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FIXED POINTS VIA INTERPOLATIVE ENRICHED CONTRACTIONS IN METRIC AND QUASI-METRIC SPACES

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Abstract. In this paper, we introduce the concept of generalized interpolative enriched Kannan to prove some fixed point results in the setting of complete convex metric spaces. We give an example to illustrate our result. As an application, we obtain a fixed point theorem in convex quasi-metric space.

Keywords: fixed point; interpolative; enriched Kannan; complete convex metric spaces; convex quasi-metric space.

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1. Introduction

Fixed point theory is a mathematical theory that deals with the study of fixed points in functions or mappings. A fixed point of a function is a point that remains unchanged when the function is applied to it. More formally, if you have a mapping T and a point x, then u is a fixed point of T if $Tx = x$.

Fixed point theory has a wide range of applications in various areas of mathematics and beyond. Some key aspects and concepts related to fixed point theory include:

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Existence and Uniqueness Theorems: Fixed point theory provides theorems and conditions that establish when fixed points exist and whether they are unique for certain functions or mappings. The Banach Fixed Point Theorem is one of the most well-known results in this area.

Contraction Mappings: Contraction mappings are a specific class of mappings for which fixed point theorems often apply. These mappings contract distances between points, and the Banach Fixed Point Theorem is a classic example that applies to contraction mappings see for example [2, 3, 4, 5, 6, 7, 8, 9, 14, 15, 16, 17, 18, 19, 21, 22, 23].

Applications in Equations and Systems: Fixed point theory can be used to study and solve equations and systems of equations. Many mathematical problems can be reformulated as fixed point problems, allowing for applying fixed point theorems to find solutions.

Topological and Metric Spaces: Fixed point theory is often applied in the context of topological spaces and metric spaces. Complete metric spaces, in particular, play a crucial role in many fixed point theorems.

Iterative Methods: Fixed point iterations are common in numerical analysis and optimization. These iterative methods are often used to approximate fixed points numerically.

Applications Beyond Mathematics: Fixed point theory has applications in physics, engineering, economics, computer science, and various other fields. For example, it's used in the study of dynamical systems, game theory, and equilibrium points in economic models

In 2018 Karapinar [1] proposed a new Kannan-type contractive mapping using the concept of interpolation and proved a fixed point theorem in metric space. This new type of mapping, called interpolative Kannan-type contractive mapping, is a generalization of Kannan's fixed point theorem.

Theorem 1. *Let us recall that an interpolative Kannan contraction on a metric space* (E,d) *is a self-mapping* $T: E \to E$ *such that there exist* $k \in [0,1)$ *and* $\alpha \in (0,1)$ *such that*

$$d(Tx,Ty) \leq k\left[d(Tx,x)\right]^{\alpha}\left[d(Ty,y)\right]^{1-\alpha}, \tag{1}$$

$(x,y) \in E \times E$ *with* $x,y \notin Fix(T)$

Then T *has a unique fixed point in* E.

This theorem has been generalized in 2023 by Edraoui et al. [15] for various types of cyclic contractions in a metric space.

Definition 2. [15] *Let (E,d) be a metric space and let X and Y be nonempty subsets of E. A cyclic map $T: X\cup Y \to X\cup Y$ is said to be an interpolative Kannan Type cyclic contraction if there exists $k\in[0,1)$ and $\alpha\in(0,1)$ such that*

$$d(Tx,Ty)\leq k\left[d(Tx,x)\right]^{\alpha}\left[d(Ty,y)\right]^{1-\alpha} \tag{2}$$

for all $(x,y)\in X\times Y$ with $x,y\notin Fix(T)$.

Theorem 3. [15] *Let (E,d) be a complete metric space and let X and Y be nonempty subsets of E and let $T: X\cup Y\to X\cup Y$ be interpolative Kannan type cyclic contraction. Then T has a unique fixed point in $X\cap Y$.*

Takahashi [3] introduced the notion of convexity within metric spaces, specifically referred to as convex metric spaces. In doing so, the investigation centred around establishing the existence of fixed points for nonexpansive mappings within these convex metric spaces.

Before providing the main result, we need to introduce some basic facts about convex metric spaces.

