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APPROXIMATING FIXED POINTS OF α-CONVEX GENERALIZED NONEXPANSIVE MAPPINGS

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Abstract. The objective of our paper is to introduce a novel category of nonlinear mappings namely α-convex generalized nonexpansive and demonstrate different existence and convergence theorems for this type of mappings in Banach spaces under various assumptions. We show that this class of mappings admits an approximate fixed point sequence in each nonempty bounded closed convex Φ-invariant subset of Banach spaces. Moreover, we provide an example to support the results presented in our study. Furthermore, we expand upon the findings reported by T. Suzuki (J. Math. Anal. Appl., 340(2): 1088–1095, 2008).

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1. Introduction

Let $(\Xi, \|.\|)$ be a Banach space, and let Υ be a nonempty subset of Ξ. A mapping $\Phi : \Upsilon \to \Upsilon$ is considered nonexpansive if it satisfies the inequality $\|\Phi(\xi) - \Phi(\sigma)\| \leq \|\xi - \sigma\|$ for all $\xi, \sigma \in \Upsilon$. A point $\xi^\dagger \in \Upsilon$ is considered a fixed point of Φ if $\Phi(\xi^\dagger) = \xi^\dagger$. Nonexpansive mappings include contractions, isometries, and the resolvents of accretive operators as examples. Nonexpansive mappings have been significant in nonlinear functional analysis, being connected to variational

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inequalities and the theory of monotone operators. In Banach spaces, the study of fixed point theory for nonexpansive mappings has been extensively researched due to the significant relationship between fixed point results and the geometric properties of the norm of the Banach space where the mappings are defined.

The fixed point property for nonexpansive mappings (FPP) is defined as every nonexpansive self-mapping of a nonempty, closed, convex, bounded subset Υ of a Banach space Ξ has a fixed point. Since 1965, the study of fixed point theory for nonexpansive mappings has been extensively researched in both reflexive and nonreflexive Banach spaces. The FPP is closely related to the geometric characteristics of the Banach space. Even for a weakly compact convex subset Υ of Ξ, a nonexpansive self-mapping may not have fixed points. However, if the norm of Ξ has appropriate geometric properties, such as uniform convexity, every nonexpansive self-mapping of a weakly compact convex subset of Ξ has a fixed point, and Ξ is said to have the weak fixed point property (WFPP). While nonexpansive mappings are a significant topic in metric fixed point theory, the literature contains a significant amount of research on more general classes of mappings than nonexpansive ones. There have been various generalizations of nonexpansive mappings, such as those by Goebel and Kirk [5], Goebel et al. [6], Suzuki [17], García-Falset *et al.* [4], Llorens-Fuster [12, 11] Aoyama and Kohsaka [1]. The aim of these generalizations is to expand the class of mappings for which the fixed point results (existence and convergence of some iteration process) are valid. A compilation of some of the significant expansions and generalizations of nonexpansive mappings can be found in [14], see also [13, 16, 15].

Recently, Llorens-Fuster [11] introduced the class of partially nonexpansive mappings which propely contains the class of Suzuki-nonexpansive mappings.

Definition 1.1. Let $\Phi : \Upsilon \to \Upsilon$ be a mapping. A mapping Φ is called as partially nonexpansive, (in short, PNE), if

$$\left\| \Phi\left(\frac{1}{2}(\xi + \Phi(\xi))\right) - \Phi(\xi) \right\| \leq \frac{1}{2}\|\xi - \Phi(\xi)\| \ \forall\, \xi \in \Upsilon.$$

PNE mappings may not possess fixed points, despite being defined on compact convex sets. Nonetheless, he demonstrated that the combination of both properties, (PNE and condition (E)), ensures the existence of fixed points in Banach spaces with a suitable geometrical property in their norm.

Motivated by Llorens-Fuster [11], we consider a new class of mappings known α-convex generalized nonexpansive mappings. We present an example to illustrate our claims. We present a number of fixed points theorems for α-convex generalized nonexpansive mappings in Banach spaces with a suitable geometrical property in their norm. In this way results in [17] are extended for this class of mappings.

2. Preliminaries

Let Ξ be a Banach space and Υ a closed convex subset of Ξ such that $\Upsilon \neq \emptyset$. We denote $F(\Phi)$ the set of all fixed points of mapping Φ, i.e., $F(\Phi) = \{\xi^{\dagger} \in \Upsilon : \ \Phi(\xi^{\dagger}) = \xi^{\dagger}\}$. Suppose $\{\xi_n\}$ is a bounded sequence in Ξ. For $\xi \in \Xi$, the asymptotic radius of $\{\xi_n\}$ at ξ can be defined as follows:

$$r(\xi, \{\xi_n\}) = \limsup_{n\to\infty} \|\xi_n - \xi\|.$$

Let

$$r \equiv r(\Upsilon, \{\xi_n\}) := \inf\{r(\xi, \{\xi_n\}) : \xi \in \Upsilon\}$$

and

$$A \equiv A(\Upsilon, \{\xi_n\}) := \{\xi \in \Upsilon : r(\xi, \{\xi_n\}) = r\}.$$

The number r is asymptotic radius and the set A is asymptotic center of $\{\xi_n\}$ relative to Υ.

