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# FIXED POINT THEOREMS SATISFYING IMPLICIT RELATION CONDITIONS IN $\mathscr{F}$-METRIC SPACES

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**Abstract.** The present research is an attempt to state and prove fixed point theorems for $\mathscr{F}$-metric spaces that satisfy the contractive condition in $\mathscr{F}$-metric spaces. Suitable examples are provided an illustrate the validity of our results.

**Keywords:** common fixed point; implicit relation; $\mathscr{F}$-metric space.

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## 1. Introduction

In recent years, fixed point theory has been a flourishing area of mathematical research because of its development through different standard metric spaces and many diverse application. Several scholars, including Czerwik, have expanded on the concept of metric spaces. as Czerwik [3], Khamsi and Hussain [5], Mlaiki et al. [6], and so on. Jleli and Samet [4] recently introduced the idea of $\mathscr{F}$-metric spaces, which is a generalization of the Banach Contraction

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Principle (BCP). By considering a general condition given by an implicit relation, many classical fixed point theorems have recently been united.

This method was pioneered by Popa's seminal papers [8, 9]. Implicit functions are useful because of their unifying capacity as well as their ability to admit new contraction conditions. Jleli and Samet (2018) proposed the $\mathscr{F}$-metric space as a new metric space. We look at some fixed points in the notion of implied relation in the context of $\mathscr{F}$-metric spaces in this paper. In the generalized setting, The researchers present some fixed point results. As a result, corresponding implicit relation fixed point theorems are derived.

## 2. Preliminaries

**Definition 2.1.** *Given a set $\chi$ and a function $d : \chi \times \chi \to \Re$, we say that the pair $(\chi, d)$ is a metric space if and only if $d(\theta, \eta)$ satisfies the following properties:*

(1)*(Non-negativeness) For all $\theta, \eta \in \chi, d(\theta, \eta) \geq 0$*

(2) *(Identification) For all $\theta, \eta \in \chi$ we have that $d(\theta, \eta) = 0 \Leftrightarrow \theta = \eta$*

(3) *(Symmetry) For all $\theta, \eta \in \chi, d(\theta, \eta) = d(\eta, \theta)$*

(4) *(Triangular inequality) For all $\theta, \eta, \nu \in \chi$ we have that*

$$d(\theta, \nu) \leq d(\theta, \eta) + d(\eta, \nu)$$

In this section, we list the following definitions and examples that we will refer to its in our main results.

**Definition 2.2.** *[4] Suppose $\mathscr{F}$ be the set of functions $f : (0, +\infty) \Longrightarrow \mathbb{R}$ satisfying the conditiones as below:*

$(\mathscr{F}_1)$ *$f$ is non-decreasing, i.e. $0 < s < \iota \Longrightarrow f(s) \leq f(\iota)$.*

$(\mathscr{F}_2)$ *For every sequence $\iota_n \subset (0, +\infty)$, there is*

$$\lim_{n \to +\infty} \iota_n = 0 \Longleftrightarrow \lim_{n \to +\infty} f(\iota_n) = -\infty.$$

The generalized concept of metric space is as follows:

**Definition 2.3.** *[4] Suppose $\chi$ be a non-empty set and let $d_{\mathscr{F}} : \chi \times \chi \longrightarrow [0, +\infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in \mathscr{F} \times [0, +\infty)$ such that*

$(d_{\mathscr{F}1})$ $(\theta,\eta)\in\chi\times\chi, d_{\mathscr{F}}(\theta,\eta)=0 \Longleftrightarrow \theta=\eta$.

$(d_{\mathscr{F}2})$ $d_{\mathscr{F}}(\theta,\eta)=d_{\mathscr{F}}(\eta,\theta)$, *for all* $(\theta,\eta)\in\chi\times\chi$.

$(d_{\mathscr{F}3})$ *For every* $(\theta,\eta)\in\chi\times\chi$, *for every* $N\in\mathbb{N}, N\geq 2$, *and for every* $(\mu_i)_i^n\subset\chi$ *with* $(\mu_1,\mu_N)=(\theta,\eta)$, *there is*

$$d_{\mathscr{F}}(\theta,\eta)>0 \Longrightarrow f(d_{\mathscr{F}}(\theta,\eta))\leq f(\sum_{i=1}^{N-1} d_{\mathscr{F}}(\mu_i,\mu_{i+1}))+\alpha.$$

Then $d_{\mathscr{F}}$ is said to be an $\mathscr{F}$-metric space on $\chi$, and the pair $(\chi,d_{\mathscr{F}})$ is said to be an $\mathscr{F}$-metric space.

**Example 2.1.** *[4] The set of real numbers $\mathbb{R}$ is an $\mathscr{F}$-metric space if we define $d_{\mathscr{F}}$ by*

$$d_{\mathscr{F}}(\theta,\eta)=\begin{cases}(\theta-\eta)^2, & \text{if}(\theta,\eta)\in[0,4]\times[0,4]\\ |\theta-\eta|, & \text{if } (\theta,\eta)\notin[0,4]\times[0,4]\end{cases}$$

*with* $f(\iota)=(\iota)$ *and* $a=\ln(4)$ *for all* $(\theta,\eta)\in\chi\times\chi$. *It can be easily seen that $d_{\mathscr{F}}$ satisfies* $(d_{\mathscr{F}1})$ , $(d_{\mathscr{F}2})$ *and* $(d_{\mathscr{F}3})$.

**Definition 2.4.** *[4] Suppose $(\chi,d_{\mathscr{F}})$ is an $\mathscr{F}$-metric space.*
$(i)$ *Suppose $\theta_n$ is a sequence in $\chi$. We say that $\{\theta_n\}$ is $\mathscr{F}$-convergent to $\theta\in\chi$ if $\{\theta_n\}$ is convergent to $\theta$ in relation to $\mathscr{F}$-metric space $d_{\mathscr{F}}$.*
$(ii)$ *A sequence $\theta_n$ is $\mathscr{F}$-Cauchy, if*

$$\lim_{n,m\to+\infty} D(\theta_n,\theta_m)=0.$$

$(iii)$ *We assert that $(\chi,d_{\mathscr{F}})$ is $\mathscr{F}$-complete, if every $\mathscr{F}$-Cauchy sequence in $\chi$ is $\mathscr{F}$-convergent to a certain element in $\chi$.*

Jleli and Samet [4] was presented the following generalization of Banach contraction principle (BCP):

**Theorem 2.1.** *[4] Suppose $(\chi,d_{\mathscr{F}})$ is an $\mathscr{F}$-metric space and $g:\chi\longrightarrow\chi$ be a given mapping. Assume that the following criteria are met:*

*(i)* $(\chi, d_{\mathscr{F}})$ *is* $\mathscr{F}$*-complete.*

*(ii) there exists* $k \in (0,1)$ *as*

$$d_{\mathscr{F}}(g(\theta), g(\eta)) \leq k d_{\mathscr{F}}(\theta, \eta), (\theta, \eta) \in \chi \times \chi.$$

*After that g has a unique fixed point* $z^* \in \chi$*. Furthermore, for every* $\theta_0 \in \chi$*, the sequence* $\{\theta_n\} \subset \chi$ *defined by*

$$\theta_{n+1} = g(\theta_n), n \in N,$$

*is* $\mathscr{F}$*-convergent.*

In (1997), According to Popa [8,9], several classical fixed point theorems and common fixed point theorems have been recently unified by considering general contractive conditions expressed by an implicit condition.

