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ON CERTAIN APPLICATIONS VIA (α,φ)-$\mathbb{K}$-CONTRACTION FIXED POINTS IN $C^{\star}$-ALGEBRA VALUED FUZZY SOFT METRIC SPACES

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Abstract: In this paper, we use common coupled fixed point results from (α,φ)-$\mathbb{K}$-type contraction mappings to address the application of the notion of C^{*}-algebra valued fuzzy soft metric to homotopy theory. In order to further illustrate our main discovery, we also offer an illustration. The obtained results build upon and apply to other studies in the literature.

Keywords: (α,φ)-$\mathbb{K}$-contraction; ω-compatible mappings, $C^{\star}$-algebra valued fuzzy soft metric; coupled fixed points.

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1. Introduction

Numerous real-world issues deal with ambiguous data and cannot be adequately described in classical mathematics. Fuzzy set theory, developed by Zadeh [1], and the theory of soft sets, developed by Molodstov [2], are two types of mathematical tools that can be used to deal with uncertainties and help with difficulties in a variety of fields.Thangaraj Beaula *et al.* defined

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fuzzy soft metric space in terms of fuzzy soft points in the cited work [3], and they supported various claims. However, numerous authors have established a great deal of findings regarding fuzzy soft sets and fuzzy soft metric spaces (see [4] -[6]).

A concept of C^*- algebra valued metric space was presented in 2006 by Ma *et al.* in [7], and certain fixed and coupled fixed point solutions for mapping under contraction conditions in these spaces were established. This line of inquiry was pursued in (see [8]-[14]).

Recently, R. P. Agarwal *et al.* [15] were introduced the idea of C^*-algebra valued fuzzy soft metric spaces and demonstrated some associated fixed point solutions on this space (see. [15]-[19]). The purpose of this article is to establish two pairs of ω-compatible mappings meeting (α,φ)-$\mathbb{K}$-contractive requirements as unique common coupled fixed point theorems in the context of $C^\star$-algebra valued fuzzy soft metric spaces. Additionally, we may provide pertinent examples and applications for homotopy.

First, let's review the important concepts of G_b-metric spaces.

2. Preliminaries

Definition 2.1: ([15]) Assume that $C \subseteq \mathscr{B}$ and $\tilde{\mathscr{B}}$ are the absolute fuzzy soft set and $\mathscr{A}_{\mathscr{B}}(x) = \tilde{1}$ for all $x \in \mathscr{B}$. Let the C^*-algebra be represented by $\tilde{C}$. The mapping $\tilde{d_{c^*}} : \tilde{\mathscr{B}} \times \tilde{\mathscr{B}} \to \tilde{C}$ satisfying the given constraints is known as the C^*-algebra valued fuzzy soft metric utilising fuzzy soft points.

(i_0) $\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(\mathscr{A}_{x_1}, \mathscr{A}_{x_2})$ for all $\mathscr{A}_{x_1}, \mathscr{A}_{x_2} \in \tilde{\mathscr{B}}$.

(i_1) $\tilde{d_{c^*}}(\mathscr{A}_{x_1}, \mathscr{A}_{x_2}) = \tilde{0}_{\tilde{C}} \Leftrightarrow \mathscr{A}_{x_1} = \mathscr{A}_{x_2}$

(i_2) $\tilde{d_{c^*}}(\mathscr{A}_{x_1}, \mathscr{A}_{x_2}) = \tilde{d_{c^*}}(\mathscr{A}_{x_2}, \mathscr{A}_{x_1})$

(i_3) $\tilde{d_{c^*}}(\mathscr{A}_{x_1}, \mathscr{A}_{x_3}) \preceq \tilde{d_{c^*}}(\mathscr{A}_{x_1}, \mathscr{A}_{x_2}) + \tilde{d_{c^*}}(\mathscr{A}_{x_2}, \mathscr{A}_{x_3})$ $\forall$ $\mathscr{A}_{x_1}, \mathscr{A}_{x_2}, \mathscr{A}_{x_3} \in \tilde{\mathscr{B}}$.

The C^*-algebra valued fuzzy soft metric space (C^*-AVFSMS) is made up of the fuzzy soft set $\tilde{\mathscr{B}}$ and the fuzzy soft metric $\tilde{d_{c^*}}$. It is represented by the symbol $(\tilde{\mathscr{B}}, \tilde{C}, \tilde{d_{c^*}})$.

Remark 2.2: ([15]) It is clear that fuzzy soft metric spaces with C^*-algebra valued fuzzy soft metrics generalise the idea of fuzzy soft metric spaces by substituting the set of fuzzy soft real numbers with $\tilde{C}_+$. The idea of a fuzzy soft metric space with $C^\star$-algebra values is similar to the definition of real metric spaces if we assume that $\tilde{C}_+ = \mathbb{R}$.

Example 2.3:([15]) If C and $\mathscr{B}$ are subsets of $\mathbb{R}$, then $\tilde{\mathscr{B}}$ is an absolute fuzzy soft set, where $\tilde{\mathscr{B}}(x) = \tilde{1}$ for every x in $\mathscr{B}$, and $\tilde{C}$ is defined as $M_2(\mathbb{R}(C)^*)$. Define $\tilde{d_{c^*}} : \tilde{\mathscr{B}} \times \tilde{\mathscr{B}} \to \tilde{C}$ by $\tilde{d_{c^*}}(\mathscr{A}_{x_1}, \mathscr{A}_{x_2}) = \begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix}$, where $\kappa = \inf\{|\mu^a_{\mathscr{A}_{x_1}}(t) - \mu^a_{\mathscr{A}_{x_2}}(t)|/t \in C\}$ and $\mathscr{A}_{x_1}, \mathscr{A}_{x_2} \in \tilde{\mathscr{B}}$.Then, by the completeness of $\mathbb{R}(C)^*$, $(\tilde{\mathscr{B}}, \tilde{C}, \tilde{d_{c^*}})$ is a complete C^* algebra valued fuzzy soft metric space. $\tilde{d_{c^*}}$ is a C^* - algebra valued fuzzy soft metric.

Definition 2.4:([15]) Assume that $(\tilde{\mathscr{B}}, \tilde{C}, \tilde{d_{c^*}})$ is a C^*-algebra valued fuzzy soft metric space. According to $\tilde{C}$ a sequence $\{\mathscr{A}_{x_k}\}$ in $\tilde{\mathscr{B}}$ is defined as:

(1) C^*-algebra valued fuzzy soft Cauchy sequence if, for each $\tilde{0}_{\tilde{C}} \prec \tilde{\varepsilon}$, there exist $\tilde{0}_{\tilde{C}} \prec \tilde{\delta}$ and a positive integer $N = N(\tilde{\varepsilon})$ such that $||\tilde{d_{c^*}}(\mathscr{A}_{x_k}, \mathscr{A}_{x_l})|| < \tilde{\delta}$ implies that $||\mu^a_{\mathscr{A}_{x_k}}(t) - \mu^a_{\mathscr{A}_{x_l}}(s)|| < \tilde{\varepsilon}$ whenever $k, l \geq N$. That is $||\tilde{d_{c^*}}(\mathscr{A}_{x_k}, \mathscr{A}_{x_l})||_{\tilde{C}} \to \tilde{0}_{\tilde{C}}$as $k, l \to \infty$.

(2) C^*-algebra valued fuzzy soft convergent to a point $\mathscr{A}_{x^{\shortmid}} \in \tilde{\mathscr{B}}$ if, for each $\tilde{0}_{\tilde{C}} \prec \tilde{\varepsilon}$, there exist $\tilde{0}_{\tilde{C}} \prec \tilde{\delta}$ and a positive integer $N = N(\tilde{\varepsilon})$ such that $||\tilde{d_{c^*}}(\mathscr{A}_{x_k}, \mathscr{A}_{x^{\shortmid}})|| < \tilde{\delta}$ implis $||\mu^a_{\mathscr{A}_{x_k}}(t) - \mu^a_{\mathscr{A}_{x^{\shortmid}}}(t)|| < \tilde{\varepsilon}$ whenever $k \geq N$. It is usually denoted as $\lim_{k\to\infty} \mathscr{A}_{x_k} = \mathscr{A}_{x^{\shortmid}}$.

(3) It is referred to as being complete when a C^*-algebra valued fuzzy soft metric space $(\tilde{\mathscr{B}}, \tilde{C}, \tilde{d_{c^*}})$ is present. If each Cauchy sequence in $\tilde{\mathscr{B}}$ converges to a fuzzy soft point in $\tilde{\mathscr{B}}$.

Definition 2.5:([18]) Let $(\tilde{\mathscr{B}}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space.
Let $S : \tilde{\mathscr{B}} \times \tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$ be a mapping. Then an element $(\mathscr{A}_{x_1}, \mathscr{V}_{x_1}) \in \tilde{\mathscr{B}} \times \tilde{\mathscr{B}}$ is called coupled fixed point of S if $S(\mathscr{A}_{x_1}, \mathscr{V}_{x_1}) = \mathscr{A}_{x_1}$ and $S(\mathscr{V}_{x_1}, \mathscr{A}_{x_1}) = \mathscr{V}_{x_1}$

Definition 2.6:([18]) Let $\tilde{\mathscr{B}}$ be absolute fuzzy soft set and $S : \tilde{\mathscr{B}} \times \tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$ and $f : \tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$ be two mappings. An element $(\mathscr{A}_{x_1}, \mathscr{V}_{x_1}) \in \tilde{\mathscr{B}} \times \tilde{\mathscr{B}}$ is called

(i) a coupled coincidence point of S and f if $f\mathscr{A}_{x_1} = S(\mathscr{A}_{x_1}, \mathscr{V}_{x_1})$ and $f\mathscr{V}_{x_1} = S(\mathscr{V}_{x_1}, \mathscr{A}_{x_1})$

(ii) a common coupled fixed point of S and f if $\mathscr{A}_{x_1} = f\mathscr{A}_{x_1} = S(\mathscr{A}_{x_1}, \mathscr{V}_{x_1})$ and $\mathscr{V}_{x_1} = f\mathscr{V}_{x_1} = S(\mathscr{V}_{x_1}, \mathscr{A}_{x_1})$.

