R under usual definitions of addition and scalar multiplication, and with the norm $\|.\|$ for an arbitrary $\eta \in B(M)$, define

$$\|\eta\| = \sup\{|\eta(\mathfrak{d})| : \mathfrak{d} \in M\}.$$

Let $D_b(\eta, \zeta) = \sup|\eta(\mathfrak{d}) - \zeta(\mathfrak{d})|$ for all $\eta, \zeta \in B(M)$. Then $(B(M), \|.\|)$ is a Banach space. As proposed in Bellman and Lee [8], the fundamental form of the functional equation in dynamic programming is

$$\omega(\mathfrak{d}) = opt_{\mathfrak{z} \in D} \mathfrak{H}(\mathfrak{d}, \mathfrak{z}, \omega(\sqsupset(\mathfrak{d}, \mathfrak{z}))), \text{for all } \mathfrak{d} \in M,$$

where $\mathfrak{d}$ and $\mathfrak{z}$ denote the state and decision vectors, respectively. $\sqsupset$ denotes the transformation of the process, $\omega(\mathfrak{d})$ denotes the optimal return function with the initial state $\mathfrak{d}$ and opt represents sup or inf.

Further, Liu et al. [21] established common fixed point theorems satisfying contractive condition of integral type and applied their results for the existence and uniqueness of common solutions to the following system of functional equations arising in dynamic programming.

$$\mathfrak{P}(\mathfrak{d}) = \text{opt}_{\mathfrak{z} \in D} \{\mathfrak{E}(\mathfrak{d}, \mathfrak{z}) + \mathfrak{H}(\mathfrak{d}, \mathfrak{z}, \mathfrak{P}(\sqsupset(\mathfrak{d}, \mathfrak{z})))\}, \text{for all } \mathfrak{d} \in M,$$

(4.1)

$$\mathfrak{T}(\mathfrak{d}) = \text{opt}_{\mathfrak{z} \in D} \{\mathfrak{E}(\mathfrak{d}, \mathfrak{z}) + \mathfrak{G}(\mathfrak{d}, \mathfrak{z}, \mathfrak{T}(\sqsupset(\mathfrak{d}, \mathfrak{z})))\}, \text{for all } \mathfrak{d} \in M,$$

where $\mathfrak{E} : M \times D \to \mathscr{R}, \sqsupset : M \times D \to M$ and $\mathfrak{H}, \mathfrak{G} : M \times D \times \mathscr{R} \to \mathscr{R}$.

Theorem 4.1. *Suppose that the following conditions are verified*

1. $\mathfrak{J}$, $\mathfrak{H}$ *and* $\mathfrak{T}$ *are bounded.*
2. *Let* $\xi_1, \xi_2, \xi_3, \xi_4$ *and* ξ_5 *be defined as in Theorem (3.1).*

$$\mathfrak{S}_1 \eta(\mathfrak{d}) = \text{opt}_{\mathfrak{z} \in D} \{\mathfrak{E}(\mathfrak{d}, \mathfrak{z}) + \mathfrak{H}_1(\mathfrak{d}, \mathfrak{z}, \eta(\sqsupset(\mathfrak{d}, \mathfrak{z})))\}, \textit{for all} (\mathfrak{d}, \eta) \in M \times B(M), \tag{4.2}$$

$$\mathfrak{Q}_1 \zeta(\mathfrak{d}) = \mathrm{opt}_{\mathfrak{z} \in D} \{\mathfrak{E}(\mathfrak{d}, \mathfrak{z}) + \mathfrak{T}_1(\mathfrak{d}, \mathfrak{z}, \zeta(\beth(\mathfrak{d}, \mathfrak{z})))\}, \textit{for all} (\mathfrak{d}, \zeta) \in M \times B(M). \tag{4.3}$$