Suppose $\Phi$ is the set of all real continuous real functions $\phi : \mathbb{R}_+^6 \to \mathbb{R}$, under which the following conditions are taken into results:

$(\phi_{1a})$ $\phi$ is non-increasing in the fifth variable and

$$\phi(\mu, \upsilon, \upsilon, \mu, \mu+\upsilon, 0) \leq 0 \quad \text{for} \mu, \upsilon \geq 0 \Longrightarrow \exists \hbar \in [0,1) \quad \text{as} \mu \leq \hbar \upsilon;$$

$(\phi_{1b})$ $\phi$ is non-increasing in the fourth variable and

$$\phi(\mu, \upsilon, 0, \mu+\upsilon, \mu, \upsilon) \leq 0 \quad \text{for} \mu, \upsilon \geq 0 \Longrightarrow \exists \hbar \in [0,1) \quad \text{as} \mu \leq \hbar \upsilon;$$

$(\phi_{1c})$ $\phi$ is non-increasing in the third variable and

$$\phi(\mu, \upsilon, \mu+\upsilon, 0, \mu, \upsilon) \leq 0 \quad \text{for} \mu, \upsilon \geq 0 \Longrightarrow \exists \hbar \in [0,1) \quad \text{as} \mu \leq \hbar \upsilon;$$

$(\phi_{1d})$ $\phi$ is non-increasing in the third variable and

$$\phi(\mu, \upsilon, \upsilon, \mu, \mu, 0) \leq 0 \quad \text{for} \mu, \upsilon \geq 0 \Longrightarrow \exists \hbar \in [0,1) \quad \text{as} \mu \leq \hbar \upsilon;$$

$(\phi_2)$ $\phi(\mu, \mu, 0, 0, \mu, \mu) > 0, \quad$ for $\mu > 0$.

**Example 2.2.** *The function* $\phi \in \Phi$*, is given by*

$$\phi(\iota_1, \iota_2, \iota_3, \iota_4, \iota_5, \iota_6) = \iota_1 - \frac{1}{3}\iota_2,$$

*where* $\frac{1}{3} \in [0,1)$*, satisfies* $(\phi_2)$ *and* $(\phi_{1a}) - (\phi_{1c})$*, with* $\hbar = \frac{1}{3}$*.*

## 3. MAIN RESULTS

We prove some fixed point results concerning implicit condition in this section of $\mathscr{F}$ -metric space.

**Theorem 3.1.** *Suppose $(\chi, d_{\mathscr{F}})$ be an $\mathscr{F}$- complete metric space and $\tau : \chi \to \chi$ is a self mapping for which there is an existence $\mathscr{F}$ fulfilling $(\mathscr{F}_1)$,as for all $\theta, \eta \in \chi$,*
*(1) $(\chi, d_{\mathscr{F}})$ be an $\mathscr{F}$- metric space*
*(2) as $\theta, \eta \in \chi$*

$$\phi(d_{\mathscr{F}}(\tau\theta, \tau\eta), d_{\mathscr{F}}(\theta, \eta), d_{\mathscr{F}}(\theta, \tau\theta), d_{\mathscr{F}}(\eta, \tau\eta), d_{\mathscr{F}}(\theta, \tau\eta), d_{\mathscr{F}}(\eta, \tau\theta)) \leq 0. \tag{3.1}$$

*then there is an existence $\bar{\theta} \in \chi$ such that $\bar{\theta} \subset \tau$ has a unique fixed point $(\chi, d_{\mathscr{F}})$*

*Proof.* Suppose $\theta_0$ is an arbitrary point in $\chi$, and $\theta_{n+1} = \tau\theta_n$,$n = 0, 1, ...$
If we take $\theta = \theta_{n-1}$ and $\eta = \theta_n$ in (3.1) and denote $\mu = d_{\mathscr{F}}(\theta_n, \theta_{n+1}), \upsilon = d_{\mathscr{F}}(\theta_{n-1}, \theta_n)$ we get,

$$\phi(d_{\mathscr{F}}(\tau\theta_{n-1}, \tau\theta_n), d_{\mathscr{F}}(\theta_{n-1}, \theta_n), d_{\mathscr{F}}(\theta_{n-1}, \tau\theta_{n-1}), d_{\mathscr{F}}(\theta_n, \tau\theta_n), d_{\mathscr{F}}(\theta_{n-1}, \tau\theta_n), d_{\mathscr{F}}(\theta_n, \tau\theta_{n-1})) \leq 0.$$

$$\phi(d_{\mathscr{F}}(\theta_n, \theta_{n+1}), d_{\mathscr{F}}(\theta_{n-1}, \theta_n), d_{\mathscr{F}}(\theta_{n-1}, \theta_n), d_{\mathscr{F}}(\theta_n, \theta_{n+1}), d_{\mathscr{F}}(\theta_{n-1}, \theta_{n+1}), d_{\mathscr{F}}(\theta_n, \theta_n)) \leq 0.$$

$$\phi(d_{\mathscr{F}}(\theta_n, \theta_{n+1}), d_{\mathscr{F}}(\theta_{n-1}, \theta_n), d_{\mathscr{F}}(\theta_{n-1}, \theta_n), d_{\mathscr{F}}(\theta_n, \theta_{n+1}), d_{\mathscr{F}}(\theta_{n-1}, \theta_{n+1}), 0) \leq 0.$$

$$\phi(\mu, \upsilon, \upsilon, \mu, \mu + \upsilon, 0) \leq 0$$

$$\phi(d_{\mathscr{F}}(\theta_n, \theta_{n+1}), d_{\mathscr{F}}(\theta_{n-1}, \theta_n), d_{\mathscr{F}}(\theta_{n-1}, \theta_n), d_{\mathscr{F}}(\theta_n, \theta_{n+1}) + d_{\mathscr{F}}(\theta_{n-1}, \theta_n), 0) \leq 0.$$