Definition 2.7:([18]) Let $\tilde{\mathscr{B}}$ be absolute fuzzy soft set and $S : \tilde{\mathscr{B}} \times \tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$ and $f : \tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$. Then $\{S, f\}$ is said to be ω-compatible pairs if
$f(S(\mathscr{A}_{x_1}, \mathscr{V}_{x_1})) = S(f\mathscr{A}_{x_1}, f\mathscr{V}_{x_1})$ and $f(S(\mathscr{V}_{x_1}, \mathscr{A}_{x_1})) = S(f\mathscr{V}_{x_1}, f\mathscr{A}_{x_1})$ for all $\mathscr{A}_{x_1}, \mathscr{V}_{x_1} \in \tilde{\mathscr{B}}$ whenever $f\mathscr{A}_{x_1} = S(\mathscr{A}_{x_1}, \mathscr{V}_{x_1})$ and $f\mathscr{V}_{x_1} = S(\mathscr{V}_{x_1}, \mathscr{A}_{x_1})$.

Definition 2.8:([17]) Let $(\tilde{\mathscr{B}}, \tilde{C}, \tilde{d_{c^*}})$ be a $C^\star$-algebra valued fuzzy soft metric space and $f : \tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$ be a given mapping. Then we say that f is triangular α-admissible mapping if there exist a function $\alpha : \tilde{\mathscr{B}} \times \tilde{\mathscr{B}} \to \tilde{C}_+$ such that

(i) if $\alpha(\mathscr{A}_{x_1}, \mathscr{A}_{x_2}) \succeq \tilde{I}_{\tilde{C}}$ implies that $\alpha(f\mathscr{A}_{x_1}, f\mathscr{A}_{x_2}) \succeq \tilde{I}_{\tilde{C}}$ for all $\mathscr{A}_{x_1}, \mathscr{A}_{x_2} \in \tilde{\mathscr{B}}$

(ii) if $\alpha(\mathscr{A}_{x_1}, \mathscr{A}_{x_3}) \succeq \tilde{I}_{\tilde{C}}$ and $\alpha(\mathscr{A}_{x_2}, \mathscr{A}_{x_3}) \succeq \tilde{I}_{\tilde{C}}$ implies $\alpha(f\mathscr{A}_{x_1}, f\mathscr{A}_{x_2}) \succeq \tilde{I}_{\tilde{C}}$ $\forall\ \mathscr{A}_{x_1}, \mathscr{A}_{x_2}, \mathscr{A}_{x_3} \in \tilde{\mathscr{B}}$.

Lemma 2.9:([15]) Let $\tilde{C}$ be a C^*-algebra with the identity element $\tilde{I}_{\tilde{C}}$ and $\tilde{\theta}$ be a positive element of $\tilde{C}$. If $\tilde{\lambda} \in \tilde{C}$ is such that $||\tilde{\lambda}|| < 1$ then for $p < q$, we have

(a) $\lim_{q\to\infty} \sum_{k=p}^{q} (\tilde{\lambda}^\star)^k \tilde{\theta} (\tilde{\lambda})^k = \tilde{I}_{\tilde{C}} ||\widetilde{(\theta)}^{\frac{1}{2}}||^2 \left(\frac{||\tilde{\lambda}||^p}{1-||\tilde{\lambda}||}\right)$.

(b) $\sum_{k=p}^{q} (\tilde{\lambda}^\star)^k \tilde{\theta} (\tilde{\lambda})^k \to \tilde{0}_{\tilde{C}}$ as $q \to \infty$.

Lemma 2.10:([15]) Suppose that $\tilde{C}$ is a unital C^*-algebra with unit $\tilde{1}$.

(i) If $\tilde{\kappa} \in \tilde{C_+}$ with $||\tilde{\kappa}|| < \frac{1}{2}$ then $\tilde{I} - \tilde{\kappa}$ is invertible and $||\tilde{\kappa}(\tilde{I} - \tilde{\kappa})^{-1}|| < 1$,

(ii) Suppose that $\tilde{\kappa}, \tilde{\lambda} \in \tilde{C}$ with $\tilde{\kappa}, \tilde{\lambda} \succeq \tilde{0}_{\tilde{C}}$ and $\tilde{\kappa}\tilde{\lambda} = \tilde{\lambda}\tilde{\kappa}$ then $\tilde{\kappa}\tilde{\lambda} \succeq \tilde{0}_{\tilde{C}}$,

(iii) Let $\tilde{C}' = \{\tilde{\kappa} \in \tilde{C} / \tilde{\kappa}\tilde{\lambda} = \tilde{\lambda}\tilde{\kappa}\ \forall\ \tilde{\lambda} \in \tilde{C}\}$. Let $\tilde{\kappa} \in \tilde{C}'$, if $\tilde{\lambda}, \tilde{\theta} \in \tilde{C}$ with $\tilde{\lambda} \succeq \tilde{\theta} \succeq \tilde{0}$ and $\tilde{I} - \tilde{\kappa} \in \tilde{C}'_+$ is an invertible operator, then $(\tilde{I} - \tilde{\kappa})^{-1}\tilde{\lambda} \succeq (\tilde{I} - \tilde{\kappa})^{-1}\tilde{\theta}$, where $\tilde{C_+}' = \tilde{C_+} \cap \tilde{C}'$.

Notice that in c^*-algebra , if $\tilde{0} \preceq \tilde{\kappa}, \tilde{\lambda}$, one can't conclude that $\tilde{0} \preceq \tilde{\kappa}\tilde{\lambda}$. Indeed, consider the c^*-algebra $M_2(\mathbb{R}(C)^*)$ and set

$$\tilde{\kappa} = \begin{bmatrix} \mathscr{A}_{x_1}(a) & \mathscr{A}_{x_2}(a) \\ \mathscr{A}_{x_2}(a) & \mathscr{A}_{x_1}(b) \end{bmatrix} = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} \text{ and } \tilde{\lambda} = \begin{bmatrix} \mathscr{A}_{x_1}(c) & \mathscr{A}_{x_2}(c) \\ \mathscr{A}_{x_2}(c) & \mathscr{A}_{x_1}(d) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 0.5 & 0.6 \end{bmatrix}$$

then clearly $\tilde{\kappa} \succeq \tilde{0}$ and $\tilde{\lambda} \succeq \tilde{0}$ but $\tilde{\kappa}, \tilde{\lambda} \in M_2(\mathbb{R}(C)^*)_+$ while $\tilde{\kappa}\tilde{\lambda}$ is not.

For more properties of a $C^\star$-algebra valued fuzzy soft metric and C^* -algebra we refer the reader to ([15], [20]).

Now we prove our main result.

3. Main Results

For (α, φ)-$\mathbb{K}$-contraction type mappings in C^*-algebra valued fuzzy soft metric spaces, we will demonstrate various coupled fixed point theorems in this section.

Definition 3.1: Let $(\tilde{\mathscr{B}}, \tilde{C}, \tilde{d_{c^*}})$ be a $C^\star$-algebra valued fuzzy soft metric space and $T : \tilde{\mathscr{B}} \times \tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$, $f : \tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$ be a given mappings.Then we say that T and f are an α-admissible mappings if there exist a function $\alpha : \tilde{\mathscr{B}}^2 \times \tilde{\mathscr{B}}^2 \to \tilde{C}_+$ such that

$$\alpha\left((f\mathscr{A}_{x_1}, f\mathscr{A}_{x_2}), (f\mathscr{V}_{x_1}, f\mathscr{V}_{x_2})\right) \succeq \tilde{I}_{\tilde{C}}$$

implies that

$$\alpha\left(\left(T\left(\mathscr{A}_{x_1},\mathscr{A}_{x_2}\right), T\left(\mathscr{A}_{x_2},\mathscr{A}_{x_1}\right)\right), \left(T\left(\mathscr{V}_{x_1},\mathscr{V}_{x_2}\right), T\left(\mathscr{V}_{x_2},\mathscr{V}_{x_1}\right)\right)\right) \succeq \tilde{I}_{\tilde{C}}\ \forall\ \mathscr{A}_{x_1},\mathscr{A}_{x_2},\mathscr{V}_{x_1},\mathscr{V}_{x_2} \in \tilde{\mathscr{B}}.$$

$\mathscr{I}$ be the class of function $\varphi : \tilde{C_+} \to \tilde{C_+}$ satisfying the following conditions:

(a) φ is non-decreasing, continuous;

(b) $\varphi(\tilde{a}) \prec \tilde{a}$ for $\tilde{a} \in \tilde{C_+}$ and $\varphi(\tilde{a}) = \tilde{0}$ iff $\tilde{a} = \tilde{0}$

Theorem 3.2: Assume that C^*-algebra valued fuzzy soft metric space $(\tilde{\mathscr{B}},\tilde{C},\tilde{d_{c^*}})$ and suppose two mappings S: $\tilde{\mathscr{B}} \times \tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$ and f: $\tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$ be satisfying (α,φ)-$\mathbb{K}$-contraction

(1) $\alpha\left((f\mathscr{A}_{x_1}, f\mathscr{A}_{x_2}), (f\mathscr{V}_{x_1}, f\mathscr{V}_{x_2})\right) \tilde{d_{c^*}}\left(S(\mathscr{A}_{x_1},\mathscr{V}_{x_1}), S(\mathscr{A}_{x_2},\mathscr{V}_{x_2})\right) \preceq \varphi\left(\tilde{\kappa}^{\star}\mathbb{K}(\mathscr{A}_{x_1},\mathscr{A}_{x_2},\mathscr{V}_{x_1},\mathscr{V}_{x_2})\tilde{\kappa}\right)$

for all $\mathscr{A}_{x_1},\mathscr{A}_{x_2},\mathscr{V}_{x_1},\mathscr{V}_{x_2} \in \tilde{\mathscr{B}}$, where $\varphi \in \mathscr{I}$ and $\tilde{\kappa} \in \tilde{C}$ with $||\tilde{\kappa}|| < 1$,

$$\mathbb{K}(\mathscr{A}_{x_1},\mathscr{A}_{x_2},\mathscr{V}_{x_1},\mathscr{V}_{x_2}) = \max\left\{\begin{array}{c} \tilde{d_{c^*}}(f\mathscr{A}_{x_1}, f\mathscr{A}_{x_2}), \tilde{d_{c^*}}(f\mathscr{V}_{x_1}, f\mathscr{V}_{x_2}), \\ \tilde{d_{c^*}}(f\mathscr{A}_{x_1}, S(\mathscr{A}_{x_1},\mathscr{V}_{x_1})), \tilde{d_{c^*}}(f\mathscr{A}_{x_2}, S(\mathscr{A}_{x_2},\mathscr{V}_{x_2})), \\ \tilde{d_{c^*}}(f\mathscr{V}_{x_1}, S(\mathscr{V}_{x_1},\mathscr{A}_{x_1})), \tilde{d_{c^*}}(f\mathscr{V}_{x_2}, S(\mathscr{V}_{x_2},\mathscr{A}_{x_2})), \\ \frac{\tilde{d_{c^*}}(f\mathscr{A}_{x_1}, S(\mathscr{A}_{x_1},\mathscr{V}_{x_1}))\tilde{d_{c^*}}(f\mathscr{A}_{x_2}, S(\mathscr{A}_{x_2},\mathscr{V}_{x_2}))}{1+\tilde{d_{c^*}}(f\mathscr{A}_{x_1}, f\mathscr{A}_{x_2})}, \\ \frac{\tilde{d_{c^*}}(f\mathscr{V}_{x_1}, S(\mathscr{V}_{x_1},\mathscr{A}_{x_1}))\tilde{d_{c^*}}(f\mathscr{V}_{x_2}, S(\mathscr{V}_{x_2},\mathscr{A}_{x_2}))}{1+\tilde{d_{c^*}}(f\mathscr{V}_{x_1}, f\mathscr{V}_{x_2})} \end{array}\right\}.$$