Definition 4. *Let E be a metric space. A continuous function $W: E\times E\times[0,1]\to E$ is said to be a convex structure on E, if for all $u,v\in E$ and $\lambda\in[0,1]$ the following inequality holds*

$$d(z,W(u,v,\lambda)) = \lambda d(z,u)+(1-\lambda)d(z,v) \text{ for any } z\in E \tag{3}$$

Example 5. *Let $E=\mathbb{R}^2$. For $u=(u_1,u_2)$, $v=(v_1,v_2)$ in E and $\lambda\in[0,1]$. Define a mapping $W: E\times E\times[0,1]\to E$ by*

$$W(u,v,\lambda)=\left(\lambda u_1+(1-\lambda)v_1, \frac{\lambda u_1u_2+(1-\lambda)v_1v_2}{\lambda u_1+(1-\lambda)v_1}\right)$$

and a metric

$$\begin{aligned} d &: E\times E\times \to [0,\infty) \\ u,v &\to d(u,v)=|u_1-v_1|+|u_1v_1-v_1v_2| \end{aligned}$$

It can be verified that E is a convex metric space.

Lemma 6. [2] *Let (E,d,W) be a convex metric space and $T:E\to E$. Define the mapping $T_\lambda : E\to E$ by*

$$T_\lambda u = W(u,Tu,\lambda)\ \ u\in E \tag{4}$$

Then, for any $\lambda\in[0,1)$, we have

$$Fix(T) = Fix(T_\lambda). \tag{5}$$

Definition 7. [2] *Let (E,d,W) be a convex metric space and $T:E\to E$. be a selfmap. If there exist $k\in[0,1)$ and $\lambda\in[0,1]$ such that*

$$d(W(u,Tu,\lambda),W(v,Tv,\lambda))\le kd(u,v) \tag{6}$$

for all $u,v\in E$. We also call T a (λ,k)-enriched Kannan type contraction.

We defne $T_\lambda : E\to E$ by $T_\lambda u = W(u,Tu,\lambda)$. Then (6) reduces to

$$d(T_\lambda u, T_\lambda v)\le kd(u,v) \text{ for all } u,v\in E. \tag{7}$$

Employing the ideas of Karapinar [1] Berinde and Pacurar , Karapinar [2] we introduce the concept of generalized interpolative enriched Kannan prove some fixed point results in the setting of complete convex metric spaces. We give an example to illustrate our result. As an application, we obtain a fixed point theorem in convex quasi-metric space.

2. Main Results

We enrich the interpolative Kannan type contraction as follows:

Definition 8. *Let (E,d,W)be a convex metric space. A self-mapping $T:E\to E$ is an generalized interpolative enriched Kannan type contraction if there exist $k\in[0,1)$, $\lambda\in[0,1)$ and $\alpha\in(0,1)$*

$$d(W(u,Tu,\lambda),W(v,Tv,\lambda))\le k\max\left\{\begin{array}{c}[d(u,W(u,Tu,\lambda))]^{\alpha}.[d(v,W(v,Tv,\lambda))]^{1-\alpha},\\ [d(u,W(u,Tu,\lambda))]^{1-\alpha}.[d(v,W(v,Tv,\lambda))]^{\alpha}\end{array}\right\} \tag{8}$$

for all $u,v\in E\backslash Fix(T)$.

Theorem 9. *Let (E,d,W) be a complete convex metric space $T:E\to E$ be a continuous generalized interpolative enriched Kannan type contraction. Then,*

1: $Fix(T) \neq \emptyset$

2: The following estimate holds

$$d(u_{n+m-1}, u^*) \leq \frac{k^m}{1-k} d(u_n, u_{n-1}), n = 1, 2, .. \text{and } m = 1, 2, .. \tag{9}$$

where $u^* = Fix(T)$

Proof. Using Lemma (6), the condition (8), is transformed to following equivalent form

$$d(T_\lambda u, T_\lambda v) \leq k \max\left\{[d(u, T_\lambda u)]^\alpha . [d(v, T_\lambda v)]^{1-\alpha}, [d(u, T_\lambda u)]^{1-\alpha} . [d(v, T_\lambda v)]^\alpha\right\}. \tag{10}$$

Namely, T_λ represents an generalized interpolative Kannan type contraction. The Picard iteration linked with T_λ corresponds to the Krasnoselskij iterative approach $\{u_n\}_{n=0}^{\infty}$ associated to T, which is defned by $u_{n+1} = W(u_n, Tu_n, \lambda)$, $i.e$, $u_{n+1} = T_\lambda u_n$, $n \geq 0$.