and consequently, in light of the assumption $(\phi_{1a})$, there exists $\hbar \in [0, 1)$ as $\mu \leq \hbar\upsilon$, that is

$$d_{\mathscr{F}}(\theta_n, \theta_{n+1}) \leq \hbar d_{\mathscr{F}}(\theta_{n-1}, \theta_n) \tag{3.2}$$

$$d_{\mathscr{F}}(\theta_{n-1}, \theta_n) \leq \hbar d_{\mathscr{F}}(\theta_{n-2}, \theta_{n-1}) \tag{3.3}$$

$$d_{\mathscr{F}}(\theta_n, \theta_{n+1}) \leq \hbar d_{\mathscr{F}}(\theta_{n-1}, \theta_n) \leq \hbar^2 d_{\mathscr{F}}(\theta_{n-2}, \theta_{n-1}) \leq ... \leq \hbar^n d_{\mathscr{F}}(\theta_0, \theta_1)$$

For all $n, m \in \mathbb{N}$,

$$\begin{aligned} d_{\mathscr{F}}(\theta_n, \theta_m) \leq & d_{\mathscr{F}}(\theta_n, \theta_{n+1}) + d_{\mathscr{F}}(\theta_{n+1}, \theta_{n+2}) + ... + d_{\mathscr{F}}(\theta_{m-1}, \theta_m) \\ \leq & \hbar^n d_{\mathscr{F}}(\theta_0, \theta_1) + \hbar^{n+1} d_{\mathscr{F}}(\theta_0, \theta_1) + ... + \hbar^{m-1} d_{\mathscr{F}}(\theta_0, \theta_1) \\ \leq & \hbar^n [1 + \hbar + ...] d_{\mathscr{F}}(\theta_0, \theta_1) \end{aligned}$$

$$\leq \frac{\hbar^n}{1-\hbar} d_{\mathscr{F}}(\theta_0, \theta_1)$$

For each, $n \in \mathbb{N}$. Let $\varepsilon > 0$ is a fixed and $(f, \alpha) \in \mathscr{F} \times [0, +\infty)$ be such as $(d_{\mathscr{F}3})$ is satisfied by $(\phi_2)$, there exists $\delta > 0$ as

$$0 < \iota < \delta \Longrightarrow \quad \text{to} f(\iota) < f(\varepsilon) - \alpha. \tag{3.4}$$

suppose $n(\varepsilon) \in \mathbb{N}$ such that $0 < \sum_{m \geq m(\varepsilon)}^{m-1} \frac{\hbar^n}{1-\hbar} d_{\mathscr{F}}(\theta_0, \theta_1) < \delta$

Consequently,by (3.4) and $(\phi_{1a})$, there is

$$f(\sum_{(i=n)}^{m-1} \frac{\hbar^n}{1-\hbar} d_{\mathscr{F}}(\theta_0, \theta_1)) \leq f(\sum_{n \geq n(\varepsilon)}^{m-1} \frac{\hbar^n}{1-\hbar} d_{\mathscr{F}}(\theta_0, \theta_1)) < f(\varepsilon) - \alpha \tag{3.5}$$

where, $n > m > m(\varepsilon)$ with $d_{\mathscr{F}}(\theta_n, \theta_m) > 0$ using $(d_{\mathscr{F}3})$ in addition (3.5)

$$f(d_{\mathscr{F}}(\theta_n, \theta_m)) \leq f(\sum_{(i=n)}^{m-1} d_{\mathscr{F}}(\theta_i, \theta_{i+1}) + \alpha \leq f(\sum_{(i=n)}^{m-1} \frac{\hbar^n}{1-\hbar} d_{\mathscr{F}}(\theta_0, \theta_1)) + \alpha < f(\varepsilon),$$

which is implied by $(\mathscr{F}_1)$ that, $d_{\mathscr{F}}(\theta_n, \theta_m) < \varepsilon, m > n > n(\varepsilon)$.

This proves that $\{\theta_n\}$ is $\mathscr{F}$- Cauchy. since $(\chi, d_{\mathscr{F}})$ is complete suppose , there exists $\theta_1 \in \chi$ as $\{\theta_n\}$ is $\mathscr{F}$- convergent to $\theta_1$.

$$\lim_{n \to \infty} d_{\mathscr{F}}(\theta_n, \theta_1) = 0 \tag{3.6}$$

Now, to prove $\theta_1$ is a fixed point of $\tau$, we start with contradiction by supposing $d_{\mathscr{F}}(\tau\theta_1, \theta_1) > 0$, $n \in \mathbb{N}$ by $(d_{\mathscr{F}3})$, there is

$$f(d_{\mathscr{F}}(\tau\theta_1, \theta_1)) \leq f(d_{\mathscr{F}}(\tau\theta_1, \theta_n) + d_{\mathscr{F}}(\theta_n, \theta_1)) + \alpha \tag{3.7}$$

using (3.1) by taking $\theta = \theta_n, \eta = \theta_1$

$$\phi(d_{\mathscr{F}}(\tau\theta_n, \tau\theta_1), d_{\mathscr{F}}(\theta_n, \theta_1), d_{\mathscr{F}}(\theta_n, \tau\theta_n), d_{\mathscr{F}}(\theta_1, \tau\theta_1), d_{\mathscr{F}}(\theta_n, \tau\theta_1), d_{\mathscr{F}}(\theta_1, \tau\theta_n)) \leq 0. \tag{3.8}$$

$$\phi(d_{\mathscr{F}}(\theta_1, \tau\theta_1), 0, 0, d_{\mathscr{F}}(\theta_1, \tau\theta_1), d_{\mathscr{F}}(\theta_1, \tau\theta_1), 0) \leq 0. \tag{3.9}$$

On the other hand, using $(\phi_2)$ and (3.6),

$$\lim_{n \to \infty} f(d_{\mathscr{F}}(\tau\theta_1, \theta_n), d_{\mathscr{F}}(\theta_n, \theta_1)) + \alpha = -\infty$$

$$\lim_{n \to \infty} f(d_{\mathscr{F}}(\tau\theta_1, \theta_n), d_{\mathscr{F}}(\theta_n, \theta_1)) + \alpha = -\infty$$

Contraction, $d_{\mathscr{F}}(\theta_1, \tau\theta_1) = 0$, that is $\tau\theta_1 = \theta_1$.