(3.1) $S(\tilde{\mathscr{B}} \times \tilde{\mathscr{B}}) \subseteq f(\tilde{\mathscr{B}})$ and $f(\tilde{\mathscr{B}})$ is complete subspace of $\tilde{\mathscr{B}}$,

(3.2) $\{S, f\}$ is ω-compatible pairs,

(3.3) S and f are α- admissible mappings,

(3.4) there exists $\mathscr{A}_{x_0}, \mathscr{V}_{x_0} \in \tilde{\mathscr{B}}$ such that
$\alpha\left((f\mathscr{A}_{x_0}, f\mathscr{V}_{x_0}), (S(\mathscr{A}_{x_0},\mathscr{V}_{x_0}), S(\mathscr{V}_{x_0},\mathscr{A}_{x_0}))\right) \succeq \tilde{I}_{\tilde{C}}$

(3.5) if $\{f\mathscr{A}_{x_n}\}, \{f\mathscr{V}_{x_n}\} \subseteq \tilde{\mathscr{B}}$ such that $\alpha\left((f\mathscr{A}_{x_n}, f\mathscr{A}_{x_{n+1}}), (f\mathscr{V}_{x_n}, f\mathscr{V}_{x_{n+1}})\right) \succeq \tilde{I}_{\tilde{C}}$ for all n and $f\mathscr{A}_{x_n} \to f\mathscr{A}_{x'}$, $f\mathscr{V}_{x_n} \to f\mathscr{V}_{x'} \in f(\tilde{\mathscr{B}})$ as $n \to \infty$, then there exist a subsequences $\left\{f\mathscr{A}_{x_{n_k}}\right\}, \left\{f\mathscr{V}_{x_{n_k}}\right\}$ of $\{f\mathscr{A}_{x_n}\}, \{f\mathscr{V}_{x_n}\}$ respectively, such that $\alpha\left((f\mathscr{A}_{x_{n_k}}, f\mathscr{A}_{x'}), (f\mathscr{V}_{x_{n_k}}, f\mathscr{V}_{x'})\right) \succeq \tilde{I}_{\tilde{C}}$ for all k.

Then, in $\tilde{\mathscr{B}}$, S and f have a unique common coupled fixed point.

Proof Let $\mathscr{A}_{x_0}, \mathscr{V}_{x_0} \in \tilde{\mathscr{B}}$. From (3.1) we can construct the sequences $\{\mathscr{A}_{x_n}\}_{n=1}^{\infty}$, $\{\mathscr{V}_{x_n}\}_{n=1}^{\infty}$, $\{\xi_{x_n}\}_{n=1}^{\infty}$, $\{\zeta_{x_n}\}_{n=1}^{\infty}$ such that

$$S(\mathscr{A}_{x_n}, \mathscr{V}_{x_n}) = f\mathscr{A}_{x_{n+1}} = \xi_{x_n} \quad S(\mathscr{V}_{x_n}, \mathscr{A}_{x_n}) = f\mathscr{V}_{x_{n+1}} = \zeta_{x_n} \text{for } n = 0, 1, 2, \ldots$$

Observes that in C^*-algebra, if $\tilde{\kappa}, \tilde{b} \in \tilde{C_+}$ and $\tilde{\kappa} \preceq \tilde{b}$ implies $\tilde{x}^\star \tilde{\kappa} \tilde{x} \preceq \tilde{x}^\star \tilde{b} \tilde{x}$ for any $\tilde{x} \in \tilde{C_+}$, we conveniently refer to the element $Q = \max\left\{\tilde{d_{c^*}}(\xi_{x_0}, \xi_{x_1}), \tilde{d_{c^*}}(\zeta_{x_0}, \zeta_{x_1})\right\}$ in $\tilde{C}$.

Now we show that S and f have common coupled fixed point in $\tilde{\mathscr{B}}$. Assume that $\tilde{0}_{\tilde{C}} \prec \tilde{d_{c^*}}(\xi_{x_n}, \xi_{x_{n+1}})$ and $\tilde{0}_{\tilde{C}} \prec \tilde{d_{c^*}}(\zeta_{x_n}, \zeta_{x_{n+1}})$ $\forall\ n$. Otherwise, there exists some positive integer n such that $\xi_{x_n} = \xi_{x_{n+1}}$, $\zeta_{x_n} = \zeta_{x_{n+1}}$ and so (ξ_{x_n}, ζ_{x_n}) is a coupled fixed point of S, f, and the proof is complete. From (3.3), S and f are α-admissible, we have

$$\alpha\left((f\mathscr{A}_{x_0}, f\mathscr{A}_{x_1}), (f\mathscr{V}_{x_0}, f\mathscr{V}_{x_1})\right) = \alpha\left((f\mathscr{A}_{x_0}, S(\mathscr{A}_{x_0}, \mathscr{V}_{x_0})), (f\mathscr{V}_{x_0}, S(\mathscr{V}_{x_0}, \mathscr{A}_{x_0}))\right) \succeq \tilde{I}_{\tilde{C}}$$

implies that

$$\alpha\left((S(\mathscr{A}_{x_0}, \mathscr{V}_{x_0}), S(\mathscr{A}_{x_1}, \mathscr{V}_{x_1})), (S(\mathscr{V}_{x_0}, \mathscr{A}_{x_0}), S(\mathscr{V}_{x_1}, \mathscr{A}_{x_1}))\right) = \alpha\left((f\mathscr{A}_{x_1}, f\mathscr{A}_{x_2}), (f\mathscr{V}_{x_1}, f\mathscr{V}_{x_2})\right) \succeq \tilde{I}_{\tilde{C}}$$

Recursively, we find that

$$\alpha\left((f\mathscr{A}_{x_n}, f\mathscr{A}_{x_{n+1}}), (f\mathscr{V}_{x_n}, f\mathscr{V}_{x_{n+1}})\right) \succeq \tilde{I}_{\tilde{C}} \Rightarrow ||\alpha\left((f\mathscr{A}_{x_n}, f\mathscr{A}_{x_{n+1}}), (f\mathscr{V}_{x_n}, f\mathscr{V}_{x_{n+1}})\right)|| \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$.

From (1), (3.3) and (3.4), we have that

$$\begin{aligned}
\tilde{d_{c^*}}\left(\xi_{x_n}, \xi_{x_{n+1}}\right) &= \tilde{d_{c^*}}\left(S(\mathscr{A}_{x_n}, \mathscr{V}_{x_n}), S(\mathscr{A}_{x_{n+1}}, \mathscr{V}_{x_{n+1}})\right) \\
&\preceq \alpha\left((f\mathscr{A}_{x_n}, f\mathscr{A}_{x_{n+1}}), (f\mathscr{V}_{x_n}, f\mathscr{V}_{x_{n+1}})\right) \tilde{d_{c^*}}\left(S(\mathscr{A}_{x_n}, \mathscr{V}_{x_n}), S(\mathscr{A}_{x_{n+1}}, \mathscr{V}_{x_{n+1}})\right) \\
&\preceq \varphi\left(\tilde{\kappa}^\star \mathbb{K}(\mathscr{A}_{x_n}, \mathscr{V}_{x_n}, \mathscr{A}_{x_{n+1}}, \mathscr{V}_{x_{n+1}}) \tilde{\kappa}\right). \qquad (2)
\end{aligned}$$