The sequence $\{\xi_n\}$ is called regular relative to Υ if for every subsequence $\{\xi_{n'}\}$ of $\{\xi_n\}$ the following equality holds

$$r(\Upsilon, \{\xi_n\}) = r(\Upsilon, \{\xi_{n'}\}).$$

Lemma 2.1. [7, 10]. *Let Υ and $\{\xi_n\}$ be the same as above. Then there exists a subsequence of $\{\xi_n\}$ which is regular relative to Υ.*

Definition 2.2. [17]. Let Ξ be a Banach space and Υ a nonempty subset of Ξ. A mapping $\Phi : \Upsilon \to \Upsilon$ is said to satisfy condition (C) if

$$\frac{1}{2}\|\xi - \Phi(\xi)\| \leq \|\xi - \sigma\| \text{ implies } \|\Phi(\xi) - \Phi(\sigma)\| \leq \|\xi - \sigma\| \ \forall\, \xi, \sigma \in \Upsilon.$$

Definition 2.3. [3]. A mapping $\Phi : \Upsilon \to \Upsilon$ is said to be a quasi-nonexpansive (QNE) if

$$\|\Phi(\xi) - \xi^\dagger\| \leq \|\xi - \xi^\dagger\| \ \forall \xi \in \Upsilon \text{ and } \xi^\dagger \in F(\Phi).$$

Corollary 2.4. [2]. *Let Υ be a nonempty convex subset of a Banach space Ξ. Then Υ has normal structure if and only if corresponding to each non-constant bounded sequence $\{\xi_n\}$ in Υ, the function $\tau(v) = \limsup_{n\to\infty} \|\xi_n - v\|$ is not constant in $conv\{\xi_n\}$.*

3. α-Convex Generalized Nonexpansive Mapping

In this section, we consider the following class of mappings:

Definition 3.1. Let Ξ be a Banach space and Υ be a convex subset of Ξ such that $\Upsilon \neq \emptyset$. A mapping $\Phi : \Upsilon \to \Upsilon$ is said to be α-convex generalized nonexpansive (α-CGNE) if there exists $\alpha \in (0,1)$ such that

$$\|\Phi((1-\alpha)\xi + \alpha\Phi(\xi)) - \Phi(\sigma)\| \leq (1-\alpha)\|\xi - \sigma\| + \alpha\|\Phi(\xi) - \sigma\| \tag{3.1}$$

for all $\xi, \sigma \in \Upsilon$.

Remark 3.2.
- If $\alpha = 0$, then condition (3.1) is same as nonexpansive mapping.
- If $\alpha = 1$, then condition (3.1) is fundamentally nonexpansive mapping [9].

Example 3.3. Let $\Upsilon = [0,4] \subset \mathbb{R}$ with the usual norm. Define $\Phi : \Upsilon \to \Upsilon$ by

$$\Phi(\xi) = \begin{cases} 0, & \text{if } \xi \neq 4 \\ \frac{36}{19}, & \text{if } \xi = 4 \end{cases}$$

First we show that Φ is α-convex generalized mapping for $\alpha = \dfrac{9}{10}$. We consider the following cases:

(1) If $\xi \neq 4$ and $\sigma \neq 4$ then the condition is trivially satisfied.

(2) If $\xi = 4$ and $\sigma \neq 4$

$$\begin{aligned} \|\Phi((1-\alpha)\xi + \alpha\Phi(\xi)) - \Phi(\sigma)\| &= \left\|\Phi\left(\left(1 - \frac{9}{10}\right) \times 4 + \frac{9}{10}\Phi(4)\right) - 0\right\| \\ &= \left\|\Phi\left(\frac{400}{190}\right)\right\| = 0 \leq (1-\alpha)\|\xi - \sigma\| + \alpha\|\Phi(\xi) - \sigma\|. \end{aligned}$$