To prove uniqueness, suppose $\theta_1 \neq \theta_2$ are two fixed point of $\tau$

$$\phi(d_{\mathscr{F}}(\tau\theta_1, \tau\theta_2), d_{\mathscr{F}}(\theta_1, \theta_2), d_{\mathscr{F}}(\theta_1, \tau\theta_1), d_{\mathscr{F}}(\theta_2, \tau\theta_2), d_{\mathscr{F}}(\theta_1, \tau\theta_2), d_{\mathscr{F}}(\theta_2, \tau\theta_1)) \leq 0$$

$$\phi(d_{\mathscr{F}}(\theta_1, \theta_2), d_{\mathscr{F}}(\theta_1, \theta_2), d_{\mathscr{F}}(\theta_1, \theta_1), d_{\mathscr{F}}(\theta_2, \theta_2), d_{\mathscr{F}}(\theta_1, \theta_2), d_{\mathscr{F}}(\theta_2, \theta_1)) \leq 0$$

$$\phi(d_{\mathscr{F}}(\theta_1, \theta_2), d_{\mathscr{F}}(\theta_1, \theta_2), 0, 0, d_{\mathscr{F}}(\theta_1, \theta_2), d_{\mathscr{F}}(\theta_2, \theta_1)) \leq 0$$

a contraction. Consequently, $\tau$ has a unique fixed point in $\chi$. □

Now, We give an example to support the generality of (3.1) over the theorem (5.1) [2].

**Example 3.1.** *Suppose $\theta = [0,1]$ is endowed with the metric $\mathscr{F}$ defined by $d_{\mathscr{F}}(\theta, \eta) = |\theta - \eta|$, It is clear that $(\chi, d_{\mathscr{F}})$ is a complete metric space. Suppose that*

$$\phi(\iota_1, \iota_2, \iota_3, \iota_4, \iota_5, \iota_6) = \iota_1 - \frac{3}{4}\iota_5,$$

*for every $\iota_1, \iota_2, \iota_3, \iota_4, \iota_5, \iota_6 \in [0, +\infty)$ It is obvious that $\phi \in \Phi$ Define a mapping $\tau$ on $\chi$ such that for all $\theta \in \chi, \tau(\theta)$ is the characteristic function for $\frac{3}{4}$ For each $\theta, \eta \in \chi$*

$$\phi(d_{\mathscr{F}}(\tau\theta, \tau\eta), d_{\mathscr{F}}(\theta, \eta), d_{\mathscr{F}}(\theta, \tau\theta), d_{\mathscr{F}}(\eta, \tau\eta), d_{\mathscr{F}}(\theta, \tau\eta), d_{\mathscr{F}}(\eta, \tau\theta)) \leq 0.$$

$$= d_{\mathscr{F}}(\tau\theta, \tau\eta) - \frac{3}{4} d_{\mathscr{F}}(\theta, \tau\eta) = \frac{3}{4} d_{\mathscr{F}}(\theta, \eta) - \frac{3}{4} d_{\mathscr{F}}(\theta, \eta) = 0$$

*The characteristic function for $0$ is the fixed point of $\tau$.*

*Since $\tau$ satisfies the condition in theorem (3.1), and also, the operator $\tau$ has a unique common fixed point, and it can be easily seen that $\tau$ does not satisfy the condition in theorem (5.1) [2].*

In (2003) [7], Popa introduced a new class of mappings $F : R_+^6 \to R$ such that the fulfillment of the inequality of type

$$\phi(d(\tau\theta, \tau\eta), d(\theta, \eta), d(\theta, \tau\eta), d(\eta, \tau\eta), d(\eta, \tau^2\theta), d(\eta, \tau\theta)) \leq 0. \tag{3.10}$$

for $\theta, \eta \in \chi$, ensures the existence and the uniqueness of a fixed point for $\tau$.

**Theorem 3.2.** *Suppose $(\chi, d_{\mathscr{F}})$ is an $\mathscr{F}$- metric space and $\tau : (\chi, d_{\mathscr{F}}) \to (\chi, d_{\mathscr{F}})$ be a mapping satisfying the inequality 3.10 for every $\theta, \eta \in \chi$, where $\phi$ satisfies condition $(\phi_{1d})$. Then $\tau$ has at most one fixed point. every $\theta \in \chi$.*

*Proof.* Suppose that $\tau$ has two fixed points $\mu$ and $\upsilon$ with $\mu \neq \upsilon$. Then by (3.10) there is successively

$$\phi(d_{\mathscr{F}}(\tau\mu,\tau\upsilon),d_{\mathscr{F}}(\mu,\upsilon),d_{\mathscr{F}}(\mu,\tau\mu),d_{\mathscr{F}}(\upsilon,\tau\upsilon),d_{\mathscr{F}}(\upsilon,\tau^2\mu),d_{\mathscr{F}}(\upsilon,\tau\mu)) \leq 0.$$

$$\phi(d_{\mathscr{F}}(\mu,\upsilon),d_{\mathscr{F}}(\mu,\upsilon),d_{\mathscr{F}}(\mu,\mu),d_{\mathscr{F}}(\upsilon,\upsilon),d_{\mathscr{F}}(\upsilon,\mu),d_{\mathscr{F}}(\upsilon,\mu)) \leq 0.$$

$$\phi(d_{\mathscr{F}}(\mu,\upsilon),d_{\mathscr{F}}(\mu,\upsilon),0,0,d_{\mathscr{F}}(\upsilon,\mu),d_{\mathscr{F}}(\upsilon,\mu)) \leq 0.$$

$$\phi(\mu,\mu,0,0,\mu,\mu) > 0.$$

Contradiction,$\tau$ has fixed points with $\mu = \upsilon$. □

**Theorem 3.3.** *Suppose $(\chi, d_{\mathscr{F}})$ is an $\mathscr{F}$- metric space and $\tau : (\chi, d_{\mathscr{F}}) \to (\chi, d_{\mathscr{F}})$ be a mapping such that there exists $\hbar \in [0,1)$ with $d_{\mathscr{F}}(\tau^2\theta,\tau\theta) \leq hd_{\mathscr{F}}(\theta,\tau\theta)$ for every $\theta \in \chi$. Then for every $\theta \in \chi$ the sequence $\{\tau^n\theta\}$ is an $\mathscr{F}$- Cauchy sequence.*