Without any loss of generality, we assume that the successive terms of $\{u_n\}$ are distinct (i,e $u_{n+1} \neq u_n$) for each nonnegative integer n. Indeed, if there exists a nonnegative integer n_0, such that $u_{n_0} = u_{n_0+1} = T_\lambda u_{n_0}$,Then u_{n_0} becomes a fixed point. Hence, we obtain.

Take $u = u_n$ and $v = u_{n-1}$ in (10), we get

$$d(T_\lambda u_n, T_\lambda u_{n-1}) \leq k \max\left\{[d(u_n, T_\lambda u_n)]^\alpha . [d(u_{n-1}, T_\lambda u_{n-1})]^{1-\alpha}, [d(u_n, T_\lambda u_n)]^{1-\alpha} . [d(u_{n-1}, T_\lambda u_{n-1})]^\alpha\right\}$$

$$d(u_{n+1}, u_n) \leq k \max\left\{[d(u_n, u_{n+1})]^\alpha . [d(u_{n-1}, u_n)]^{1-\alpha}, [d(u_n, u_{n+1})]^{1-\alpha} [.d(u_{n-1}, u_n)]^\alpha\right\}$$

If

$$\begin{aligned} &\max\left\{[d(u_n, u_{n+1})]^\alpha . [d(u_{n-1}, u_n)]^{1-\alpha}, [d(u_n, u_{n+1})]^{1-\alpha} . [d(u_{n-1}, u_n)]^\alpha\right\} \\ &= [d(u_n, u_{n+1})]^\alpha . [d(u_{n-1}, u_n)]^{1-\alpha} \end{aligned}$$

Then

$$d(u_{n+1}, u_n) \leq kd[(u_n, u_{n+1})]^\alpha . [d(u_{n-1}, u_n)]^{1-\alpha}$$

which further gives

$$d[(u_{n+1}, u_n)]^{1-\alpha} \leq k. [d(u_{n-1}, u_n)]^{1-\alpha}$$

We deduce that

$$\begin{aligned} d(u_{n+1}, u_n) &\leq k.d(u_{n-1}, u_n) \\ &\leq k.d(u_{n-2}, u_{n-1}) \end{aligned} \tag{11}$$

$$\begin{aligned} & \vdots \\ \leq \; & k^{n}.d\left(u_{0},u_{1}\right) \end{aligned}$$

If

$$\max\left\{\left[d\left(u_{n},u_{n+1}\right)\right]^{\alpha}.\left[d\left(u_{n-1},u_{n}\right)\right]^{1-\alpha},\left[d\left(u_{n},u_{n+1}\right)\right]^{1-\alpha}.\left[d\left(u_{n-1},u_{n}\right)\right]^{\alpha}\right\}$$
$$=\left[d\left(u_{n},u_{n+1}\right)\right]^{1-\alpha}.\left[d\left(u_{n-1},u_{n}\right)\right]^{\alpha},$$

then

$$d\left(u_{n+1},u_{n}\right)\leq k\left[d\left(u_{n},u_{n+1}\right)\right]^{1-\alpha}.\left[d\left(u_{n-1},u_{n}\right)\right]^{\alpha}.$$

which further gives

$$d\left[\left(u_{n+1},u_{n}\right)\right]^{\alpha}\leq k.\left[d\left(u_{n-1},u_{n}\right)\right]^{\alpha}.$$

We deduce that

$$\begin{aligned} d\left(u_{n+1},u_{n}\right) & \leq k.d\left(u_{n-1},u_{n}\right) \\ & \leq k.d\left(u_{n-2},u_{n-1}\right) \\ & \;\;\vdots \\ & \leq k^{n}.d\left(u_{0},u_{1}\right) \end{aligned} \tag{12}$$