$$\phi|\mathfrak{H}_1(\mathfrak{d}, \mathfrak{z}, \eta(\beth(\mathfrak{d}, \mathfrak{z}))) - \mathfrak{T}_1(\mathfrak{d}, \mathfrak{z}, \zeta(\beth(\mathfrak{d}, \mathfrak{z})))| \leq \mathfrak{F}(\phi(\daleth(\eta, \zeta))) \tag{4.4}$$

where

$$\daleth(\eta, \zeta) = \xi_1 \frac{|(\eta - \mathfrak{S}_1\eta)|}{1 + |(\eta - \mathfrak{S}_1\eta)|} + \xi_2 \frac{|(\zeta - \mathfrak{Q}_1\zeta)|}{1 + |(\zeta - \mathfrak{Q}_1\zeta)|}$$

$$+ \xi_3 \frac{|(\eta - \mathfrak{Q}_1\zeta)|}{1 + |(\eta - \mathfrak{Q}_1\zeta)|} + \xi_4 \frac{|(\zeta - \mathfrak{S}_1\eta)|}{1 + |(\zeta - \mathfrak{S}_1\eta)|} + \xi_5 |(\eta - \zeta)|$$

where $\eta, \zeta \in B(M), \mathfrak{d} \in M, \mathfrak{z} \in D, \psi \in \Psi, \phi \in \Phi$. Then the system of functional equations (4.1) has a unique common solution in $B(M)$.

Proof. Assume that the conditions (1) and (2). Then the system of functional equations (4.1) have a unique bounded solution if and only if the operator (4.2) and (4.3) have a common fixed point.

Let $\mathfrak{d} \in M$, for all $\eta, \zeta \in B(M)$. Suppose that opt $t_{\mathfrak{z} \in D} = \sup_{\mathfrak{z} \in D}$. For each $\varepsilon > 0$. Then using (4.2) and (4.3) we can find $\mathfrak{z}_1, \mathfrak{z}_2 \in D$ such that

$$\mathfrak{S}_1\eta(\mathfrak{d}) < \mathfrak{E}(\mathfrak{d}, \mathfrak{z}_1) + \mathfrak{H}_1(\mathfrak{d}, \mathfrak{z}_1, \eta_1(\beth(\mathfrak{d}, \mathfrak{z}_1))) + (\varepsilon), \tag{4.5}$$

$$\mathfrak{Q}_1\zeta(\mathfrak{d}) < \mathfrak{E}(\mathfrak{d}, \mathfrak{z}_2) + \mathfrak{T}_1(\mathfrak{d}, \mathfrak{z}_2, \zeta(\beth(\mathfrak{d}, \mathfrak{z}_2))) + (\varepsilon), \tag{4.6}$$

$$\mathfrak{S}_1\eta(\mathfrak{d}) \geq \mathfrak{E}(\mathfrak{d}, \mathfrak{z}_2) + \mathfrak{H}_1(\mathfrak{d}, \mathfrak{z}_2, \eta(\beth(\mathfrak{d}, \mathfrak{z}_2))), \tag{4.7}$$

$$\mathfrak{Q}_1\zeta(\mathfrak{d}) \geq \mathfrak{E}(\mathfrak{d}, \mathfrak{z}_1) + \mathfrak{T}_1(\mathfrak{d}, \mathfrak{z}_1, \zeta(\beth(\mathfrak{d}, \mathfrak{z}_1))). \tag{4.8}$$

From (4.5) and (4.8), we get

$$\begin{aligned}\mathfrak{S}_1\eta(\mathfrak{d}) - \mathfrak{Q}_1\zeta(\mathfrak{d}) &< \mathfrak{H}_1(\mathfrak{d}, \mathfrak{z}_1, \eta(\beth(\mathfrak{d}, \mathfrak{z}_1))) - \mathfrak{T}_1(\mathfrak{d}, \mathfrak{z}_1, \zeta(\beth(\mathfrak{d}, \mathfrak{z}_1))) + (\varepsilon) \\ &\leq |\mathfrak{H}_1(\mathfrak{d}, \mathfrak{z}_1, \eta(\beth(\mathfrak{d}, \mathfrak{z}_1))) - \mathfrak{T}_1(\mathfrak{d}, \mathfrak{z}_1, \zeta(\beth(\mathfrak{d}, \mathfrak{z}_1)))| + (\varepsilon).\end{aligned}$$