*Proof.* Suppose $\theta$ be arbitrary in $\chi$. We shall show that the sequence defined by $\theta_{n+1} = \tau^n\theta$, there is

$$\phi(d_{\mathscr{F}}(\tau\theta_n,\tau\theta_{n+1}),d_{\mathscr{F}}(\theta_n,\theta_{n+1}),d_{\mathscr{F}}(\theta_n,\tau\theta_n),d_{\mathscr{F}}(\theta_{n+1},\tau\theta_{n+1}),d_{\mathscr{F}}(\theta_{n+1},\tau^n\theta_{n+1}),d_{\mathscr{F}}(\theta_{n+1},\tau\theta_n)) \leq 0.$$

$$\phi(d_{\mathscr{F}}(\tau\theta_{n+1},\theta_{n+2}),d_{\mathscr{F}}(\theta_n,\theta_{n+1}),d_{\mathscr{F}}(\theta_n,\theta_{n+1}),d_{\mathscr{F}}(\theta_{n+1},\theta_{n+2}),d_{\mathscr{F}}(\theta_{n+1},\theta_{n+2}),0) \leq 0.$$

$$\phi(d_{\mathscr{F}}(\tau^{n+1}\theta,\tau^n\theta),d_{\mathscr{F}}(\theta_n,\theta_{n+1}),d_{\mathscr{F}}(\theta_n,\theta_{n+1}),d_{\mathscr{F}}(\tau^{n+1}\theta,\tau^n\theta),d_{\mathscr{F}}(\tau^{n+1}\theta,\tau^n\theta),0) \leq 0.$$

Since, $d_{\mathscr{F}}(\tau^2\theta,\tau\theta) \leq \hbar d_{\mathscr{F}}(\theta,\tau\theta)$.
By induction, there is $d_{\mathscr{F}}(\tau^{n+1}\theta,\tau^n\theta) \leq \hbar^n d_{\mathscr{F}}(\theta,\tau\theta)$
For each $n,m \in \mathbb{N}$,

$$\begin{aligned} d_{\mathscr{F}}(\tau^{n+1}\theta,\tau^{m+1}\theta) \leq& \hbar^n d_{\mathscr{F}}(\theta,\tau\theta) + \hbar^{n+1} d_{\mathscr{F}}(\theta,\tau\theta) + \ldots + \hbar^{m-1} d_{\mathscr{F}}(\theta,\tau\theta) \\ \leq& \hbar^n[1+\hbar+\ldots]d_{\mathscr{F}}(\theta,\tau\theta) \\ \leq& \frac{\hbar^n}{1-\hbar} d_{\mathscr{F}}(\theta,\tau\theta) \end{aligned}$$

For each, $n \in \mathbb{N}$. Suppose $\varepsilon > 0$ be a fixed and $(f,\alpha) \in \mathscr{F} \times [0,+\infty)$ be such that $(d_{\mathscr{F}3})$ is satisfied by $(\phi_2)$, there exists $\delta > 0$
such as

$$0 < \iota < \delta \implies \quad \text{to} f(\iota) < f(\varepsilon) - \alpha \tag{3.11}$$

suppose $n(\varepsilon) \in \mathbb{N}$ such that $0 < \sum_{m \geq m(\varepsilon)}^{m-1} \frac{\hbar^n}{1-\hbar} d_{\mathscr{F}}(\theta, \tau\theta) < \delta$

Hence,by (3.11) and $(\phi_{1a})$, there exists

$$f(\sum_{i=n)}^{m-1} \frac{\hbar^n}{1-\hbar} d_{\mathscr{F}}(\theta, \tau\theta)) \leq f(\sum_{n \geq n(\varepsilon)}^{m-1} \frac{\hbar^n}{1-\hbar} d_{\mathscr{F}}(\theta, \tau\theta)) < f(\varepsilon) - \alpha \tag{3.12}$$

where, $n > m > m(\varepsilon)$ with $d_{\mathscr{F}}(\tau^{n+1}\theta, \tau^{m+1}\theta) > 0$ using $(d_{\mathscr{F}3})$ and (3.12)

$$f(d_{\mathscr{F}}(\tau^{n+1}\theta, \tau^{m+1}\theta)) \leq f(\sum_{(i=n)}^{m-1} d_{\mathscr{F}}(\theta_i, \tau\theta_{i+1}) + \alpha \leq f(\sum_{(i=n)}^{m-1} \frac{\hbar^n}{1-\hbar} d_{\mathscr{F}}(\theta, \tau\theta)) + \alpha < f(\varepsilon),$$

which is implied by $(\mathscr{F}_1)$ that, $d_{\mathscr{F}}(\tau^{n+1}\theta, \tau^{m+1}\theta) < \varepsilon, n > m > m(\varepsilon)$.

this proves that $\{\tau^n\theta\}$ is $\mathscr{F}$- Cauchy. since $(\chi, d_{\mathscr{F}})$ is complete. ☐

**Theorem 3.4.** *Suppose $(\chi, d_{\mathscr{F}})$ is a complete $\mathscr{F}$-metric space and $\tau : (\chi, d_{\mathscr{F}}) \to (\chi, d_{\mathscr{F}})$ a mapping satisfying the inequality (3.10) for every $\theta, \eta \in X$ where $\phi \in \Phi$. Then $\tau$ has a unique fixed point*

*Proof.* Suppose $\theta$ is arbitrary in $\chi$. From (3.10) for $\eta = \tau\theta, \theta_{n+1} = \tau\theta_n$ there is

$$\phi(d_{\mathscr{F}}(\tau\theta, \tau^2\theta), d_{\mathscr{F}}(\theta, \tau\theta), d_{\mathscr{F}}(\theta, \tau^2\theta), d_{\mathscr{F}}(\tau\theta, \tau^2\theta), d_{\mathscr{F}}(\tau\theta, \tau^2\theta), d_{\mathscr{F}}(\tau\theta, \tau\theta)) \leq 0.$$

$$\phi(d_{\mathscr{F}}(\tau\theta, \tau^2\theta), d_{\mathscr{F}}(\theta, \tau\theta), d_{\mathscr{F}}(\theta, \tau\theta), d_{\mathscr{F}}(\tau\theta, \tau^2\theta), d_{\mathscr{F}}(\tau\theta, \tau^2\theta), 0) \leq 0.$$

By $(\phi_{1d})$

$$\phi(d_{\mathscr{F}}(\tau\theta, \tau^2\theta), d_{\mathscr{F}}(\theta, \tau\theta), d_{\mathscr{F}}(\theta, \tau\theta), d_{\mathscr{F}}(\tau\theta, \tau^2\theta), d_{\mathscr{F}}(\tau\theta, \tau^2\theta), 0) \leq 0.$$