It follows from (11) and (12), we get

$$d\left(u_{n+1},u_{n}\right)\leq k^{n}.d\left(u_{0},u_{1}\right). \tag{13}$$

For $m>0$. Using the triangular inequality, we obtain

$$\begin{aligned} d\left(u_{n},u_{n+m}\right) & \leq d\left(u_{n},u_{n+1}\right)+d\left(u_{n+1},u_{n+2}\right)+...+d\left(u_{n+m-1},u_{n+m}\right) \\ & \leq k^{n}d\left(u_{0},u_{1}\right)+k^{n+1}d\left(u_{0},u_{1}\right)+...+k^{n+m-1}d\left(u_{0},u_{1}\right) \\ & \leq \left(k^{n}+k^{n+1}+...+k^{n+m-1}\right)d\left(u_{0},u_{1}\right) \\ & \leq \left(k^{n}+k^{n+1}+...+k^{n+m-1}\right)d\left(u_{0},u_{1}\right) \\ & \leq \frac{k^{n}}{1-k}\left(1-k^{m}\right)d\left(u_{0},u_{1}\right). \end{aligned}$$

We obtain

$$d(u_n, u_{n+m}) \leq \frac{k^n}{1-k}(1-k^m)\, d(u_0, u_1) \tag{14}$$

Taking $n \to +\infty$ in the inequality above, we derive that $\{u_n\}$ is a Cauchy sequence. since (E, d, W), is a convex complete metric space, there exists a $u^* \in X$ such that $\lim_{n\to\infty} d(u_n, u^*) = 0$.

Taking $u = u_n$ and $v = u^*$in (10), Using the continuity of the metric in its both variables we can prove that x^* ia a fixed point of T as follows

$$d(T_\lambda u_n, T_\lambda u^*) \leq k \max\left\{[d(u_n, T_\lambda u_n)]^\alpha . [d(u^*, T_\lambda u^*)]^{1-\alpha}, [d(u_n, T_\lambda u_n)]^{1-\alpha} . [d(u^*, T_\lambda u^*)]^\alpha\right\}$$

Letting $n \to +\infty$ we get $d(T_\lambda u^*, u^*) = 0$ that is $T_\lambda u^* = u^*$,i.e $u^* \in Fix(T_\lambda) = Fix(T)$

To establish the final statement of our theorem, we first derive the subsequent result from (13)

$$d(u_{n+m}, u_n) \leq \frac{k}{1-k} d(u_n, u_{n-1}). \tag{15}$$

Now, letting $m \to +\infty$ in(14) and (15), we get

$$d(u_n, u^*) \leq \frac{k^n}{1-k} d(u_0, u_1),\ n \geq 1, \tag{16}$$

and

$$d(u_n, u^*) \leq \frac{k}{1-k} d(u_n, u_{n-1}),\ n \geq 1. \tag{17}$$

respectively. From (16) and (17), we get the unifying error estimate (9) □

Example 10. *Let $E = [0,5]$. For any $u, v \in E$ and $\lambda \in [0,1)$. Define a mapping $W : E \times E \times [0,1] \to E$ by $W(u, v, \lambda) = \lambda u + (1-\lambda) v$ and a metric $d : E \times E \to [0, \infty)$ by $d(u, v) = |u - v|$.*

Let $T : E \to E$ be given by $Tx = \frac{5-u}{9}$.

For any $u, v, z \in E$, we have

$$\begin{aligned} d(z, W(u, v, \lambda)) &= |\lambda(z-u) + (1-\lambda)(z-v)| \\ &\leq \lambda |(z-u)| + (1-\lambda)|(z-v)| \\ &= \lambda d(z, u) + (1-\lambda) d(z, v). \end{aligned}$$

Hence, (E, d, W) is a convex metric space. Also for $\lambda = \frac{1}{10}$, then $T_\lambda u = W(u, Tu, \lambda) = \lambda u + \lambda Tu = \frac{1}{10}u + \left(1 - \frac{1}{10}\right)\frac{5-u}{9} = \frac{1}{2}$.

So, we have $d(T_\lambda u, T_\lambda v) = 0$ where $u, v \in E \backslash Fix(T)$ Therefore, for any $\alpha \in (0,1)$, T generalized interpolative enriched Kannan type contraction. Thus, by Theorem 9, T has a fixed point which is $\frac{1}{2}$.