Hence we get

$$\mathfrak{S}_1\eta(\mathfrak{d}) - \mathfrak{Q}_1\zeta(\mathfrak{d}) < |\mathfrak{H}_1(\mathfrak{d}, \mathfrak{z}_1, \eta(\beth(\mathfrak{d}, \mathfrak{z}_1))) - \mathfrak{T}_1(\mathfrak{d}, \mathfrak{z}_1, \zeta(\beth(\mathfrak{d}, \mathfrak{z}_1)))| + (\varepsilon) \tag{4.9}$$

$$\leq \xi_1 \frac{|(\eta - \mathfrak{S}_1\eta)|}{1+|(\eta - \mathfrak{S}_1\eta)|} + \xi_2 \frac{|(\zeta - \mathfrak{Q}_1\zeta)|}{1+|(\zeta - \mathfrak{Q}_1\zeta)|}$$

$$+ \xi_3 \frac{|(\eta - \mathfrak{Q}_1\zeta)|}{1+|(\eta - \mathfrak{Q}_1\zeta)|} + \xi_4 \frac{|(\zeta - \mathfrak{S}_1\eta)|}{1+|(\zeta - \mathfrak{S}_1\eta)|} + \xi_5|(\eta - \zeta)| + (\varepsilon)$$

From (4.6) and (4.7), we get

$$\begin{aligned} \mathfrak{Q}_1\zeta(\mathfrak{d}) - \mathfrak{S}_1\eta(\mathfrak{d}) &< \mathfrak{T}_1(\mathfrak{d}, \mathfrak{z}_2, \zeta(\sqsupset(\mathfrak{d}, \mathfrak{z}_2))) - \mathfrak{H}_1(\mathfrak{d}, \mathfrak{z}_2, \eta(\sqsupset(\mathfrak{d}, \mathfrak{z}_2))) + (\varepsilon) \\ &\leq |\mathfrak{T}_1(\mathfrak{d}, \mathfrak{z}_2, \zeta(\sqsupset(\mathfrak{d}, \mathfrak{z}_2))) - \mathfrak{H}_1(\mathfrak{d}, \mathfrak{z}_2, \eta(\sqsupset(\mathfrak{d}, \mathfrak{z}_2)))| + (\varepsilon). \end{aligned} \tag{4.10}$$

$$\leq \xi_1 \frac{|(\eta - \mathfrak{S}_1\eta)|}{1+|(\eta - \mathfrak{S}_1\eta)|} + \xi_2 \frac{|(\zeta - \mathfrak{Q}_1\zeta)|}{1+|(\zeta - \mathfrak{Q}_1\zeta)|}$$
$$+ \xi_3 \frac{|(\eta - \mathfrak{Q}_1\zeta)|}{1+|(\eta - \mathfrak{Q}_1\zeta)|} + \xi_4 \frac{|(\zeta - \mathfrak{S}_1\eta)|}{1+|(\zeta - \mathfrak{S}_1\eta)|} + \xi_5|(\eta - \zeta)| + (\varepsilon)$$

From (4.9) and (4.10), we get

$$\begin{aligned} \phi(|\mathfrak{S}_1\eta(\mathfrak{d}) - \mathfrak{Q}_1\zeta(\mathfrak{d})|) &< \mathfrak{F}(\phi(|\mathfrak{H}_1(\mathfrak{d}, \mathfrak{z}_1, \eta(\sqsupset(\mathfrak{d}, \mathfrak{z}_1))) - \mathfrak{T}_1(\mathfrak{d}, \mathfrak{z}_1, \zeta(\sqsupset(\mathfrak{d}, \mathfrak{z}_1)))|)) + \phi(\varepsilon) \\ &\leq \mathfrak{F}(\phi(\xi_1 \frac{|(\eta - \mathfrak{S}_1\eta)|}{1+|(\eta - \mathfrak{S}_1\eta)|} + \xi_2 \frac{|(\zeta - \mathfrak{Q}_1\zeta)|}{1+|(\zeta - \mathfrak{Q}_1\zeta)|} \\ &+ \xi_3 \frac{|(\eta - \mathfrak{Q}_1\zeta)|}{1+|(\eta - \mathfrak{Q}_1\zeta)|} + \xi_4 \frac{|(\zeta - \mathfrak{S}_1\eta)|}{1+|(\zeta - \mathfrak{S}_1\eta)|} + \xi_5|(\eta - \zeta)|)) + \phi(\varepsilon) \end{aligned}$$