Here

$$\begin{aligned}
\mathbb{K}(\mathscr{A}_{x_n},\mathscr{V}_{x_n},\mathscr{A}_{x_{n+1}},\mathscr{V}_{x_{n+1}}) &= \max\left\{\begin{array}{c}
\tilde{d_{c^*}}(f\mathscr{A}_{x_n},f\mathscr{A}_{x_{n+1}}),\tilde{d_{c^*}}(f\mathscr{V}_{x_n},f\mathscr{V}_{x_{n+1}}),\\
\tilde{d_{c^*}}(f\mathscr{A}_{x_n},S(\mathscr{A}_{x_n},\mathscr{V}_{x_n}),\tilde{d_{c^*}}(f\mathscr{A}_{x_{n+1}},S(\mathscr{A}_{x_{n+1}},\mathscr{V}_{x_{n+1}}),\\
\tilde{d_{c^*}}(f\mathscr{V}_{x_n},S(\mathscr{V}_{x_n},\mathscr{A}_{x_n}),\tilde{d_{c^*}}(f\mathscr{V}_{x_{n+1}},S(\mathscr{V}_{x_{n+1}},\mathscr{A}_{x_{n+1}}),\\
\frac{\tilde{d_{c^*}}(f\mathscr{A}_{x_n},S(\mathscr{A}_{x_n},\mathscr{V}_{x_n})\tilde{d_{c^*}}(f\mathscr{A}_{x_{n+1}},S(\mathscr{A}_{x_{n+1}},\mathscr{V}_{x_{n+1}})}{1+\tilde{d_{c^*}}(f\mathscr{A}_{x_n},f\mathscr{A}_{x_{n+1}})},\\
\frac{\tilde{d_{c^*}}(f\mathscr{V}_{x_n},S(\mathscr{V}_{x_n},\mathscr{A}_{x_n})\tilde{d_{c^*}}(f\mathscr{V}_{x_{n+1}},S(\mathscr{V}_{x_{n+1}},\mathscr{A}_{x_{n+1}})}{1+\tilde{d_{c^*}}(f\mathscr{V}_{x_n},f\mathscr{V}_{x_{n+1}})}
\end{array}\right\}\\
&= \max\left\{\begin{array}{c}
\tilde{d_{c^*}}(\xi_{x_{n-1}},\xi_{x_n}),\tilde{d_{c^*}}(\zeta_{x_{n-1}},\zeta_{x_n}),\\
\tilde{d_{c^*}}(\xi_{x_{n-1}},\xi_{x_n}),\tilde{d_{c^*}}(\xi_{x_n},\xi_{x_{n+1}}),\\
\tilde{d_{c^*}}(\zeta_{x_{n-1}},\zeta_{x_n}),\tilde{d_{c^*}}(\zeta_{x_n},\zeta_{x_{n+1}}),\\
\frac{\tilde{d_{c^*}}(\xi_{x_{n-1}},\xi_{x_n})\tilde{d_{c^*}}(\xi_{x_n},\xi_{x_{n+1}})}{1+\tilde{d_{c^*}}(\xi_{x_{n-1}},\xi_{x_n})},\\
\frac{\tilde{d_{c^*}}(\zeta_{x_{n-1}},\zeta_{x_n})\tilde{d_{c^*}}(\zeta_{x_n},\zeta_{x_{n+1}})}{1+\tilde{d_{c^*}}(\zeta_{x_{n-1}},\zeta_{x_n})}
\end{array}\right\}\\
&\preceq \max\left\{\begin{array}{c}
\tilde{d_{c^*}}(\xi_{x_{n-1}},\xi_{x_n}),\tilde{d_{c^*}}(\zeta_{x_{n-1}},\zeta_{x_n}),\\
\tilde{d_{c^*}}(\xi_{x_n},\xi_{x_{n+1}}),\tilde{d_{c^*}}(\zeta_{x_n},\zeta_{x_{n+1}})
\end{array}\right\}.
\end{aligned}$$

From (2), we have

$$\tilde{d_{c^*}}\left(\xi_{x_n},\xi_{x_{n+1}}\right) \preceq \varphi\left(\tilde{\kappa}^\star \max\left\{\begin{array}{c}\tilde{d_{c^*}}(\xi_{x_{n-1}},\xi_{x_n}),\tilde{d_{c^*}}(\zeta_{x_{n-1}},\zeta_{x_n}),\\ \tilde{d_{c^*}}(\xi_{x_n},\xi_{x_{n+1}}),\tilde{d_{c^*}}(\zeta_{x_n},\zeta_{x_{n+1}})\end{array}\right\}\tilde{\kappa}\right).$$

(3)

Similarly, we can prove that

$$\tilde{d_{c^*}}\left(\zeta_{x_n},\zeta_{x_{n+1}}\right) \preceq \varphi\left(\tilde{\kappa}^\star \max\left\{\begin{array}{c}\tilde{d_{c^*}}(\xi_{x_{n-1}},\xi_{x_n}),\tilde{d_{c^*}}(\zeta_{x_{n-1}},\zeta_{x_n}),\\ \tilde{d_{c^*}}(\xi_{x_n},\xi_{x_{n+1}}),\tilde{d_{c^*}}(\zeta_{x_n},\zeta_{x_{n+1}})\end{array}\right\}\tilde{\kappa}\right).$$

(4)

Combining (3) and (4), we get

$$\max\left\{\tilde{d_{c^*}}\left(\xi_{x_n},\xi_{x_{n+1}}\right),\tilde{d_{c^*}}\left(\zeta_{x_n},\zeta_{x_{n+1}}\right)\right\} \preceq \varphi\left(\tilde{\kappa}^\star \max\left\{\begin{array}{c}\tilde{d_{c^*}}(\xi_{x_{n-1}},\xi_{x_n}),\\ \tilde{d_{c^*}}(\zeta_{x_{n-1}},\zeta_{x_n}),\\ \tilde{d_{c^*}}(\xi_{x_n},\xi_{x_{n+1}}),\\ \tilde{d_{c^*}}(\zeta_{x_n},\zeta_{x_{n+1}})\end{array}\right\}\tilde{\kappa}\right),$$

if $\tilde{d_{c^*}}(\xi_{x_{n-1}},\xi_{x_n}) \preceq \tilde{d_{c^*}}(\xi_{x_n},\xi_{x_{n+1}})$ and $\tilde{d_{c^*}}(\zeta_{x_{n-1}},\zeta_{x_n}) \preceq \tilde{d_{c^*}}(\zeta_{x_n},\zeta_{x_{n+1}})$, then we have

$$\begin{aligned}
\max\left\{\tilde{d_{c^*}}\left(\xi_{x_n},\xi_{x_{n+1}}\right),\tilde{d_{c^*}}\left(\zeta_{x_n},\zeta_{x_{n+1}}\right)\right\} &\preceq \varphi\left(\tilde{\kappa}^\star \max\left\{\begin{array}{c}\tilde{d_{c^*}}(\xi_{x_n},\xi_{x_{n+1}}),\\ \tilde{d_{c^*}}(\zeta_{x_n},\zeta_{x_{n+1}})\end{array}\right\}\tilde{\kappa}\right)\\
&\prec \tilde{\kappa}^\star \max\left\{\begin{array}{c}\tilde{d_{c^*}}(\xi_{x_n},\xi_{x_{n+1}}),\\ \tilde{d_{c^*}}(\zeta_{x_n},\zeta_{x_{n+1}})\end{array}\right\}\tilde{\kappa}
\end{aligned}$$

a contradiction. Accordingly, we conclude that

$$\max\left\{\tilde{d_{c^*}}\left(\xi_{x_n},\xi_{x_{n+1}}\right),\tilde{d_{c^*}}\left(\zeta_{x_n},\zeta_{x_{n+1}}\right)\right\} \preceq \varphi\left(\tilde{\kappa}^\star \max\left\{\begin{array}{c}\tilde{d_{c^*}}(\xi_{x_{n-1}},\xi_{x_n}),\\ \tilde{d_{c^*}}(\zeta_{x_{n-1}},\zeta_{x_n})\end{array}\right\}\tilde{\kappa}\right)$$

By the definition of φ, we have

$$\begin{aligned}
\max\left\{\begin{array}{c}\tilde{d_{c^*}}(\xi_{x_n},\xi_{x_{n+1}}),\\ \tilde{d_{c^*}}(\zeta_{x_n},\zeta_{x_{n+1}}),\end{array}\right\} &\preceq \tilde{\kappa}^\star \max\left\{\begin{array}{c}\tilde{d_{c^*}}(\xi_{x_{n-1}},\xi_{x_n}),\\ \tilde{d_{c^*}}(\zeta_{x_{n-1}},\zeta_{x_n})\end{array}\right\}\tilde{\kappa}\\
&\preceq (\tilde{\kappa}^\star)^2 \max\left\{\begin{array}{c}\tilde{d_{c^*}}(\xi_{x_{n-2}},\xi_{x_{n-1}}),\\ \tilde{d_{c^*}}(\zeta_{x_{n-2}},\zeta_{x_{n-1}})\end{array}\right\}\tilde{\kappa}^2\\
&\preceq \cdots\\
&\preceq (\tilde{\kappa}^\star)^n \max\left\{\begin{array}{c}\tilde{d_{c^*}}(\xi_{x_0},\xi_{x_1}),\\ \tilde{d_{c^*}}(\zeta_{x_0},\zeta_{x_1})\end{array}\right\}\tilde{\kappa}^n\\
&\preceq (\tilde{\kappa}^\star)^n Q\tilde{\kappa}^n.
\end{aligned}$$

So for $n+1>m$

$$\begin{aligned}
\tilde{d_{c^*}}\left(\xi_{x_{n+1}},\xi_{x_m}\right) &\preceq \tilde{d_{c^*}}\left(\xi_{x_{n+1}},\xi_{x_n}\right)+\tilde{d_{c^*}}\left(\xi_{x_n},\xi_{x_{n-1}}\right)+\cdots+\tilde{d_{c^*}}\left(\xi_{x_{m+1}},\xi_{x_m}\right)\\
&\preceq (\tilde{\kappa}^\star)^n Q\tilde{\kappa}^n+(\tilde{\kappa}^\star)^{n-1}Q\tilde{\kappa}^{n-1}+\cdots+(\tilde{\kappa}^\star)^m Q\tilde{\kappa}^m\\
&\preceq \sum_{k=m}^{n}(\tilde{\kappa}^\star)^k Q\tilde{\kappa}^k=\sum_{k=m}^{n}(\tilde{\kappa}^\star)^k Q^{\frac{1}{2}}Q^{\frac{1}{2}}\tilde{\kappa}^k\\
&\preceq \sum_{k=m}^{n}(\tilde{\kappa}^k Q^{\frac{1}{2}})^\star(Q^{\frac{1}{2}}\tilde{\kappa}^k)=\sum_{k=m}^{n}|Q^{\frac{1}{2}}\tilde{\kappa}^k|^2\\
&\preceq \|\sum_{k=m}^{n}|Q^{\frac{1}{2}}\tilde{\kappa}^k|^2\|\tilde{I}_{\tilde{C}}\preceq\sum_{k=m}^{n}\|Q^{\frac{1}{2}}\|^2\|\tilde{\kappa}\|^{2k}\tilde{I}_{\tilde{C}}\\
&\preceq \|Q^{\frac{1}{2}}\|^2\sum_{k=m}^{n}\|\tilde{\kappa}\|^{2k}\tilde{I}_{\tilde{C}}\preceq\|Q\|\frac{||\tilde{\kappa}||^{2m}}{1-||\tilde{\kappa}||}\tilde{I}_{\tilde{C}}\to\tilde{0}_{\tilde{C}} \text{ as } m\to\infty.
\end{aligned}$$