Suppose, $\tau^n\theta = \theta_{n+1}, \tau^2\theta = \tau\mu$

$$\phi(d_{\mathscr{F}}(\tau\theta_n, \tau\mu), d_{\mathscr{F}}(\theta_n, \mu), d_{\mathscr{F}}(\theta_n, \tau\theta_n), d_{\mathscr{F}}(\mu, \tau\mu), d_{\mathscr{F}}(\mu, \tau^2\theta_n), d_{\mathscr{F}}(\mu, \tau\theta_n)) \leq 0.$$

$$\phi(d_{\mathscr{F}}(\theta_{n+1}, \tau\mu), d_{\mathscr{F}}(\theta_n, \mu), d_{\mathscr{F}}(\theta_n, \theta_{n+1}), d_{\mathscr{F}}(\mu, \tau\mu), d_{\mathscr{F}}(\mu, \theta_{n+2}), d_{\mathscr{F}}(\mu, \theta_{n+1})) \leq 0.$$

Suppose $n \to \infty$, there is

$$\phi(d_{\mathscr{F}}(\mu, \tau\mu), 0, 0, , d_{\mathscr{F}}(\mu, \tau\mu), d_{\mathscr{F}}(\mu, \tau\mu), 0) \leq 0.$$

which implies by $(\phi_{1d})$ that $\mu = \tau\mu$. By Theorem (3.2) $\mu$ is the unique fixed point of $\tau$. ☐

**Theorem 3.5.** *If the inequality*

$$\phi(d_{\mathscr{F}}(\tau\theta,\tau\eta),d_{\mathscr{F}}(\theta,\eta),d_{\mathscr{F}}(\theta,\tau\theta),d_{\mathscr{F}}(\eta,\tau\eta),d_{\mathscr{F}}(\theta,\tau^2\eta),d_{\mathscr{F}}(\theta,\tau\eta))\leq 0.$$

*For each $\theta,\eta\in\chi$, where $\phi\in\Phi$, then $\phi$ has a unique fixed point.*

*Proof.* The proof is similar to the proof of Theorem (3.2). Suppose $\Phi$ be the family of all continuous mappings $\phi:\mathbb{R}_+^6\longrightarrow\mathbb{R}_+$ satisfying the following properties: $(\phi_{1aa})$ $\phi$ is non-increasing in the $5^{th}$ coordinate variables, and

$$\int_0^{\phi\left(\int_0^\mu\sigma(\varsigma)d\varsigma,\int_0^\upsilon\sigma(\varsigma)d\varsigma,\int_0^\upsilon\sigma(\varsigma)d\varsigma,\int_0^\mu\sigma(\varsigma)d\varsigma,\int_0^{\mu+\upsilon}\sigma(\varsigma)d\varsigma,0\right)}\psi(\rho)d\rho\leq 0$$

there exists $\hbar\in[0,1)$ such that for every $\int_0^\mu\sigma(\varsigma)d\varsigma\leq\hbar\int_0^\upsilon\sigma(\varsigma)d\varsigma$ . $(\phi_{1bb})$ $\phi$ is non-increasing in the $4^{th}$ coordinate variables,and

$$\int_0^{\phi\left(\int_0^\mu\sigma(\varsigma)d\varsigma,\int_0^\upsilon\sigma(\varsigma)d\varsigma,0,\int_0^{\mu+\upsilon}\sigma(\varsigma)d\varsigma,\int_0^\mu\sigma(\varsigma)d\varsigma,\int_0^\upsilon\sigma(\varsigma)d\varsigma\right)}\psi(\rho)d\rho\leq 0$$

implies

$$\int_0^\mu\sigma(\varsigma)d\varsigma\leq\hbar\int_0^\upsilon\sigma(\varsigma)d\varsigma.$$

$(\phi_{1cc})$ $\phi$ is non-increasing in the $3^{rd}$ coordinate variables, and

$$\int_0^{\phi\left(\int_0^\mu\sigma(\varsigma)d\varsigma,\int_0^\upsilon\sigma(\varsigma)d\varsigma,\int_0^{\mu+\upsilon}\sigma(\varsigma)d\varsigma,0,\int_0^\mu\sigma(\varsigma)d\varsigma,0,\int_0^v\sigma(\varsigma)d\varsigma\right)}\psi(\rho)d\rho\leq 0$$

implies

$$\int_0^\mu\sigma(\varsigma)d\varsigma\precsim\hbar\int_0^v\sigma(\varsigma)d\varsigma.$$

$(\phi_{1d})$ $\phi$ is non-increasing in the $3^{rd}$ coordinate variables, and

$$\int_0^{\phi\left(\int_0^\mu\sigma(\varsigma)d\varsigma,\int_0^\upsilon\sigma(\varsigma)d\varsigma,\int_0^\upsilon\sigma(\varsigma)d\varsigma,\int_0^\mu\sigma(\varsigma)d\varsigma,\int_0^u\sigma(\varsigma)d\varsigma,0\right)}\psi(\rho)d\rho\leq 0$$

implies

$$\int_0^\mu\sigma(\varsigma)d\varsigma\precsim\hbar\int_0^\upsilon\sigma(\varsigma)d\varsigma.$$

$(\phi_{22})$

$$\int_0^{\phi\left(\int_0^\mu\sigma(\varsigma)d\varsigma,\int_0^\mu\sigma(\varsigma)d\varsigma,0,0,\int_0^\mu\sigma(\varsigma)d\varsigma,0,\int_0^\mu\sigma(\varsigma)d\varsigma\right)}\psi(\rho)d\rho>0\forall\mu>0.$$

Where $\psi, \sigma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a summable non negative Lebesgue integrable function such as for each $\varepsilon \geq 0$, $\int_0^\varepsilon \psi(\rho)d\rho \geq 0$ and $\int_0^\varepsilon \sigma(\varsigma)d\varsigma \geq 0$. □

**Corollary 3.1.** *suppose $(\chi, d_{\mathscr{F}})$ be an $\mathscr{F}$- complete metric space and $\tau : \chi \to \chi$ be a self mapping for which there exists $\mathscr{F}$ satisfying $(\mathscr{F}_1)$,such that for all $\theta, \eta \in \chi$,*

(1) *$(\chi, d_{\mathscr{F}})$ be an $\mathscr{F}$- metric space*

(2) *For each $\theta, \eta \in \chi$*

$$(3.13)\int_0^\phi \begin{pmatrix} \int_0^{d_{\mathscr{F}}(\tau\theta,\tau\eta)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta,\eta)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta,\tau\theta)} \sigma(\varsigma)d\varsigma, \\ \\ \int_0^{d_{\mathscr{F}}(\eta,\tau\eta)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta,\tau\eta)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\eta,\tau\theta)} \sigma(\varsigma)d\varsigma \end{pmatrix} \psi(\rho)d\rho \leq 0.$$