3. Application to Quasi-Metric Spaces

Definition 11. *Let E be a nonempty set and let $d : E \times E \to [0, +\infty[$ be a functiion satisfying the following conditions : for all $u, v, w \in E$*

(C_1) $d(u,u) = 0$

(C_2) $d(u,v) \leq d(u,w) + d(w,v)$

Then d is called a quasi-pseudometric on E if d satisfies the additional condition

(C_4) $d(u,v) = 0 \Longrightarrow u = v$

then d is said to be T_0-quasi metric.

Example 12. *Let $E = [0, +\infty[$ and $d(u,v) = max\{u - v, 0\}$ for all $u, v \in E$. Then, (E,d) it's a T_0-quasi-metric space but it's not a metric space.*

Fach quasi-metric d on E induces a T_0 topology τ_d on E which has as a base the family of open balls $B_d(u, \varepsilon)$ for all $u \in E$.

$$B_d(u, \varepsilon) = \{v \in E : d(u,v) < \varepsilon\}$$

If the quasi-metric τ_d satisfies the separation axiom T_1 (or T_2) on E, This is termed (E,d) as a T_1 (or Hausdorff) quasi-metric space. It's important to observe that a T_0-quasi-metric space (E,d) qualifies as T_1 only if for each $u, v \in E$, the condition $d(u,v) = 0$ implies $u = v$.

Definition 13. *Let (E,d) be a quasi-metric space. The fonction $d^s$and $d^{-1} : E \times E \to \mathbb{R}^+$ defined by : for all $u, v \in E$*

$$d^s(u,v) = max\{d(u,v), d(v,u)\} \quad d^{-1}(u,v) = d(v,u).$$

If (E,d) is a quasi metric on E, then (E,d^{-1}) is also a quasi metric and (E,d^s) is an ordinary metric on E.

A quasi-metric space (E,d) is called bicomplete if the metric space (E,d^s) is complete.

The convergence of a sequence $\{u_n\}$ to u with respect to τ_d called d-convergence and denoted by $u_n \overset{d}{\mapsto} u$, is defined

$$u_n \overset{d}{\mapsto} u \Leftrightarrow d(u,u_n) = 0 \text{ as } n \to \infty.$$

Similarly, the convergence of a sequence$\{u_n\}$ to u with respect to $\tau_{d^{-1}}$ called $d^{-1}-$-convergence and denoted by $u_n \overset{d^{-1}}{\mapsto} u$ is defined

$$u_n \overset{d^{-1}}{\mapsto} u \Leftrightarrow d(u,u_n) = 0 \text{ as } n \to \infty.$$

Finally, the convergence of a sequence $\{u_n\}$to u with respect to τ_{d^s} called d^s-convergence and denoted by $u_n \overset{d^s}{\mapsto} u$, is defined

$$u_n \overset{d^s}{\mapsto} u \Leftrightarrow d(u,u_n) = 0 \text{ as } n \to \infty.$$

It is clear that $u_n \overset{d^s}{\mapsto} u \Leftrightarrow u_n \overset{d}{\mapsto} u$ and $u_n \overset{d^{-1}}{\mapsto} u$.

Additional and comprehensive insights into the significant characteristics of quasi-metric spaces and their associated topological structures are available in references [5, 6, 7, 8].

The analysis conducted in the preceding section indicates, in a straightforward manner, the following concepts.

Definition 14. *Let (E,d,W) be a convex quasi-metric space. A self-mapping $T: E \to E$ is a $d-$interpolative enriched Kannan type contraction if there exist $k \in [0,1)$, $\lambda \in [0,1)$ and $\alpha \in (0,1)$*

$$d(W(u,Tu,\lambda),W(v,Tv,\lambda)) \leq k\left[d(W(u,Tu,\lambda),u)\right]^{\alpha}.\left[d(v,W(v,Tv,\lambda))\right]^{1-\alpha}, \tag{18}$$

for all $u,v \in E \backslash Fix(T)$.