As a result, $\{\xi_{x_n}\}$ is a Cauchy sequence in $\tilde{\mathscr{B}}$ with regard to $\tilde{C}$. We can also demonstrate that $\{\zeta_{x_n}\}$ is a Cauchy sequence with regard to $\tilde{C}$. Let's say $f(\tilde{\mathscr{B}})$ the complete subspace of $(\tilde{\mathscr{B}},\tilde{C},\tilde{d_{c^*}})$. Then the sequences $\{\xi_{x_n}\}$ and $\{\zeta_{x_n}\}$ are converge to $\xi_{x'},\zeta_{x'}$ respectively in $f(\tilde{\mathscr{B}})$. Thus there exist $\mathscr{A}_{x'},\mathscr{V}_{x'}$ in $f(\tilde{\mathscr{B}})$ Such that

$$\lim_{n\to\infty}\xi_{x_n}=\lim_{n\to\infty}f\mathscr{A}_{x_{n+1}}=\xi_{x'}=f\mathscr{A}_{x'} \text{ and } \lim_{n\to\infty}\zeta_{x_n}=\lim_{n\to\infty}f\mathscr{V}_{x_{n+1}}=\zeta_{x'}=f\mathscr{V}_{x'} \tag{5}$$

Now we claim that $S(\mathscr{A}_{x'},\mathscr{V}_{x'})=\xi_{x'}$ and $S(\mathscr{V}_{x'},\xi_{x'})=\zeta_{x'}$. Since by condition (3.5) and (5), we have $\alpha\left((f\mathscr{A}_{x_{n_k}},f\mathscr{A}_{x'}),(f\mathscr{V}_{x_{n_k}},f\mathscr{V}_{x'})\right)\succeq\tilde{I}_{\tilde{C}}$ for all k, then by the use of triangle inequality and (1) we obtain

$$\begin{aligned}
\tilde{0}_{\tilde{C}} &\preceq \tilde{d_{c^*}}(\xi_{x'},S(\mathscr{A}_{x'},\mathscr{V}_{x'}))\preceq\tilde{d_{c^*}}(\xi_{x'},\xi_{x_{n_k}})+\tilde{d_{c^*}}(\xi_{x_{n_k}},S(\mathscr{A}_{x'},\mathscr{V}_{x'}))\\
&\preceq \tilde{d_{c^*}}(\xi_{x'},\xi_{x_{n_k}})+\alpha\left((f\mathscr{A}_{x_{n_k}},f\mathscr{A}_{x'}),(f\mathscr{V}_{x_{n_k}},f\mathscr{V}_{x'})\right)\tilde{d_{c^*}}(S(\mathscr{A}_{x_{n_k}},\mathscr{V}_{x_{n_k}}),S(\mathscr{A}_{x'},\mathscr{V}_{x'}))\\
&\preceq \tilde{d_{c^*}}(\xi_{x'},\xi_{x_{n_k}})+\varphi\left(\tilde{\kappa}^\star\mathbb{K}(\mathscr{A}_{x_{n_k}},\mathscr{V}_{x_{n_k}},\mathscr{A}_{x'},\mathscr{V}_{x'})\tilde{\kappa}\right)
\end{aligned}$$

If we assume that the relation's limit is $n\to\infty$, we get

$$\tilde{0}_{\tilde{C}}\preceq\tilde{d_{c^*}}(\xi_{x'},S(\mathscr{A}_{x'},\mathscr{V}_{x'}))\preceq\lim_{n\to\infty}\varphi\left(\tilde{\kappa}^\star\mathbb{K}(\mathscr{A}_{x_{n_k}},\mathscr{V}_{x_{n_k}},\mathscr{A}_{x'},\mathscr{V}_{x'})\tilde{\kappa}\right).$$

Here

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{K}(\mathscr{A}_{x_{n_k}},\mathscr{V}_{x_{n_k}},\mathscr{A}_{x'},\mathscr{V}_{x'}) &= \lim_{n\to\infty}\max\left\{\begin{array}{c}
\tilde{d_{c^*}}(f\mathscr{A}_{x_{n_k}},f\mathscr{A}_{x'}),\tilde{d_{c^*}}(f\mathscr{V}_{x_{n_k}},f\mathscr{V}_{x'}),\\
\tilde{d_{c^*}}(f\mathscr{A}_{x_{n_k}},S(\mathscr{A}_{x_{n_k}},\mathscr{V}_{x_{n_k}}),\tilde{d_{c^*}}(f\mathscr{A}_{x'},S(\mathscr{A}_{x'},\mathscr{V}_{x'}),\\
\tilde{d_{c^*}}(f\mathscr{V}_{x_{n_k}},S(\mathscr{V}_{x_{n_k}},\mathscr{A}_{x_{n_k}}),\tilde{d_{c^*}}(f\mathscr{V}_{x'},S(\mathscr{V}_{x'},\mathscr{A}_{x'}),\\
\frac{\tilde{d_{c^*}}(f\mathscr{A}_{x_{n_k}},S(\mathscr{A}_{x_{n_k}},\mathscr{V}_{x_{n_k}}))\tilde{d_{c^*}}(f\mathscr{A}_{x'},S(\mathscr{A}_{x'},\mathscr{V}_{x'}))}{1+\tilde{d_{c^*}}(f\mathscr{A}_{x_{n_k}},f\mathscr{A}_{x'})},\\
\frac{\tilde{d_{c^*}}(f\mathscr{V}_{x_{n_k}},S(\mathscr{V}_{x_{n_k}},\mathscr{A}_{x_{n_k}}))\tilde{d_{c^*}}(f\mathscr{V}_{x'},S(\mathscr{V}_{x'},\mathscr{A}_{x'}))}{1+\tilde{d_{c^*}}(f\mathscr{V}_{x_{n_k}},f\mathscr{V}_{x'})}
\end{array}\right\}\\
&= \max\left\{\begin{array}{c}
\tilde{d_{c^*}}(f\mathscr{A}_{x'},S(\mathscr{A}_{x'},\mathscr{V}_{x'}),\\
\tilde{d_{c^*}}(f\mathscr{V}_{x'},S(\mathscr{V}_{x'},\mathscr{A}_{x'})
\end{array}\right\}\\
&= \max\left\{\begin{array}{c}
\tilde{d_{c^*}}(\xi_{x'},S(\mathscr{A}_{x'},\mathscr{V}_{x'}),\\
\tilde{d_{c^*}}(\zeta_{x'},S(\mathscr{V}_{x'},\mathscr{A}_{x'})
\end{array}\right\}
\end{aligned}$$

Therefore,

$$\tilde{0}_{\tilde{C}}\preceq\tilde{d_{c^*}}(\xi_{x'},S(\mathscr{A}_{x'},\mathscr{V}_{x'}))\preceq\varphi\left(\tilde{\kappa}^\star\max\left\{\begin{array}{c}
\tilde{d_{c^*}}(\xi_{x'},S(\mathscr{A}_{x'},\mathscr{V}_{x'}),\\
\tilde{d_{c^*}}(\zeta_{x'},S(\mathscr{V}_{x'},\mathscr{A}_{x'})
\end{array}\right\}\tilde{\kappa}\right) \tag{6}$$

Similarly, we can prove that

$$\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(\zeta_{x'}, S(\mathscr{V}_{x'}, \mathscr{A}_{x'})) \preceq \varphi\left(\tilde{\kappa}^{\star} \max\left\{\begin{array}{c} \tilde{d_{c^*}}(\xi_{x'}, S(\mathscr{A}_{x'}, \mathscr{V}_{x'}), \\ \tilde{d_{c^*}}(\zeta_{x'}, S(\mathscr{V}_{x'}, \mathscr{A}_{x'}) \end{array}\right\} \tilde{\kappa}\right) \tag{7}$$

Combining (6) and (7), we get

$$\begin{aligned} \tilde{0}_{\tilde{C}} \preceq \max\left\{\begin{array}{c} \tilde{d_{c^*}}(\xi_{x'}, S(\mathscr{A}_{x'}, \mathscr{V}_{x'})), \\ \tilde{d_{c^*}}(\zeta_{x'}, S(\mathscr{V}_{x'}, \mathscr{A}_{x'})) \end{array}\right\} &\preceq \varphi\left(\tilde{\kappa}^{\star} \max\left\{\begin{array}{c} \tilde{d_{c^*}}(\xi_{x'}, S(\mathscr{A}_{x'}, \mathscr{V}_{x'}), \\ \tilde{d_{c^*}}(\zeta_{x'}, S(\mathscr{V}_{x'}, \mathscr{A}_{x'}) \end{array}\right\} \tilde{\kappa}\right) \\ &\prec \tilde{\kappa}^{\star} \max\left\{\begin{array}{c} \tilde{d_{c^*}}(\xi_{x'}, S(\mathscr{A}_{x'}, \mathscr{V}_{x'}), \\ \tilde{d_{c^*}}(\zeta_{x'}, S(\mathscr{V}_{x'}, \mathscr{A}_{x'}) \end{array}\right\} \tilde{\kappa}, \end{aligned}$$

then we have

$$\begin{aligned} 0 \leq \max\left\{\begin{array}{c} ||\tilde{d_{c^*}}(\xi_{x'}, S(\mathscr{A}_{x'}, \mathscr{V}_{x'}))||, \\ ||\tilde{d_{c^*}}(\zeta_{x'}, S(\mathscr{V}_{x'}, \mathscr{A}_{x'}))|| \end{array}\right\} &< ||\tilde{\kappa}^{\star}|| \max\left\{\begin{array}{c} ||\tilde{d_{c^*}}(\xi_{x'}, S(\mathscr{A}_{x'}, \mathscr{V}_{x'})||, \\ ||\tilde{d_{c^*}}(\zeta_{x'}, S(\mathscr{V}_{x'}, \mathscr{A}_{x'})|| \end{array}\right\} ||\tilde{\kappa}|| \\ &< ||\tilde{\kappa}||^2 \max\left\{\begin{array}{c} ||\tilde{d_{c^*}}(\xi_{x'}, S(\mathscr{A}_{x'}, \mathscr{V}_{x'})||, \\ ||\tilde{d_{c^*}}(\zeta_{x'}, S(\mathscr{V}_{x'}, \mathscr{A}_{x'})|| \end{array}\right\} \\ &< \max\left\{\begin{array}{c} ||\tilde{d_{c^*}}(\xi_{x'}, S(\mathscr{A}_{x'}, \mathscr{V}_{x'})||, \\ ||\tilde{d_{c^*}}(\zeta_{x'}, S(\mathscr{V}_{x'}, \mathscr{A}_{x'})|| \end{array}\right\}. \end{aligned}$$

It is impossible and hence $\tilde{d_{c^*}}(\xi_{x'}, S(\mathscr{A}_{x'}, \mathscr{V}_{x'})) = \tilde{0}_{\tilde{C}}$, $\tilde{d_{c^*}}(\zeta_{x'}, S(\mathscr{V}_{x'}, \mathscr{A}_{x'})) = \tilde{0}_{\tilde{C}}$ implies that $S(\mathscr{A}_{x'}, \mathscr{V}_{x'}) = \xi_{x'}$ and $S(\mathscr{V}_{x'}, \mathscr{A}_{x'}) = \zeta_{x'}$.