(3) If $\xi \neq 4$ and $\sigma = 4$, then

$$\begin{aligned}
\|\Phi((1-\alpha)\xi + \alpha\Phi(\xi)) - \Phi(\sigma)\| &= \left\|\Phi\left(\left(1-\frac{9}{10}\right)\times\xi + \frac{9}{10}\times 0\right) - \Phi(4)\right\| \\
&= \left\|\Phi\left(\frac{\xi}{10}\right) - \frac{36}{19}\right\| = \frac{36}{19} \\
&\leq \frac{36}{10} \leq \frac{1}{10}\|\xi - 4\| + \frac{9}{10}\|0-4\| \\
&= (1-\alpha)\|\xi-\sigma\| + \alpha\|\Phi(\xi)-\sigma\|.
\end{aligned}$$

(4) If $\xi = 4$ and $\sigma = 4$, then

$$\begin{aligned}
\|\Phi((1-\alpha)\xi + \alpha\Phi(\xi)) - \Phi(\sigma)\| &= \left\|\Phi\left(\left(1-\frac{9}{10}\right)\times 4 + \frac{9}{10}\Phi(4)\right) - \frac{36}{19}\right\| \\
&= \left\|\Phi\left(\frac{400}{190}\right) - \frac{36}{19}\right\| = \frac{36}{19} = \frac{1}{10}\|4-4\| + \frac{9}{10}\left\|\frac{36}{19} - 4\right\| \\
&= (1-\alpha)\|\xi-\sigma\| + \alpha\|\Phi(\xi)-\sigma\|.
\end{aligned}$$

On the other hand, Φ does not satisfy condition (C). Indeed, at $\xi = 2.9$ and $\sigma = 4$

$$\frac{1}{2}\|\sigma - \Phi(\sigma)\| = \frac{1}{2}\left\|4 - \frac{36}{19}\right\| = \frac{20}{19} < 1.1 = \|\xi - \sigma\|$$

and

$$\|\Phi(\xi) - \Phi(\sigma)\| = \frac{36}{19} > 1.1 = \|\xi - \sigma\|.$$

Hence Φ is not nonexpansive mapping.

Proposition 3.4. *Let $\Phi : \Upsilon \to \Upsilon$ be an α-CGNE mapping with $F(\Phi) \neq \emptyset$. Then Φ is a quasi-nonexpansive.*

Proof. Let $\xi \in F(\Phi)$ and $\sigma \in \Upsilon$ Then

$$\begin{aligned}
\|\Phi((1-\alpha)\xi + \alpha\Phi(\xi)) - \Phi(\sigma)\| &= \|\Phi((1-\alpha)\xi + \alpha\xi) - \Phi(\sigma)\| \\
&= \|\xi - \Phi(\sigma)\| \leq (1-\alpha)\|\xi-\sigma\| + \alpha\|\Phi(\xi)-\sigma\| \\
&= (1-\alpha)\|\xi-\sigma\| + \alpha\|\xi-\sigma\| = \|\xi-\sigma\|.
\end{aligned}$$

Thus, $\|\xi - \Phi(\sigma)\| \leq \|\xi - \sigma\|$ for all $\xi \in F(\Phi)$ and $\sigma \in \Upsilon$. □

Example 3.5. Let Φ be a mapping on $[0,4]$ defined by

$$\Phi(\xi) = \begin{cases} 0 & \text{if } \xi \neq 4 \\ 3 & \text{if } \xi = 4 \end{cases}$$

It can be seen that $F(\Phi) = \{0\} \neq \emptyset$ and Φ is quasinonexpansive. The mapping Φ is not α-CGNE for any α. Indeed at $\xi = 4$ and $\sigma = 4$

$$\begin{aligned} \|\Phi((1-\alpha)\xi + \alpha\Phi(\xi)) - \Phi(\sigma)\| &= \|\Phi((1-\alpha)4 + \alpha\Phi(4)) - \Phi(4)\| \\ &= \|\Phi(4-\alpha) - 3\| \\ &= 3 > \alpha = (1-\alpha)\|4-4\| + \alpha\|\Phi(4) - 4\| \\ &= (1-\alpha)\|\xi - \sigma\| + \alpha\|\Phi(\xi) - \sigma\| \end{aligned}$$

hold.

Lemma 3.6. *Let Ξ be a Banach space and Υ a closed subset of Ξ such that $\Upsilon \neq \emptyset$. Let $\Phi : \Upsilon \to \Upsilon$ be an α-CGNE. Then $F(S)$ is closed in Υ. If Ξ is strictly convex and Υ is convex then $F(\Phi)$ is convex.*

Proof. Suppose there is a sequence $\{\xi_n^\dagger\}$ in $F(\Phi)$ that converges to a point $\xi^\dagger \in C$. From Proposition 3.4

$$\begin{aligned} \limsup_{n\to\infty} \left\|\xi_n^\dagger - \Phi(\xi^\dagger)\right\| &= \limsup_{n\to\infty} \left\|\Phi(\xi_n^\dagger) - \Phi(\xi^\dagger)\right\| \\ &\leq \limsup_{n\to\infty} \left\|\xi_n^\dagger - \xi^\dagger\right\| = 0. \end{aligned}$$