A self-mapping $T: E \to E$ is a $d^{-1}-$interpolative enriched Kannan type contraction if there exist $k \in [0,1)$, $\lambda \in [0,1)$ and $\alpha \in (0,1)$

$$d(W(u,Tu,\lambda),W(v,Tv,\lambda)) \leq k\left[d(u,W(u,Tu,\lambda))\right]^{\alpha}.\left[d(W(v,Tv,\lambda)),v\right]^{1-\alpha}, \tag{19}$$

for all $u,v \in E \backslash Fix(T)$.

Lemma 15. *Let (E,d,W) be a T_0-quasi-metric space. If T is a d-interpolative enriched Kannan contraction or a $d^{-1}-$interpolative enriched Kannan contraction on (E,d,W), then*

(I)**:** *T is a generalized interpolative enriched Kannan contractionon the metric space (E,d^s,W).*

(II)**:** *The sequence $\{T_\lambda^n u_0\}_{n=0}^\infty$ obtained from the iterative process $T_\lambda u_n = W(u_n, Tu_n, \lambda)$, $n \geq 0$ is a Cauchy sequence in (E,d^s,W).*

Proof. Suppose that T is a d- interpolative enriched contraction in the space (E,d,W). Using Lemma (6), the condition (18), is transformed to following equivalent form

$$\begin{aligned} d(T_\lambda u, T_\lambda v) &\leq [d(T_\lambda u, u)]^\alpha . [d(T_\lambda v, v)]^{1-\alpha} \\ &\leq k[d^s(v, T_\lambda v)]^\alpha . [d^s(T_\lambda u, u)]^{1-\alpha} \end{aligned} \tag{20}$$

for all $u,v \in E \backslash Fix(T)$. Similarly, given $x,y \in E$, We also possess.

$$\begin{aligned} d(T_\lambda v, T_\lambda u) &\leq k[d(T_\lambda v, v)]^\alpha . [d(u, T_\lambda u)]^{1-\alpha} \\ &\leq k[d^s(u, T_\lambda u)]^\alpha . [d^s(T_\lambda v, v)]^{1-\alpha} \end{aligned} \tag{21}$$

for all $u,v \in E \backslash Fix(T)$.

Thus, given $u,v \in E$

(22)

$$d^s(d(T_\lambda u, T_\lambda v)) \leq k \max\left\{[d^s(u, T_\lambda u)]^\alpha . [d^s(T_\lambda v, v)]^{1-\alpha}, [d^s(v, T_\lambda v)]^\alpha . [d^s(T_\lambda u, u)]^{1-\alpha}\right\}.$$

We have

$$d^s(W(u,Tu,\lambda), W(v,Tv,\lambda)) \leq k \max\left\{\begin{array}{c} [d^s(u, W(u,Tu,\lambda))]^\alpha . [d^s(W(v,Tv,\lambda), v)]^{1-\alpha}, \\ [d^s(v, W(v,Tv,\lambda))]^\alpha . [d^s(W(u,Tu,\lambda), u)]^{1-\alpha} \end{array}\right\}.$$

Then, T is a generalized interpolative enriched Kannan contraction on (E,d^s,W).

Let $u_0 \in E$ and $\{u_n\}_{n=0}^\infty$, $u_n = T_\lambda u_{n-1} = \cdots = T_\lambda^n u_0$.

Let T_λ be a d-interpolative enriched Kannan contraction. By the statement (I), it is a T_λ is a generalized interpolative enriched Kannan contraction on (E,d^s,W). Then, the proof of interpolative enriched Kannan that $\left(T_\lambda^n u_0\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in (E,d^s,W) □

By employing the proposition mentioned earlier, three variations of the generalized interpolative enriched Kannan principle can be readily derived.