Using (4.4) and above inequality we have

$$\phi|\mathfrak{S}_1\eta(\mathfrak{d}) - \mathfrak{Q}_1\zeta(\mathfrak{d})| \leq \mathfrak{F}(\phi(\daleth(\eta, \zeta))) + \phi(\varepsilon)$$

where

$$\daleth(\eta, \zeta) = \xi_1 \frac{|(\eta - \mathfrak{S}_1\eta)|}{1+|(\eta - \mathfrak{S}_1\eta)|} + \xi_2 \frac{|(\zeta - \mathfrak{Q}_1\zeta)|}{1+|(\zeta - \mathfrak{Q}_1\zeta)|}$$

$$+ \xi_3 \frac{|(\eta - \mathfrak{Q}_1\zeta)|}{1+|(\eta - \mathfrak{Q}_1\zeta)|} + \xi_4 \frac{|(\zeta - \mathfrak{S}_1\eta)|}{1+|(\zeta - \mathfrak{S}_1\eta)|} + \xi_5|(\eta - \zeta)|$$

Since the above inequality is true for all $\mathfrak{d} \in M$ and $\varepsilon \to +\infty$, and using $\mathfrak{F} \in \Psi$, and $\phi \in \Phi$ we get

$$\phi D_b(\mathfrak{S}_1\eta, \mathfrak{Q}_1\zeta) \leq \mathfrak{F}(\phi(\daleth(\eta, \zeta)))$$

where

$$\begin{aligned}\urcorner(\eta,\zeta) &= \xi_1\frac{D_b(\eta,\mathfrak{S}_1\eta)}{1+D_b(\eta,\mathfrak{S}_1\eta)}+\xi_2\frac{D_b(\zeta,\mathfrak{Q}_1\zeta)}{1+D_b(\zeta,\mathfrak{Q}_1\zeta)}\\ &+\xi_3\frac{D_b(\eta,\mathfrak{Q}_1\zeta)}{1+D_b(\eta,\mathfrak{Q}_1\zeta)}+\xi_4\frac{D_b(\zeta,\mathfrak{S}_1\eta)}{1+D_b(\zeta,\mathfrak{S}_1\eta)}+\xi_5 D_b(\eta,\zeta)\end{aligned}$$

If $\mathfrak{I}=\mathfrak{S}_1,$ and $\mathfrak{L}=\mathfrak{Q}_1$, it is easy to observe that all the hypotheses of Theorem (3.1) are satisfied. Hence the mappings $\mathfrak{S}_1$ and $\mathfrak{Q}_1$ have a unique common fixed point in $B(M)$, and hence the system of functional equations (4.1) has a unique bounded common solution. □

5. Conclusions

In Theorem (3.1) we have formulated a new contractive conditions to modify and extend some common fixed point theorem $(\phi,\mathfrak{F})$-contraction mapping in cone b-metric space over Banach algebra. We consider the existence and uniqueness of fixed points for the contraction in the framework of b-metric spaces. As an application, we find existence and uniqueness of solutions for system of functional equations arising in dynamic programming are demonstrated with the help of our main results. We have also given examples which satisfies the condition of our main result. Our new result may be the vision for other authors to extend and improve several results in such spaces and applications to other related areas.

Conflict of Interests

The authors declare that there is no conflict of interests.

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