That is, $\{\xi_n^\dagger\}$ converges to $\Phi(\xi^\dagger)$. This implies $\Phi(\xi^\dagger) = \xi^\dagger$. Therefore $F(\Phi)$ is closed. We make the assumption that E is strictly convex and that C is a convex set. Let $\lambda \in (0,1)$ and $\xi, \sigma \in F(\Phi)$ with $\xi \neq \sigma$, and put $\xi^\dagger := \lambda\xi + (1-\lambda)\sigma \in C$. Then we have

$$\begin{aligned} \|\xi - \sigma\| &\leqslant \|\xi - \Phi(\xi^\dagger)\| + \|\sigma - \Phi(\xi^\dagger)\| = \|\Phi(\xi) - \Phi(\xi^\dagger)\| + \|\Phi(\sigma) - \Phi(\xi^\dagger)\| \\ &\leqslant \|\xi - \xi^\dagger\| + \|\sigma - \xi^\dagger\| = \|\xi - \sigma\|. \end{aligned}$$

By virtue of E being strictly convex, there exists some $\mu \in [0,1]$ such that $\Phi(\xi^\dagger) = \mu\xi + (1-\mu)\sigma$. Since

$$(1-\mu)\|\xi - \sigma\| = \|\Phi(\xi) - \Phi(\xi^\dagger)\| \leqslant \|\xi - \xi^\dagger\| = (1-\lambda)\|\xi - \sigma\|$$

and

$$\mu\|\xi-\sigma\| = \|\Phi(\sigma)-\Phi(\xi^{\dagger})\| \leqslant \|\sigma-\xi^{\dagger}\| = \lambda\|\xi-\sigma\|,$$

it follows that $1-\mu \leqslant 1-\lambda$ and $\mu \leqslant \lambda$, which implies that $\lambda = \mu$. Hence, we can conclude that $\xi^{\dagger} \in F(\Phi)$. □

The following lemma illustrates the fact that α-PNE mapping has an approximate fixed point sequence in each nonempty bounded convex Φ-invariant subset Υ of a Banach space Ξ.

Lemma 3.7. *Let Ξ be a Banach space and Υ a bounded convex subset of Ξ such that $\Upsilon \neq \emptyset$. Let $\Phi : \Upsilon \to \Upsilon$ be an α-CGNE mapping. Let $\xi_0 \in \Upsilon$ and define,*

$$\xi_{n+1} = (1-\alpha)\xi_n + \alpha\Phi(\xi_n)\ \forall\ n \in \mathbb{N}\cup\{0\}. \tag{3.2}$$

Then $\lim\limits_{n\to\infty} \|\xi_n - \Phi(\xi_n)\| = 0.$

Proof. From (3.2), we have

$$\xi_{n+1} - \xi_n = \alpha(\xi_n - \Phi(\xi_n)). \tag{3.3}$$

From (3.2), using the fact that Φ is α-CGNE,

$$\begin{aligned} \|\Phi(\xi_{n+1}) - \Phi(\xi_n)\| &= \|\Phi((1-\alpha)\xi_n + \alpha\Phi(\xi_n)) - \Phi(\xi_n)\| \\ &\leq (1-\alpha)\|\xi_n - \xi_n\| + \alpha\|\xi_n - \Phi(\xi_n)\| \\ &= \alpha\|\xi_n - \Phi(\xi_n)\|. \end{aligned}$$

From (3.3)

$$\|\Phi(\xi_{n+1}) - \Phi(\xi_n)\| \leq \|\xi_{n+1} - \xi_n\|.$$

Hence, in view of [17, Lemma 3], (see [8]), it implies that $\lim\limits_{n\to\infty} \|\xi_n - \Phi(\xi_n)\| = 0$. □

Lemma 3.8. *Let Ξ be a Banach space and Υ a convex subset of Ξ such that $\Upsilon \neq \emptyset$. Let $\Phi : \Upsilon \to \Upsilon$ be an α-CGNE mapping. Then*

$$\|\xi - \Phi(\sigma)\| \leq 2\alpha\|\xi - \Phi(\xi)\| + \|(1-\alpha)\xi + \alpha\Phi(\xi) - \Phi((1-\alpha)\xi + \alpha\Phi(\xi))\| + \|\xi - \sigma\|$$

holds for all $\xi, \sigma \in \Upsilon$.