*then there exists $\theta \in \chi$ such that $\theta \subset \tau$ has a unique fixed point.*

*Proof.* Suppose $\theta_0$ be an arbitrary point in $\chi$, and $\theta_{n+1} = \tau\theta_n$,$n = 0, 1, ...$

If we take $\theta = \theta_{n-1}$ and $\eta = \theta_n$ in (3.13) and denote

$$\int_0^\mu \sigma(\varsigma)d\varsigma = \int_0^{d_{\mathscr{F}}(\theta_n,\theta_{n+1})} \sigma(\varsigma)d\varsigma, \int_0^{\upsilon\upsilon} = \int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_n)} \sigma(\varsigma)d\varsigma$$

we get,

$$\int_0^\phi \begin{pmatrix} \int_0^{d_{\mathscr{F}}(\tau\theta_{n-1},\tau\theta_n)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_n)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_{n-1},\tau\theta_{n-1})} \sigma(\varsigma)d\varsigma, \\ \\ \int_0^{d_{\mathscr{F}}(\theta_n,\tau\theta_n)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_{n-1},\tau\theta_n)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_n,\tau\theta_{n-1})} \sigma(\varsigma)d\varsigma \end{pmatrix} \psi(\rho)d\rho \leq 0.$$

$$\int_0^\phi \begin{pmatrix} \int_0^{d_{\mathscr{F}}(\theta_n,\theta_{n+1})} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_n)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_n)} \sigma(\varsigma)d\varsigma, \\ \\ \int_0^{d_{\mathscr{F}}(\theta_n,\theta_{n+1})} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_{n+1})} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_n,\theta_n)} \sigma(\varsigma)d\varsigma \end{pmatrix} \psi(\rho)d\rho \leq 0.$$

$$\int_0^\phi \begin{pmatrix} \int_0^{d_{\mathscr{F}}(\theta_n,\theta_{n+1})} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_n)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_n)} \sigma(\varsigma)d\varsigma, \\ \\ \int_0^{d_{\mathscr{F}}(\theta_n,\theta_{n+1})} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_{n+1})} \sigma(\varsigma)d\varsigma, 0 \end{pmatrix} \psi(\rho)d\rho \leq 0.$$

$$\int_0^\phi \begin{pmatrix} \int_0^{d_{\mathscr{F}}(\theta_n,\theta_{n+1})} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_n)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_n)} \sigma(\varsigma)d\varsigma, \\ \\ \int_0^{d_{\mathscr{F}}(\theta_n,\theta_{n+1})+d_{\mathscr{F}}(\theta_{n-1},\theta_{n+1})} \sigma(\varsigma)d\varsigma, 0 \end{pmatrix} \psi(\rho)d\rho \leq 0.$$

$$\exists\hbar \in [0,1)\ :\ \int_0^{d_{\mathscr{F}}(\theta_n,\theta_{n+1})} \sigma(\varsigma)d\varsigma \leq \hbar \int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_n)} \sigma(\varsigma)d\varsigma$$

$$\int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_n)} \sigma(\varsigma)d\varsigma \leq \hbar \int_0^{d_{\mathscr{F}}(\theta_{n-2},\theta_{n-1})} \sigma(\varsigma)d\varsigma$$

$$\int_0^{d_{\mathscr{F}}(\theta_n,\theta_{n+1})} \sigma(\varsigma)d\varsigma \leq \hbar \int_0^{d_{\mathscr{F}}(\theta_{n-1},\theta_n)} \sigma(\varsigma)d\varsigma \leq ... \leq \hbar^n \int_0^{d_{\mathscr{F}}(\theta_0,\theta_1)} \sigma(\varsigma)d\varsigma$$

For each $n,m \in \mathbb{N}$,

$$\begin{aligned}\int_0^{d_{\mathscr{F}}(\tau^{n+1}\theta,\tau^{m+1}\theta)} \sigma(\varsigma)d\varsigma \leq& \hbar^n \int_0^{d_{\mathscr{F}}(\theta,\tau\theta)} \sigma(\varsigma)d\varsigma + \hbar^{n+1} \int_0^{d_{\mathscr{F}}(\theta,\tau\theta)} \sigma(\varsigma)d\varsigma + ... + \hbar^{m-1} \int_0^{d_{\mathscr{F}}(\theta,\tau\theta)} \sigma(\varsigma)d\varsigma \\ \leq& \hbar^n [1+\hbar+...] \int_0^{d_{\mathscr{F}}(\theta,\tau\theta)} \sigma(\varsigma)d\varsigma \\ \leq& \frac{\hbar^n}{1-\hbar} \int_0^{d_{\mathscr{F}}(\theta,\tau\theta)} \sigma(\varsigma)d\varsigma\end{aligned}$$

For each, $n \in \mathbb{N}$. Suppose $\varepsilon > 0$ be a fixed and $(f,\alpha) \in \mathscr{F} \times [0,+\infty)$ be such that $(d_{\mathscr{F}}3)$ is satisfied by $(\phi_{22})$, there exists $\delta > 0$
such that

$$0 < \iota < \delta \Longrightarrow \quad \text{to} f(\iota) < f(\varepsilon) - \alpha \tag{3.14}$$

suppose $n(\varepsilon) \in \mathbb{N}$ as

$$0 < \sum_{m \geq m(\varepsilon)}^{m-1} \frac{\hbar^n}{1-\hbar} \int_0^{d_{\mathscr{F}}(\theta_0,\theta_1)} \sigma(\varsigma)d\varsigma < \delta$$

Consequently , by (3.14) and $(\phi_{1a})$, we have suppose $n(\varepsilon) \in \mathbb{N}$ such as

$$0 < \sum_{m \geq m(\varepsilon)}^{m-1} \frac{\hbar^n}{1-\hbar} \int_0^{d_{\mathscr{F}}(\theta_0,\theta_1)} \sigma(\varsigma)d\varsigma < \delta$$