Therefore, it follows $S(\mathscr{A}_{x'}, \mathscr{V}_{x'}) = \xi_{x'} = f\mathscr{A}_{x'}$ and $S(\mathscr{V}_{x'}, \mathscr{A}_{x'}) = \zeta_{x'} = f\mathscr{V}_{x'}$. Since $\{S, f\}$ is ω-compatible pair, we have $S(\xi_{x'}, \zeta_{x'}) = f\xi_{x'}$ and $S(\zeta_{x'}, \xi_{x'}) = f\zeta_{x'}$.

Now to prove that $f\xi_{x'} = \xi_{x'}$ and $f\zeta_{x'} = \zeta_{x'}$. Since S and f are α-admissible, we have $\alpha\left((f\xi_{x'}, f\mathscr{A}_{x_{n+1}}), (f\zeta_{x'}, f\mathscr{V}_{x_{n+1}})\right) \succeq \tilde{I}_{\tilde{C}}$ for all n,

$$\begin{aligned} \tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(f\xi_{x'}, \xi_{x_{n+1}}) &= \tilde{d_{c^*}}(S(\xi_{x'}, \zeta_{x'}), S(\mathscr{A}_{x_{n+1}}, \mathscr{V}_{x_{n+1}})) \\ &\preceq \alpha\left((f\xi_{x'}, f\mathscr{A}_{x_{n+1}}), (f\zeta_{x'}, f\mathscr{V}_{x_{n+1}})\right) \tilde{d_{c^*}}(S(\xi_{x'}, \zeta_{x'}), S(\mathscr{A}_{x_{n+1}}, \mathscr{V}_{x_{n+1}})) \\ &\preceq \varphi\left(\tilde{\kappa}^{\star} \mathbb{K}(\xi_{x'}, \zeta_{x'}, \mathscr{A}_{x_{n+1}}, \mathscr{V}_{x_{n+1}}) \tilde{\kappa}\right) \end{aligned}$$

By the definition of φ and taking the limit as $n \to \infty$ in the above relation, we obtain

$$\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(f\xi_{x'}, \xi_{x'}) \prec \lim_{n\to\infty} \tilde{\kappa}^{\star} \mathbb{K}(\xi_{x'}, \zeta_{x'}, \mathscr{A}_{x_{n+1}}, \mathscr{V}_{x_{n+1}}) \tilde{\kappa}$$

here

$$\lim_{n\to\infty}\mathbb{K}(\xi_{x'},\zeta_{x'},\mathscr{A}_{x_{n+1}},\mathscr{V}_{x_{n+1}}) \tag{8}$$
$$= \lim_{n\to\infty}\max\left\{\begin{array}{c} \tilde{d_{c^*}}(f\xi_{x'},f\mathscr{A}_{x_{n+1}}),\tilde{d_{c^*}}(f\zeta_{x'},f\mathscr{V}_{x_{n+1}}), \\ \tilde{d_{c^*}}(f\xi_{x'},S(\xi_{x'},\zeta_{x'}),\tilde{d_{c^*}}(f\mathscr{A}_{x_{n+1}},S(\mathscr{A}_{x_{n+1}},\mathscr{V}_{x_{n+1}}), \\ \tilde{d_{c^*}}(f\zeta_{x'},S(\zeta_{x'},\xi_{x'}),\tilde{d_{c^*}}(f\mathscr{V}_{x_{n+1}},S(\mathscr{V}_{x_{n+1}},\mathscr{A}_{x_{n+1}}), \\ \frac{\tilde{d_{c^*}}(f\xi_{x'},S(\xi_{x'},\zeta_{x'}))\tilde{d_{c^*}}(f\mathscr{A}_{x_{n+1}},S(\mathscr{A}_{x_{n+1}},\mathscr{V}_{x_{n+1}}))}{1+\tilde{d_{c^*}}(f\xi_{x'},f\mathscr{A}_{x_{n+1}})}, \\ \frac{\tilde{d_{c^*}}(f\zeta_{x'},S(\zeta_{x'},\xi_{x'}))\tilde{d_{c^*}}(f\mathscr{V}_{x_{n+1}},S(\mathscr{V}_{x_{n+1}},\mathscr{A}_{x_{n+1}}))}{1+\tilde{d_{c^*}}(f\zeta_{x'},f\mathscr{V}_{x_{n+1}})} \end{array}\right\}$$
$$= \max\left\{\ \tilde{d_{c^*}}(f\xi_{x'},\xi_{x'}),\tilde{d_{c^*}}(f\zeta_{x'},\zeta_{x'})\ \right\}.$$

Therefore,

$$\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(f\xi_{x'},\xi_{x'}) \preceq \tilde{\kappa}^{\star}\max\left\{\ \tilde{d_{c^*}}(f\xi_{x'},\xi_{x'}),\tilde{d_{c^*}}(f\zeta_{x'},\zeta_{x'})\ \right\}\tilde{\kappa}.$$

Similarly, we can prove that

$$\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(f\zeta_{x'},\zeta_{x'}) \preceq \tilde{\kappa}^{\star}\max\left\{\ \tilde{d_{c^*}}(f\xi_{x'},\xi_{x'}),\tilde{d_{c^*}}(f\zeta_{x'},\zeta_{x'})\ \right\}\tilde{\kappa}$$

Thus

$$\tilde{0}_{\tilde{C}} \preceq \max\left\{\ \tilde{d_{c^*}}(f\xi_{x'},\xi_{x'}),\tilde{d_{c^*}}(f\zeta_{x'},\zeta_{x'})\ \right\} \preceq \tilde{\kappa}^{\star}\max\left\{\ \tilde{d_{c^*}}(f\xi_{x'},\xi_{x'}),\tilde{d_{c^*}}(f\zeta_{x'},\zeta_{x'})\ \right\}\tilde{\kappa}$$

we have

$$\begin{aligned} 0 &\leq \max\left\{\ ||\tilde{d_{c^*}}(f\xi_{x'},\xi_{x'})||,||\tilde{d_{c^*}}(f\zeta_{x'},\zeta_{x'})||\ \right\} \\ &\leq ||\tilde{\kappa}^{\star}||\max\left\{\ ||\tilde{d_{c^*}}(f\xi_{x'},\xi_{x'})||,||\tilde{d_{c^*}}(f\zeta_{x'},\zeta_{x'})||\ \right\}||\tilde{\kappa}|| \\ &\leq ||\tilde{\kappa}||^2\max\left\{\ ||\tilde{d_{c^*}}(f\xi_{x'},\xi_{x'})||,||\tilde{d_{c^*}}(f\zeta_{x'},\zeta_{x'})||\ \right\} \\ &< \max\left\{\ ||\tilde{d_{c^*}}(f\xi_{x'},\xi_{x'})||,||\tilde{d_{c^*}}(f\zeta_{x'},\zeta_{x'})||\ \right\} \end{aligned}$$

It is impossible. So $\tilde{d_{c^*}}(f\xi_{x'},\xi_{x'})=0$ and $\tilde{d_{c^*}}(f\zeta_{x'},\zeta_{x'})=0$ implies that $f\xi_{x'}=\xi_{x'}$ and $f\zeta_{x'}=\zeta_{x'}$. Therefore, $S(\xi_{x'},\zeta_{x'})=f\xi_{x'}=\xi_{x'}$ and $S(\zeta_{x'},\xi_{x'})=f\zeta_{x'}=\zeta_{x'}$.

Thus $(\xi_{x'},\zeta_{x'})$ is common coupled fixed point of S and f. The following will demonstrate the distinctness of the common coupled fixed point in $\tilde{\mathscr{B}}$. Take into account that there is a second coupled fixed point $(\xi_{x''},\zeta_{x''})$ for S and f. Then

$$\begin{aligned}
\tilde{d_{c^*}}(\xi_{x'},\xi_{x''}) &= \tilde{d_{c^*}}(S(\xi_{x'},\zeta_{x'}),S(\xi_{x''},\zeta_{x''})) \\
&\preceq \alpha\left((f\xi_{x'},f\xi_{x''}),(f\zeta_{x'},f\zeta_{x''})\right)\tilde{d_{c^*}}(S(\xi_{x'},\zeta_{x'}),S(\xi_{x''},\zeta_{x''})) \\
&\preceq \varphi\left(\tilde{\kappa}^{\star}\mathbb{K}(\xi_{x'},\xi_{x''},\zeta_{x'},\zeta_{x''})\tilde{\kappa}\right) \\
&\prec \tilde{\kappa}^{\star}\max\left\{\ \tilde{d_{c^*}}(\xi_{x'},\xi_{x''}),\tilde{d_{c^*}}(\zeta_{x'},\zeta_{x''})\ \right\}\tilde{\kappa}
\end{aligned}$$

Therefore,

$$\max\left\{\ \tilde{d_{c^*}}(\xi_{x'},\xi_{x''}),\tilde{d_{c^*}}(\zeta_{x'},\zeta_{x''})\ \right\} \prec \tilde{\kappa}^{\star}\max\left\{\ \tilde{d_{c^*}}(\xi_{x'},\xi_{x''}),\tilde{d_{c^*}}(\zeta_{x'},\zeta_{x''})\ \right\}\tilde{\kappa}$$

which further induces that

$$\begin{aligned}
\max\left\{\ ||\tilde{d_{c^*}}(\xi_{x'},\xi_{x''})||,||\tilde{d_{c^*}}(\zeta_{x'},\zeta_{x''})||\ \right\} &\leq\ ||\tilde{\kappa}||^2\max\left\{\ ||\tilde{d_{c^*}}(\xi_{x'},\xi_{x''})||,||\tilde{d_{c^*}}(\zeta_{x'},\zeta_{x''})||\ \right\} \\
&<\ \max\left\{\ ||\tilde{d_{c^*}}(\xi_{x'},\xi_{x''})||,||\tilde{d_{c^*}}(\zeta_{x'},\zeta_{x''})||\ \right\}
\end{aligned}$$