Proof. By the triangle inequality, we have

$$\begin{aligned}
\|\xi-\Phi(\sigma)\| &\leq \|\xi-\Phi((1-\alpha)\xi+\alpha\Phi(\xi))\|+\|\Phi((1-\alpha)\xi+\alpha\Phi(\xi))-\Phi(\sigma)\| \\
&\leq \|\xi-(1-\alpha)\xi+\alpha\Phi(\xi)\|+\|(1-\alpha)\xi+\alpha\Phi(\xi)-\Phi((1-\alpha)\xi+\alpha\Phi(\xi))\| \\
&\quad +\|\Phi((1-\alpha)\xi+\alpha\Phi(\xi))-\Phi(\sigma)\| \\
&= \alpha\|\xi-\Phi(\xi)\|+\|(1-\alpha)\xi+\alpha\Phi(\xi)-\Phi((1-\alpha)\xi+\alpha\Phi(\xi))\| \\
&\quad +\|\Phi((1-\alpha)\xi+\alpha\Phi(\xi))-\Phi(\sigma)\|.
\end{aligned}$$

Again by the condition on the mapping Φ, we get

$$\begin{aligned}
\|\xi-\Phi(\sigma)\| &\leq \alpha\|\xi-\Phi(\xi)\|+\|(1-\alpha)\xi+\alpha\Phi(\xi)-\Phi((1-\alpha)\xi+\alpha\Phi(\xi))\| \\
&\quad +(1-\alpha)\|\xi-\sigma\|+\alpha\|\Phi(\xi)-\sigma\| \\
&\leq \alpha\|\xi-\Phi(\xi)\|+\|(1-\alpha)\xi+\alpha\Phi(\xi)-\Phi((1-\alpha)\xi+\alpha\Phi(\xi))\| \\
&\quad +(1-\alpha)\|\xi-\sigma\|+\alpha\|\xi-\sigma\|+\alpha\|\xi-\Phi(\xi)\| \\
&\leq 2\alpha\|\xi-\Phi(\xi)\|+\|(1-\alpha)\xi+\alpha\Phi(\xi)-\Phi((1-\alpha)\xi+\alpha\Phi(\xi))\|+\|\xi-\sigma\|.
\end{aligned}$$

This completes the proof. $\square$

Theorem 3.9. *Let Ξ be a Banach space having normal structure. Let Υ be a weakly compact convex subset of Ξ such that $\Upsilon \neq \emptyset$. Let $\Phi : \Upsilon \to \Upsilon$ be an α-CGNE mapping. Then Φ admits a fixed point.*

Proof. Let

$$\Sigma := \{\mathscr{E} \subseteq \Upsilon : \text{ where } \mathscr{E} \text{ is weakly compact convex } \Phi\text{-invariant }\}.$$

Note that $\Upsilon \in \Sigma$. By reverse set inclusion, we define a partially order, that is, $\mathscr{E}_1 \preceq \mathscr{E}_2 \Leftrightarrow \mathscr{E}_1 \supseteq \mathscr{E}_2$. Consider chain $\Gamma \subset \Sigma$ and the set

$$\mathscr{E}^o = \bigcap_{\mathscr{E}\in\Gamma} \mathscr{E}.$$

Weak compactness of Υ implies that Γ has the finite intersection property, so $\mathscr{E}^o \neq \emptyset$. Moreover $\mathscr{E}^o$ is the upper bound for the chain Γ. By Zorn's lemma there is a maximal element $\mathscr{F} \in \Sigma$. In other words, there is a nonempty weakly compact convex Φ-invariant subset $\mathscr{F}$ of Υ with no proper subsets. Let $\xi_0 \in \mathscr{F}$ and define,

$$\xi_{n+1} = (1-\alpha)\xi_n + \alpha\Phi(\xi_n) \ \forall\ n \in \mathbb{N} \cup \{0\}.$$

By the convexity of set $\mathscr{F}$, the sequence $\{\xi_n\}$ in $\mathscr{F}$. In view of Lemma 3.7 it follows that $\|\xi_n - \Phi(\xi_n)\| \to 0$ as $n \to \infty$. The sequence is either constant and therefore comprises a fixed point of Φ, or it is non-constant. Define a function $\tau : \Upsilon \to [0, \infty)$ such that

$$\tau(v) = \limsup_{n\to\infty} \|\xi_n - v\|.$$

Since the sequence $\{\xi_n\}$ is bounded and the space Ξ has normal structure property, from Corollary 2.4 function τ is not constant on $\operatorname{conv}\{\xi_n\}$. Let us assume that τ admits at least two distinct real values. More precisely, let $v_1, v_2 \in \operatorname{conv}\{\xi_n\} \subseteq \Upsilon$ such that