Theorem 16. *Let (E,d,W) be a bicomplete T_0-quasi-metric space. Every $d-$(resp. every d^{-1}) interpolative enriched Kannan contraction on (E,d,W) has a unique fixed point.*

Proof. Let T be a d^{-1} or $d-$interpolative enriched Kannan contraction. by lemma (6)(I), T is a generalized interpolative Kannan on (E,d^s,W). Since (E,d^s,W) is a complete metric space. So by Theorem (9) T has a fixed point. □

Theorem 17. *Every d-interpolative enriched Kannan contraction on a Hausdorff d-sequentially complete T_0-quasi-metric space (E,d,W) has a fixed point.*

Proof. Let T be a d-interpolative enriched Kannan contraction on the Hausdorff d-sequentially complete T_0-quasi-metric space (E,d,w). Fix an $u_0 \in E$. by lemma (6)(II), $\left(T_\lambda^n u_0\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in (E,d^s,W). Hence, there is $u^* \in E$ such that $T_\lambda^n u_0 \overset{d}{\to} u^*$. Since T is d-interpolative enriched Kannan contraction there exist $k \in [0,1)$, $\alpha \in (0,1)$ and $\lambda \in [0,1)$ for which

$$d\left(T_\lambda^{n+1}u_0, Tu^*\right) \le k\left[d\left(T_\lambda^{n+1}u_0, T_\lambda^n u_0\right)\right]^\alpha . \left[d\left(u^*, T_\lambda u^*\right)\right]^{1-\alpha}. \tag{23}$$

Consequently, $\lim_{n\to\infty} d\left(T_\lambda^{n+1}u_0, Tu^*\right) = 0$, From Hausdorffness of (E,d,W), we get $u^* \in Fix(T_\lambda) = Fix(T)$. □

Theorem 18. *Every d^{-1}-interpolative enriched Kannan contraction on a $T_1 d$-sequentially complete T_0-quasi-metric space (E,d,W) has a fixed point.*

Proof. Let T be d^{-1}-interpolative enriched Kannan contraction on T_1d-sequentially complete T_0-quasi-metric space (E,d,W).

$Fix\, u_0 \in E$. As in the proof of ,Teorem(18) (see lemma (15)) $\left(T_\lambda^n u_0\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in (E,d^s,W). Hence, there is $u^* \in E$ such that $T_\lambda^n u_0 \overset{d}{\to} u^*$,i.e $d\left(u^*, T_\lambda^n u_0\right) = 0$ as $n \to \infty$.Since T is a d^{-1}-interpolative enriched Kannan contraction there exist $k \in [0,1)$, $\lambda \in [0,1)$ and $\alpha \in (0,1)$ for which

$$d\left(T_\lambda^{n+1}u_0, Tu^*\right) \le k\left[d\left(T_\lambda^n u_0, T_\lambda^{n+1}u_0\right)\right]^\alpha . \left[d\left(T_\lambda u^*, u^*\right)\right]^{1-\alpha}.$$

Consequently, $\lim_{n\to\infty} d\left(T_\lambda^{n+1}u_0, Tu^*\right) = 0$.Using the triangular inequality, we obtain,

$$d\left(u^*, T_\lambda u^*\right) \le d\left(u^*, T_\lambda^n u_0\right) + d\left(T_\lambda^n u_0, T_\lambda u^*\right).$$

We deduce $d(u^*, T_\lambda u^*) = 0$. i.e, $u^* \in Fix(T_\lambda) = Fix(T)$. □

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] E. Karapinar, Revisiting the Kannan type contractions via interpolation, Adv. Theory Nonlinear Anal. Appl. 2 (2018), 85–87. https://doi.org/10.31197/atnaa.431135.

[2] V. Berinde, M. Pacurar, Existence and approximation of fixed points of enriched contractions and enriched φ-contractions, Symmetry. 13 (2021), 498. https://doi.org/10.3390/sym13030498.

[3] W. Takahashi, A convexity in metric space and nonexpansive mappings, I. Kodai Math. Sem. Rep. 22 (1970), 142–149. https://doi.org/10.2996/kmj/1138846111.

[4] H.P.A. Künzi, F. Yıldız, Convexity structures in T_0-quasi-metric spaces, Topol. Appl. 200 (2016), 2–18. https://doi.org/10.1016/j.topol.2015.12.009.

[5] Ş. Cobzaş, Functional analysis in asymmetric normed spaces, Springer, Basel, 2013. https://doi.org/10.1007/978-3-0348-0478-3.

[6] J.C. Kelly, Bitopological spaces, Proc. Lond. Math. Soc. s3-13 (1963), 71–89. https://doi.org/10.1112/plms/s3-13.1.71.