It is impossible. So $\tilde{d_{c^*}}(\xi_{x'},\xi_{x''})=0$ and $\tilde{d_{c^*}}(\zeta_{x'},\zeta_{x''})=0$ implies $\xi_{x'}=\xi_{x''}$ and $\zeta_{x'}=\zeta_{x''}$ and hence $(\xi_{x'},\zeta_{x'})=(\xi_{x''},\zeta_{x''})$ which means the coupled fixed point is unique. In order to prove that S and f have a unique fixed point, we only have to prove $\xi_{x'}=\zeta_{x'}$. we have

$$\begin{aligned}
\tilde{d_{c^*}}(\xi_{x'},\zeta_{x'}) &= \tilde{d_{c^*}}(S(\xi_{x'},\zeta_{x'}),S(\zeta_{x'},\xi_{x'})) \\
&\preceq \alpha\left((f\xi_{x'},f\zeta_{x'}),(f\zeta_{x'},f\xi_{x'})\right)\tilde{d_{c^*}}(S(\xi_{x'},\zeta_{x'}),S(\zeta_{x'},\xi_{x'})) \\
&\preceq \varphi\left(\tilde{\kappa}^{\star}\tilde{d_{c^*}}(\xi_{x'},\zeta_{x'})\tilde{\kappa}\right)
\end{aligned}$$

By the definition of φ, which further induces that

$$||\tilde{d_{c^*}}(\xi_{x'},\zeta_{x'})|| \leq ||\tilde{\kappa}^{\star}\tilde{d_{c^*}}(\xi_{x'},\zeta_{x'})\tilde{\kappa}|| \leq ||\tilde{\kappa}||^2||\tilde{d_{c^*}}(\xi_{x'},\zeta_{x'})||$$

It follows from the fact $||\tilde{\kappa}||<1$ that $||\tilde{d_{c^*}}(\xi_{x'},\zeta_{x'})||=0$, thus $\xi_{x'}=\zeta_{x'}$. which means that S and f have a unique fixed point in $\tilde{\mathscr{B}}$.

Corollary 3.3: Assume that complete C^*-algebra valued fuzzy soft metric space $(\tilde{\mathscr{B}},\tilde{C},\tilde{d_{c^*}})$ and $S\colon \tilde{\mathscr{B}}\times\tilde{\mathscr{B}}\to\tilde{\mathscr{B}}$ be satisfying (α,φ)-$\mathbb{K}$-contraction

$$\alpha\left((\mathscr{A}_{x_1},\mathscr{A}_{x_2}),(\mathscr{V}_{x_1},\mathscr{V}_{x_2})\right)\tilde{d_{c^*}}\left(S(\mathscr{A}_{x_1},\mathscr{V}_{x_1}),S(\mathscr{A}_{x_2},\mathscr{V}_{x_2})\right) \preceq \varphi\left(\tilde{\kappa}^{\star}\mathbb{K}(\mathscr{A}_{x_1},\mathscr{A}_{x_2},\mathscr{V}_{x_1},\mathscr{V}_{x_2})\tilde{\kappa}\right)$$

for all $\mathscr{A}_{x_1},\mathscr{A}_{x_2},\mathscr{V}_{x_1},\mathscr{V}_{x_2}\in\tilde{\mathscr{B}}$, where $\varphi\in\mathscr{I}$ and $\tilde{\kappa}\in\tilde{C}$ with $||\tilde{\kappa}||<1$,

$$\mathbb{K}(\mathscr{A}_{x_1},\mathscr{A}_{x_2},\mathscr{V}_{x_1},\mathscr{V}_{x_2})=\max\left\{\begin{array}{c}\tilde{d_{c^*}}(\mathscr{A}_{x_1},\mathscr{A}_{x_2}),\tilde{d_{c^*}}(\mathscr{V}_{x_1},\mathscr{V}_{x_2}),\\ \tilde{d_{c^*}}(\mathscr{A}_{x_1},S(\mathscr{A}_{x_1},\mathscr{V}_{x_1}),\tilde{d_{c^*}}(\mathscr{A}_{x_2},S(\mathscr{A}_{x_2},\mathscr{V}_{x_2}),\\ \tilde{d_{c^*}}(\mathscr{V}_{x_1},S(\mathscr{V}_{x_1},\mathscr{A}_{x_1}),\tilde{d_{c^*}}(\mathscr{V}_{x_2},S(\mathscr{V}_{x_2},\mathscr{A}_{x_2}),\\ \frac{\tilde{d_{c^*}}(\mathscr{A}_{x_1},S(\mathscr{A}_{x_1},\mathscr{V}_{x_1}))\tilde{d_{c^*}}(\mathscr{A}_{x_2},S(\mathscr{A}_{x_2},\mathscr{V}_{x_2}))}{1+\tilde{d_{c^*}}(\mathscr{A}_{x_1},\mathscr{A}_{x_2})},\\ \frac{\tilde{d_{c^*}}(\mathscr{V}_{x_1},S(\mathscr{V}_{x_1},\mathscr{A}_{x_1}))\tilde{d_{c^*}}(\mathscr{V}_{x_2},S(\mathscr{V}_{x_2},\mathscr{A}_{x_2}))}{1+\tilde{d_{c^*}}(\mathscr{V}_{x_1},\mathscr{V}_{x_2})}\end{array}\right\}.$$

(3.1) S is α- admissible mappings,

(3.2) for $\mathscr{A}_{x_0},\mathscr{V}_{x_0}\in\tilde{\mathscr{B}}$ such that $\alpha\left((\mathscr{A}_{x_0},\mathscr{V}_{x_0}),(S(\mathscr{A}_{x_0},\mathscr{V}_{x_0}),S(\mathscr{V}_{x_0},\mathscr{A}_{x_0}))\right)\succeq\tilde{I}_{\tilde{C}}$

(3.3) if $\{\mathscr{A}_{x_n}\},\{\mathscr{V}_{x_n}\}\subseteq\tilde{\mathscr{B}}$ such that $\alpha\left((\mathscr{A}_{x_n},\mathscr{A}_{x_{n+1}}),(\mathscr{V}_{x_n},\mathscr{V}_{x_{n+1}})\right)\succeq\tilde{I}_{\tilde{C}}$ for all n and $\mathscr{A}_{x_n}\to\mathscr{A}_{x'}$, $\mathscr{V}_{x_n}\to\mathscr{V}_{x'}\in\tilde{\mathscr{B}}$ as $n\to\infty$, then there exist a subsequences $\left\{\mathscr{A}_{x_{n_k}}\right\},\left\{\mathscr{V}_{x_{n_k}}\right\}$ of $\{\mathscr{A}_{x_n}\},\{\mathscr{V}_{x_n}\}$ respectively, such that $\alpha\left((\mathscr{A}_{x_{n_k}},\mathscr{A}_{x'}),(\mathscr{V}_{x_{n_k}},\mathscr{V}_{x'})\right)\succeq\tilde{I}_{\tilde{C}}$ for all k.

Then, S has a unique coupled fixed point in $\tilde{\mathscr{B}}$.

Corollary 3.4: Assume that complete C^*-algebra valued fuzzy soft metric space $(\tilde{\mathscr{B}},\tilde{C},\tilde{d_{c^*}})$ and $S\colon\tilde{\mathscr{B}}\times\tilde{\mathscr{B}}\to\tilde{\mathscr{B}}$ be satisfying (α,φ)-contraction

$$\alpha\left((\mathscr{A}_{x_1},\mathscr{A}_{x_2}),(\mathscr{V}_{x_1},\mathscr{V}_{x_2})\right)\tilde{d_{c^*}}\left(S(\mathscr{A}_{x_1},\mathscr{V}_{x_1}),S(\mathscr{A}_{x_2},\mathscr{V}_{x_2})\right)\preceq\varphi\left(\tilde{\kappa}^{\star}\tilde{d_{c^*}}(\mathscr{A}_{x_1},\mathscr{A}_{x_2})\tilde{\kappa}+\tilde{\kappa}^{\star}\tilde{d_{c^*}}(\mathscr{V}_{x_1},\mathscr{V}_{x_2})\tilde{\kappa}\right)$$

for all $\mathscr{A}_{x_1},\mathscr{A}_{x_2},\mathscr{V}_{x_1},\mathscr{V}_{x_2}\in\tilde{\mathscr{B}}$, where $\varphi\in\mathscr{I}$ and $\tilde{\kappa}\in\tilde{C}$ with $||\sqrt{2}\tilde{\kappa}||<1$,

(3.1) S is α- admissible mappings,

(3.2) for $\mathscr{A}_{x_0},\mathscr{V}_{x_0}\in\tilde{\mathscr{B}}$ such that $\alpha\left((\mathscr{A}_{x_0},\mathscr{V}_{x_0}),(S(\mathscr{A}_{x_0},\mathscr{V}_{x_0}),S(\mathscr{V}_{x_0},\mathscr{A}_{x_0}))\right)\succeq\tilde{I}_{\tilde{C}}$

(3.3) if $\{\mathscr{A}_{x_n}\},\{\mathscr{V}_{x_n}\}\subseteq\tilde{\mathscr{B}}$ such that $\alpha\left((\mathscr{A}_{x_n},\mathscr{A}_{x_{n+1}}),(\mathscr{V}_{x_n},\mathscr{V}_{x_{n+1}})\right)\succeq\tilde{I}_{\tilde{C}}\ \forall\, n$ and $\mathscr{A}_{x_n}\to\mathscr{A}_{x'}$, $\mathscr{V}_{x_n}\to\mathscr{V}_{x'}\in\tilde{\mathscr{B}}$ as $n\to\infty$, then $\alpha\left((\mathscr{A}_{x_n},\mathscr{A}_{x'}),(\mathscr{V}_{x_n},\mathscr{V}_{x'})\right)\succeq\tilde{I}_{\tilde{C}}\ \forall\, n$.

Then, S has a unique coupled fixed point in $\tilde{\mathscr{B}}$.