$$\lambda_1 := \tau(v_1) < \tau(v_2) =: \lambda_2.$$

Take $\lambda = \dfrac{\lambda_1 + \lambda_2}{2}$ and define a set

$$\mathscr{H} := \{w \in \mathscr{F} : \tau(w) \leq \lambda\}$$

we observe that $v_1 \in \mathscr{H}$ and $v_2 \notin \mathscr{H}$. Now we claim that $\mathscr{H}$ is closed and convex subset of $\mathscr{F}$. Let $\{w_n\}$ be a sequence in $\mathscr{H}$ such that $\{w_n\}$ converges strongly to a point $\xi \in \mathscr{F}$. Then for each arbitrary $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|w_n - \xi\| \leq \varepsilon$ for all $n \geq n_0$. For all $n \geq n_0$, we have

$$\begin{aligned} \|\xi_n - \xi\| &\leq \|\xi_n - w_n\| + \|w_n - \xi\| \\ &< \|\xi_n - w_n\| + \varepsilon, \end{aligned}$$

which implies that $\tau(\xi) \leq \lambda$. Therefore $\xi \in \mathscr{H}$ and $\mathscr{H}$ is closed. Now let $w_1, w_2 \in \mathscr{H}$ and $t \in [0,1]$. Then we have

$$\|\xi_n - (tw_1 + (1-t)w_2)\| \leq t\|\xi_n - w_1\| + (1-t)\|\xi_n - w_2\|,$$

which implies that $\tau(tw_1+(1-t)w_2) \le t\tau(w_1)+(1-t)\tau(w_2)$. Thus $tw_1+(1-t)w_2 \in \mathscr{H}$ and $\mathscr{H}$ is convex. Let $\sigma \in \mathscr{H}$, from the Lemma 3.8, we have

$$\begin{aligned}\|\xi_n-\Phi(\sigma)\| &\le 2\alpha\|\xi_n-\Phi(\xi_n)\|+\|(1-\alpha)\xi_n+\alpha\Phi(\xi_n)-\Phi((1-\alpha)\xi_n+\alpha\Phi(\xi_n))\| \\ &\quad +\|\xi_n-\sigma\| \\ &\le 2\alpha\|\xi_n-\Phi(\xi_n)\|+\|\xi_{n+1}-\Phi(\xi_{n+1})\|+\|\xi_n-\sigma\|.\end{aligned}$$

Thus

$$\begin{aligned}\limsup\|\xi_n-\Phi(\xi)\| &\le \limsup\left(2\alpha\|\xi_n-\Phi(\xi_n)\|+\|\xi_{n+1}-\Phi(\xi_{n+1})\|+\|\xi_n-\sigma\|\right) \\ &= \limsup\|\xi_n-\xi\|.\end{aligned}$$

Thus, $\tau(\Phi(\sigma)) \le \tau(\sigma) \le \lambda$. Thus $\mathscr{H}$ is a nonempty closed convex Φ-invariant subset of $\mathscr{F}$ with $\mathscr{H} \neq \mathscr{F}$ and this contradict the minimality of the set $\mathscr{F}$. Hence Φ admits a fixed point. $\square$

Corollary 3.10. *Let Ξ be a uniformly convex in every direction (in short, UCED) Banach space and Υ a weakly compact convex subset of Ξ such that $\Upsilon \neq \emptyset$. Let $\Phi : \Upsilon \to \Upsilon$ be an α-CGNE mapping. Then Φ has a fixed point.*

Theorem 3.11. *Let Ξ be Banach space and Υ a compact convex subset of Ξ such that $\Upsilon \neq \emptyset$. Assume that Φ is same as in Theorem 3.9. Define a sequence $\{\xi_n\}$ in Υ by $\xi_0 \in \Upsilon$ and*

$$\xi_{n+1} = (1-\alpha)\xi_n+\alpha\Phi(\xi_n)\ \forall\, n \in \mathbb{N}\cup\{0\}.$$

Then $\{\xi_n\}$ converges strongly to a fixed point of Φ.

Proof. From Lemma 3.7 it follows that $\lim_{n\to\infty}\|\Phi(\xi_n)-\xi_n\| = 0$. Due to the compactness of Υ, it is guaranteed that there exists a subsequence $\{\xi_{n_j}\}$ of $\{\xi_n\}$ and an element $\xi^\dagger \in \Upsilon$ such that $\{\xi_{n_j}\}$ converges to $\xi^\dagger$. From the Lemma 3.8, we have