[7] J. Marín, S. Romaguera, P. Tirado, Weakly contractive multivalued maps and ω-distances on complete quasi-metric spaces, Fixed Point Theory Appl. 2011 (2011), 2. https://doi.org/10.1186/1687-1812-2011-2.

[8] H.A. Hançer, On two types almost (α, F_d) -contractions on quasi metric space, Commun. Fac. Sci. Univ. Ank. Ser. A Math. Stat. 68 (2019), 1819–1830. https://doi.org/10.31801/cfsuasmas.419821.

[9] E. Karapınar, I.M. Erhan, Best proximity point on different type contractions, Appl. Math. Inform. Sci. 5 (2011), 558–569.

[10] E. Karapınar, O. Alqahtani, H. Aydi, On interpolative Hardy-Rogers type contractions, Symmetry. 11 (2018), 8. https://doi.org/10.3390/sym11010008.

[11] Y.U. Gaba, E. Karapınar, A new approach to the interpolative contractions, Axioms. 8 (2019), 110. https://doi.org/10.3390/axioms8040110.

[12] E. Karapınar, R.P. Agarwal, Interpolative Rus-Reich-Cirić type contractions via simulation functions, An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat. 27 (2019), 137–152. https://doi.org/10.2478/auom-2019-0038.

[13] E. Karapınar, R. Agarwal, H. Aydi, Interpolative Reich–Rus–Cirić type contractions on partial metric spaces, Mathematics. 6 (2018), 256. https://doi.org/10.3390/math6110256.

[14] E. Mohamed, E.K. Amine, A. Mohamed, Best proximity point theorems for proximal pointwise tricyclic contraction, Adv. Fixed Point Theory, 13 (2023), 22. https://doi.org/10.28919/afpt/8194.

[15] M. Edraoui, A. El koufi, S. Semami, Fixed points results for various types of interpolative cyclic contraction, Appl. Gen. Topol. 24 (2023), 247–252. https://doi.org/10.4995/agt.2023.19515.

[16] E. Mohamed, A. Mohamed, L. Samih, Relatively cyclic and noncyclic P-contractions in locally K-convex space, Axioms. 8 (2019), 96. https://doi.org/10.3390/axioms8030096.

[17] M. Edraoui, M. Aamri, S. Lazai, Fixed point theorem in locally K-convex space, Int. J. Math. Anal. 12 (2018), 485–490. https://doi.org/10.12988/ijma.2018.8753.

[18] M. Edraoui, M. Aamri, S. Lazaiz, Fixed point theorems for set valued Caristi type mappings in locally convex space, Adv. Fixed Point Theory, 7 (2017), 500–511.

[19] E. Mohamed, A. Mohamed, L. Samih, Fixed point theorem for $p\alpha$-nonexpansive wrt orbits in locally convex space. J. Math. Comput. Sci. 10 (2020), 1582–1589. https://doi.org/10.28919/jmcs/4645.

[20] M. Edraoui, A. Baiz, A. Faiz, et al. Best proximity point theorems for tricyclic diametrically contractive mappings, Adv. Fixed Point Theory, 14 (2024), 4. https://doi.org/10.28919/afpt/8314.

[21] A. Kari, H. Emadifar, A. Baiz, Fixed point theorems for generalized θ-Ω-contraction on metric spaces, JP J. Fixed Point Theory Appl. 19 (2023), 7–33.

[22] A. Kari, H. Hammad, A. Baiz, et al. Best proximity point of generalized (F -τ)-proximal non-self contractions in generalized metric spaces, Appl. Math. Inform. Sci. 16 (2022), 853–861. https://doi.org/10.18576/amis/16 0601.

[23] A. Baiz, J. Mouline, A. Kari, Fixed point theorems for generalized τ-ψ-contraction mappings in rectangular quasi b-metric spaces, Adv. Fixed Point Theory, 13 (2023), 10. https://doi.org/10.28919/afpt/8114.

[24] A. Baiz, J. Mouline, Y. El Bekri, Existence and uniqueness of fixed point for α-contractions in rectangular quasi b-metric spaces, Adv. Fixed Point Theory, 13 (2023), 16. https://doi.org/10.28919/afpt/8152.