Consequently, by (3.14) and $(\phi_{1a})$, we have

$$f(\sum_{(i=n)}^{m-1} \frac{\hbar^n}{1-\hbar} \int_0^{d_{\mathscr{F}}(\theta_0,\theta_1)} \sigma(\varsigma)d\varsigma) \leq f(\sum_{n \geq n(\varepsilon)}^{m-1} \frac{\hbar^n}{1-\hbar} \int_0^{d_{\mathscr{F}}(\theta_0,\theta_1)} \sigma(\varsigma)d\varsigma) < f(\varepsilon) - \alpha \tag{3.15}$$

$$\int_0^{f(d_{\mathscr{F}}(x_n,x_m}} \sigma(\varsigma)d\varsigma) \leq f(\sum_{(i=n)}^{m-1} \int_0^{d_{\mathscr{F}}(x_i,x_{i+1})+\alpha} \sigma(\varsigma)d\varsigma) \leq f(\sum_{i=n}^{m-1} \frac{\hbar^n}{1-\hbar} \int_0^{d_{\mathscr{F}}(x_0,x_1)+\alpha} \sigma(\varsigma)d\varsigma) < f(\varepsilon),$$

which signifies $(\mathscr{F}_1)$ that, $\int_0^{d_{\mathscr{F}}(\theta_n,\theta_m)} \sigma(\varsigma)d\varsigma < \varepsilon, n > m > m(\varepsilon)$. this proves that $\{\theta_n\}$ is $\mathscr{F}$-Cauchy. since $(\chi, d_{\mathscr{F}})$ is complete , there exists $\theta_1 \in \chi$ such as $\{\theta_n\}$ is $\mathscr{F}$- convergent to $\theta_1$.

$$\lim_{n\to\infty} \int_0^{d_{\mathscr{F}}(\theta_n,\theta_1)} \sigma(\varsigma)d\varsigma = 0 \tag{3.16}$$

Now, to prove $\theta_1$ is a fixed point of $\tau$, we start with contradiction by supposing $\int_0^{d_{\mathscr{F}}(\tau\theta_1,\theta_1)} \sigma(\varsigma)d\varsigma > 0$, $n \in \mathbb{N}$ by $(d_{\mathscr{F}}3)$, there is

$$\int_0^{f(d_{\mathscr{F}}(\tau\theta_1,\theta_1))} \sigma(\varsigma)d\varsigma \le \int_0^{f(d_{\mathscr{F}}(\tau\theta_1,\theta_n)+d_{\mathscr{F}}(\theta_n,\theta_1))+\alpha} \sigma(\varsigma)d\varsigma \tag{3.17}$$

using (3.13) by taking $\theta = \theta_n, \eta = \theta_1$

$$\int_0^{\phi\begin{pmatrix} \int_0^{d_{\mathscr{F}}(\tau\theta_n,\tau\theta_1)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_n,\theta_1)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_n,\tau\theta_n)} \sigma(\varsigma)d\varsigma, \\ \int_0^{d_{\mathscr{F}}(\theta_1,\tau\theta_1)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_n,\tau\theta_1),d_{\mathscr{F}}(\theta_1,\tau\theta_n)} \sigma(\varsigma)d\varsigma \end{pmatrix}} \psi(\rho)d\rho \le 0 \tag{3.18}$$

$$\int_0^{\phi\left(\int_0^{d_{\mathscr{F}}(\theta_1,\tau\theta_1)} \sigma(\varsigma)d\varsigma,0,0,\int_0^{d_{\mathscr{F}}(\theta_1,\tau\theta_1)} \sigma(\varsigma)d\varsigma,\int_0^{d_{\mathscr{F}}(\theta_1,\tau\theta_1)} \sigma(\varsigma)d\varsigma,0.\right)} \psi(\rho)d\rho \le 0 \tag{3.19}$$

On the other hand, using $(\phi_{22})$ and (3.16),

$$\lim_{n\to\infty} \int_0^{f(d_{\mathscr{F}}(\tau\theta_1,\theta_n),d_{\mathscr{F}}(\theta_n,\theta_1))+\alpha} \sigma(\varsigma)d\varsigma = -\infty$$

$$\lim_{n\to\infty} \int_0^{f(d_{\mathscr{F}}(\tau\theta_1,\theta_n),d_{\mathscr{F}}(x_n,\theta_1))+\alpha} \sigma(\varsigma)d\varsigma = -\infty$$

Contraction, $\int_0^{d_{\mathscr{F}}(\theta_1,\tau\theta_1)} \sigma(\varsigma)d\varsigma = 0$, that is $\tau\theta_1 = \theta_1$ to prove uniqueness, suppose $\theta_1 \neq \theta_2$ are two fixed point of $\tau$

$$\int_0^{\phi\begin{pmatrix} \int_0^{d_{\mathscr{F}}(\tau\theta_1,\tau\theta_2)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_1,\theta_2)} \sigma(\varsigma)d\varsigma \int_0^{d_{\mathscr{F}}(\theta_1,\tau\theta_1)} \sigma(\varsigma)d\varsigma, \\ \int_0^{d_{\mathscr{F}}(\theta_2,\tau\theta_2)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_1,\tau\theta_2)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_2,\tau\theta_1)} \sigma(\varsigma)d\varsigma \end{pmatrix}} \psi(\rho)d\rho \le 0$$

$$\int_0^{\phi\begin{pmatrix} \int_0^{d_{\mathscr{F}}(\theta_1,\theta_2)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_1,\theta_2)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_1,\theta_1)} \sigma(\varsigma)d\varsigma, \\ \int_0^{d_{\mathscr{F}}(\theta_2,\theta_2)\sigma(\varsigma)d\varsigma}, \int_0^{d_{\mathscr{F}}(\theta_1,\theta_2)} \sigma(\varsigma)d\varsigma, \int_0^{d_{\mathscr{F}}(\theta_2,\theta_1)} \sigma(\varsigma)d\varsigma \end{pmatrix}} \psi(\rho)d\rho \le 0$$

$$\int_0^{\phi\left(\int_0^{d_{\mathscr{F}}(\theta_1,\theta_2)}\sigma(\varsigma)d\varsigma,\int_0^{d_{\mathscr{F}}(\theta_1,\theta_2)}\sigma(\varsigma)d\varsigma,0,0,\int_0^{d_{\mathscr{F}}(\theta_1,\theta_2)}\sigma(\varsigma)d\varsigma,\int_0^{d_{\mathscr{F}}(\theta_2,\theta_1)}\sigma(\varsigma)d\varsigma\right)}\psi(\rho)d\rho \leq 0$$

a contraction. Hence, $\tau$ has a unique fixed point in $\chi$. □

## 4. Conclusion

We have suggested an application in an integral type contractive condition based on fixed points theorems of $\mathscr{F}$ metric space. We have also discussed the existence of fixed points for implicit relation $\mathscr{F}$ metric space as a generalization for some fixed point theorems on metric space.

### Conflict of Interests

The authors declare that there is no conflict of interests.

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