$$\begin{aligned}\|\xi_{n_j}-\Phi(\xi^\dagger)\| &\le 2\alpha\|\xi_{n_j}-\Phi(\xi_{n_j})\|+\|(1-\alpha)\xi_{n_j}+\alpha\Phi(\xi_{n_j})-\Phi((1-\alpha)\xi_{n_j}+\alpha\Phi(\xi_{n_j}))\| \\ &\quad +\|\xi_n-\xi^\dagger\| \\ &\le 2\alpha\|\xi_{n_j}-\Phi(\xi_n)\|+\|\xi_{n_j+1}-\Phi(\xi_{n_j+1})\|+\|\xi_{n_j}-\xi^\dagger\|.\end{aligned}$$

for all $j \in \mathbb{N}$. Thus $\{\xi_{n_j}\}$ converges to $\Phi(\xi^\dagger)$ which implies $\Phi(\xi^\dagger) = \xi^\dagger$, that is $\xi^\dagger \in F(\Phi)$. Again, from Proposition 3.4

$$\left\|\xi_{n+1} - \xi^\dagger\right\| \leq \alpha \left\|\Phi(\xi_n) - \xi^\dagger\right\| + (1-\alpha)\left\|\xi_n - \xi^\dagger\right\| \leq \left\|\xi_n - \xi^\dagger\right\|$$

for $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|\xi_n - \xi^\dagger\|$ exists. Hence $\{\xi_n\}$ converges to $\xi^\dagger$. $\square$

Theorem 3.12. *Let Ξ be a Banach space having the Opial property and Υ a weakly compact convex subset of Ξ such that $\Upsilon \neq \emptyset$. Assume that Φ and $\{\xi_n\}$ are same as in Theorem 3.11. Then $\{\xi_n\}$ converges weakly to a fixed point of Φ.*

Proof. From Lemma 3.7 it follows that $\lim_{n\to\infty} \|\Phi(\xi_n) - \xi_n\| = 0$. Given that Υ is weakly compact, there exists a subsequence $\{\xi_{n_j}\}$ of $\{\xi_n\}$ and a point $\xi^\dagger \in \Upsilon$, such that $\{\xi_{n_j}\}$ weakly converges to $\xi^\dagger$. From the Lemma 3.8, we have

$$\begin{aligned}
\|\xi_{n_j} - \Phi(\xi^\dagger)\| &\leq 2\alpha\|\xi_{n_j} - \Phi(\xi_{n_j})\| + \|(1-\alpha)\xi_{n_j} + \alpha\Phi(\xi_{n_j}) - \Phi((1-\alpha)\xi_{n_j} + \alpha\Phi(\xi_{n_j}))\| \\
&\quad + \|\xi_{n_j} - \xi^\dagger\| \\
&\leq 2\alpha\|\xi_{n_j} - \Phi(\xi_{n_j})\| + \|\xi_{n_j+1} - \Phi(\xi_{n_j+1})\| + \|\xi_{n_j} - \xi^\dagger\|.
\end{aligned}$$

Thus,

$$\liminf_{n\to\infty} \|\xi_{n_j} - \Phi(\xi^\dagger)\| \leq \liminf_{n\to\infty} \|\xi_{n_j} - \xi^\dagger\|.$$

In view of Opial property, we get

$$\liminf_{n\to\infty} \|\xi_{n_j} - \Phi(\xi^\dagger)\| \leq \liminf_{n\to\infty} \|\xi_{n_j} - \xi^\dagger\| < \liminf_{n\to\infty} \|\xi_{n_j} - \Phi(\xi^\dagger)\|$$

a contradiction, thus $\Phi(\xi^\dagger) = \xi^\dagger$.. Following the same proof technique as in Theorem 3.11, one can show that $\{\left\|\xi_n - \xi^\dagger\right\|\}$ is a nonincreasing sequence. Employing a strategy of assuming the opposite and deriving a contradiction, suppose $\{\xi_n\}$ does not converge weakly to $\xi^\dagger$. Then $\{\xi_{n_k}\}$ of $\{\xi_n\}$ and $w \in \Upsilon$ such that $\{\xi_{n_k}\}$ converges weakly to w and $\xi^\dagger \neq w$. We note $\Phi(w) = w$. By Opial property,

$$\begin{aligned}
\lim_{n\to\infty} \left\|\xi_n - \xi^\dagger\right\| &= \lim_{j\to\infty} \left\|\xi_{n_j} - \xi^\dagger\right\| < \lim_{j\to\infty} \|\xi_{n_j} - w\| = \lim_{n\to\infty} \|\xi_n - w\| \\
&= \lim_{k\to\infty} \|\xi_{n_k} - w\| < \lim_{k\to\infty} \left\|\xi_{n_k} - \xi^\dagger\right\| = \lim_{n\to\infty} \left\|\xi_n - \xi^\dagger\right\|.
\end{aligned}$$

This is a contradiction. This completes the proof. $\square$

Theorem 3.13. *Let Ξ and Υ be same as in Corollary 3.10. Let S be a family of commuting α-convex generalized nonexpansive self-mappings on Υ. Then S has a common fixed point.*

Proof. Suppose $\Phi_1, \Phi_2, \dots, \Phi_l \subset S$. By Corollary 3.10, there exists a fixed point of Φ_1 in Υ and $F(\Phi_1) \neq \emptyset$. Using Lemma 3.6, we know that $F(\Phi_1)$ is a closed and convex set. Let $A := \bigcap_{j=1}^{k-1} F(\Phi_j)$, where k is a positive integer such that $1 < k \leq l$. Since A is the intersection of closed convex sets, it is nonempty, closed, and convex.

Take $\xi \in A$ and $j \in \mathbb{N}$ such that $1 \leq j < k$. Since $\Phi_k \circ \Phi_j = \Phi_j \circ \Phi_k$, we have $\Phi_k(\xi) = \Phi_k \circ \Phi_j(\xi) = \Phi_j \circ \Phi_k(\xi)$, which implies that $\Phi_k(\xi)$ is a fixed point of Φ_j. Therefore, $\Phi_k(\xi) \in A$, and $\Phi_k(A) \subset A$. By Corollary 3.10, Φ_k admits a fixed point in A, and we have $A \cap F(\Phi_k) = \bigcap_{j=1}^{k} F(\Phi_j) \neq \emptyset$. Since $A \cap F(\Phi_k)$ is a closed and convex set, it follows from induction that $\bigcap_{j=1}^{l} F(\Phi_j) \neq \emptyset$. Thus, we have shown that $\{F(\Phi) : \Phi \in S\}$ has the finite intersection property. Since Υ is weakly compact and $F(\Phi)$ is weakly closed for every $\Phi \in S$, it follows that $\bigcap_{\Phi \in S} F(\Phi) \neq \emptyset$. $\square$

Theorem 3.14. *Let Ξ be a Banach space and Υ a bounded closed convex subset of Ξ such that $\Upsilon \neq \emptyset$. Let $\Phi : \Upsilon \to \Upsilon$ be an α-CGNE. If any bounded sequence in Ξ has an asymptotic center in Υ that is both non-empty and compact, then Φ must have a fixed point.*

Proof. Let $\{\xi_n\}$ be a sequence defined in 3.2, by Lemma 3.7 $\{\xi_n\}$ is an a.f.p sequence for Φ and $\lim_{n\to\infty} \|\Phi(\xi_n) - \xi_n\| = 0$. Let $A(\Upsilon, \{\xi_n\})$ be the asymptotic center of $\{\xi_n\}$ with respect to Υ. Based on the given hypothesis, $A(\Upsilon, \{\xi_n\})$ is not an empty set and is compact. Suppose $\xi^\dagger \in A(\Upsilon, \{\xi_n\})$. In view of the definition of asymptotic radius of $\{\xi_n\}$ and from Lemma 3.8, we have

$$
\begin{aligned}
r\left(\Phi(\xi^\dagger), \{\xi_n\}\right) &= \limsup_{n\to\infty} \left\|\xi_n - \Phi(\xi^\dagger)\right\| \\
&\leq \limsup_{n\to\infty} 2\alpha \|\Phi(\xi_n) - \xi_n\| + \limsup_{n\to\infty} \|(1-\alpha)\xi_n + \alpha\Phi(\xi_n) - \Phi((1-\alpha)\xi_n \\
&\quad + \alpha\Phi(\xi_n))\| + \limsup_{n\to\infty} \left\|\xi_n - \xi^\dagger\right\| \\
&= \limsup_{n\to\infty} 2\alpha \|\Phi(\xi_n) - \xi_n\| + \limsup_{n\to\infty} \|\xi_{n+1} - \Phi(\xi_{n+1})\| \\
&\quad + \limsup_{n\to\infty} \left\|\xi_n - \xi^\dagger\right\|
\end{aligned}
$$

$$= \limsup_{n\to\infty} \left\| \xi_n - \xi^\dagger \right\|$$
$$= r\left(\xi^\dagger, \{\xi_n\}\right).$$

It follows that $\Phi(\xi^\dagger) \in A(\Upsilon, \{\xi_n\})$. Thus $A(\Upsilon, \{\xi_n\})$ is invariant under Φ. Therefore, the proof is now complete because Theorem 3.11 guarantees the existence of at least one fixed point of Φ in $A(\Upsilon, \{\xi_n\})$. $\square$

Remark 3.15. Banach spaces that are either uniformly convex or k-uniformly rotund can be included in the scope of Theorem 3.14, as these spaces fulfill the assumption required in the theorem.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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