Corollary 3.5: Assume that complete C^*-algebra valued fuzzy soft metric space $(\tilde{\mathscr{B}},\tilde{C},\tilde{d_{c^*}})$ and suppose $S\colon\tilde{\mathscr{B}}\times\tilde{\mathscr{B}}\to\tilde{\mathscr{B}}$ be satisfying

$\tilde{d_{c^*}}\left(S(\mathscr{A}_{x_1},\mathscr{V}_{x_1}),S(\mathscr{A}_{x_2},\mathscr{V}_{x_2})\right)\preceq\varphi\left(\tilde{\kappa}^{\star}\tilde{d_{c^*}}(\mathscr{A}_{x_1},\mathscr{A}_{x_2})\tilde{\kappa}+\tilde{\kappa}^{\star}\tilde{d_{c^*}}(\mathscr{V}_{x_1},\mathscr{V}_{x_2})\tilde{\kappa}\right)$

for all $\mathscr{A}_{x_1},\mathscr{A}_{x_2},\mathscr{V}_{x_1},\mathscr{V}_{x_2}\in\tilde{\mathscr{B}}$, where $\varphi\in\mathscr{I}$ and $\tilde{\kappa}\in\tilde{C}$ with $||\sqrt{2}\tilde{\kappa}||<1$, Then, S has a unique coupled fixed point in $\tilde{\mathscr{B}}$.

Example 3.5: Let $U=[0,\infty)$ and $\mathscr{B}=C=[0,1]$, let $\tilde{\mathscr{B}}$ be absolute fuzzy soft set, that is

$\tilde{\mathscr{B}}(x) = \tilde{1}$ for all $x \in \mathscr{B}$, and $\tilde{C} = M_2(\mathbb{R}(C)^*)$, be the C^*-algebra.

Define $\tilde{d_{c^*}}: \tilde{\mathscr{B}} \times \tilde{\mathscr{B}} \to \tilde{C}$ by $\tilde{d_{c^*}}(\mathscr{A}_{x_1}, \mathscr{A}_{x_2}) = \begin{pmatrix} \inf\{|\mathscr{A}_{x_1}(x) - \mathscr{A}_{x_2}(x)|/x \in U\} & 0 \end{pmatrix}$ then obviously $(\tilde{\mathscr{B}}, \tilde{C}, \tilde{d_{c^*}})$ is a complete C^*-algebra valued fuzzy soft metric space.

We define $S\colon \tilde{\mathscr{B}} \times \tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$ by $S(\mathscr{A}_{x_1}, \mathscr{V}_{x_1}) = \begin{cases} \frac{\mathscr{A}_{x_1}+2\mathscr{V}_{x_1}+3}{12} & \text{for} \quad \mathscr{A}_{x_1}, \mathscr{V}_{x_1} \in [0, \frac{1}{2}) \cup (\frac{5}{6}, 1] \\ 0 & \text{for} \quad \mathscr{A}_{x_1}, \mathscr{V}_{x_1} \in \{\frac{1}{2}, \frac{5}{6}\} \end{cases}$

$f\colon \tilde{\mathscr{B}} \to \tilde{\mathscr{B}}$ by $f\mathscr{A}_{x_1} = \frac{2\mathscr{A}_{x_1}+1}{5}$ $\forall$ $\mathscr{A}_{x_1} \in \tilde{\mathscr{B}}$. Now, we define the mapping $\alpha : \tilde{\mathscr{B}}^2 \times \tilde{\mathscr{B}}^2 \to \tilde{C}_+$ as

$\alpha\left((f\mathscr{A}_{x_1}, f\mathscr{A}_{x_2}), (f\mathscr{V}_{x_1}, f\mathscr{V}_{x_2})\right) = \begin{cases} \tilde{I}_{\tilde{C}} & \text{for} \quad \mathscr{A}_{x_1}, \mathscr{A}_{x_2}, \mathscr{V}_{x_1}, \mathscr{V}_{x_2} \in [0, \frac{1}{2}) \cup (\frac{5}{6}, 1] \\ \tilde{0}_{\tilde{C}} & \text{for} \quad \text{otherwise} \end{cases}$ and $\varphi : \tilde{C_+} \to \tilde{C_+}$ as $\varphi(\tilde{\kappa}) = \frac{\tilde{\kappa}}{2}$ for all $\tilde{\kappa} \in \tilde{C_+}$. Then obviously, $S(\tilde{\mathscr{B}} \times \tilde{\mathscr{B}}) \subseteq f(\tilde{\mathscr{B}})$ and $\{S, f\}$ is ω-compatible pair. Now observer that S and f are α-admissible mappings.

Let $\mathscr{A}_{x_1}, \mathscr{A}_{x_2}, \mathscr{V}_{x_1}, \mathscr{V}_{x_2} \in \tilde{\mathscr{B}}$,

if $\alpha\left((f\mathscr{A}_{x_1}, f\mathscr{A}_{x_2}), (f\mathscr{V}_{x_1}, f\mathscr{V}_{x_2})\right) \succeq \tilde{I}_{\tilde{C}}$ implies $||\alpha\left((f\mathscr{A}_{x_1}, f\mathscr{A}_{x_2}), (f\mathscr{V}_{x_1}, f\mathscr{V}_{x_2})\right)|| \geq 1$

then $\mathscr{A}_{x_1}, \mathscr{A}_{x_2}, \mathscr{V}_{x_1}, \mathscr{V}_{x_2} \in [0, \frac{1}{2}) \cup (\frac{5}{6}, 1]$ which gives us $S(\mathscr{A}_{x_1}, \mathscr{V}_{x_1}) \preceq \tilde{1}$, $S(\mathscr{A}_{x_2}, \mathscr{V}_{x_2}) \preceq \tilde{1}$, $S(\mathscr{V}_{x_1}, \mathscr{A}_{x_1}) \preceq \tilde{1}$ and $S(\mathscr{V}_{x_2}, \mathscr{A}_{x_2}) \preceq \tilde{1}$ then

$||\alpha\left((S(\mathscr{A}_{x_1}, \mathscr{V}_{x_1}), S(\mathscr{A}_{x_2}, \mathscr{V}_{x_2})), (S(\mathscr{V}_{x_1}, \mathscr{A}_{x_1}), S(\mathscr{V}_{x_2}, \mathscr{A}_{x_2}))\right)|| \geq 1$ implies

$\alpha\left((S(\mathscr{A}_{x_1}, \mathscr{V}_{x_1}), S(\mathscr{A}_{x_2}, \mathscr{V}_{x_2})), (S(\mathscr{V}_{x_1}, \mathscr{A}_{x_1}), S(\mathscr{V}_{x_2}, \mathscr{A}_{x_2}))\right) \succeq \tilde{I}_{\tilde{C}}$. Therefore the assertion holds. In reason of the above arguments,

$\alpha\left((f\frac{1}{3}, f\frac{1}{3}), (S(\frac{1}{3}, \frac{1}{3}), S(\frac{1}{3}, \frac{1}{3}))\right) \succeq \tilde{1} \Rightarrow ||\alpha\left((f\frac{1}{3}, f\frac{1}{3}), (S(\frac{1}{3}, \frac{1}{3}), S(\frac{1}{3}, \frac{1}{3}))\right)|| \geq 1.$

If $\{f\mathscr{A}_{x_n}\}, \{f\mathscr{V}_{x_n}\} \subseteq \tilde{\mathscr{B}}$ such that $\alpha\left((f\mathscr{A}_{x_n}, f\mathscr{A}_{x_{n+1}}), (f\mathscr{V}_{x_n}, f\mathscr{V}_{x_{n+1}})\right) \succeq \tilde{I}_{\tilde{C}}$ for all $n = 0, 1, 2 \cdots$ and $f\mathscr{A}_{x_n} \to f\mathscr{A}_{x'}$, $f\mathscr{V}_{x_n} \to f\mathscr{V}_{x'} \in f(\tilde{\mathscr{B}})$ as $n \to \infty$, then $\{f\mathscr{A}_{x_n}\}, \{f\mathscr{V}_{x_n}\} \subseteq [0, \frac{1}{2}) \cup (\frac{5}{6}, 1]$ and hence $f\mathscr{A}_{x'}, f\mathscr{V}_{x'} \subseteq [0, \frac{1}{2}) \cup (\frac{5}{6}, 1]$, by the by the definition of α, there exist a subsequences $\left\{f\mathscr{A}_{x_{n_k}}\right\}, \left\{f\mathscr{V}_{x_{n_k}}\right\} \subseteq [0, \frac{1}{2}) \cup (\frac{5}{6}, 1]$ such that $\alpha\left((f\mathscr{A}_{x_{n_k}}, f\mathscr{A}_{x'}), (f\mathscr{V}_{x_{n_k}}, f\mathscr{V}_{x'})\right) \succeq \tilde{I}_{\tilde{C}}$.

Now for all $\mathscr{A}_{x_1}, \mathscr{A}_{x_2}, \mathscr{V}_{x_1}, \mathscr{V}_{x_2} \in [0, \frac{1}{2}) \cup (\frac{5}{6}, 1]$, we have

$$\begin{aligned} \tilde{d}_{c^*}(S(\mathscr{A}_{x_1}, \mathscr{V}_{x_1}), S(\mathscr{A}_{x_2}, \mathscr{V}_{x_2})) &= \begin{pmatrix} \inf\{|S(\mathscr{A}_{x_1}, \mathscr{V}_{x_1})(a) - S(\mathscr{A}_{x_2}, \mathscr{V}_{x_2})(a)|/a \in U\} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \inf\{|\frac{\mathscr{A}_{x_1}(a)+2\mathscr{V}_{x_1}(a)+3}{12} - \frac{\mathscr{A}_{x_2}(a)+2\mathscr{V}_{x_2}(a)+3}{12}|/a \in U\} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \inf\{|\frac{2\mathscr{A}_{x_1}(a)-2\mathscr{A}_{x_2}(a)}{24} + \frac{2\mathscr{V}_{x_1}(a)-2\mathscr{V}_{x_2}(a)}{12}|/a \in U\} & 0 \end{pmatrix} \\ &\preceq \begin{pmatrix} \frac{1}{\sqrt{5}} \inf\{|\frac{2\mathscr{A}_{x_1}(a)-2\mathscr{A}_{x_2}(a)}{5}|/a \in U\} & 0 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&+\begin{pmatrix} \frac{1}{\sqrt{5}}\inf\{|\frac{2\mathscr{V}_{x_1}(a)-2\mathscr{V}_{x_2}(a)}{5}|/a\in U\} & 0 \end{pmatrix}\\
&\preceq \frac{1}{\sqrt{5}}\left(\tilde{d_{c^*}}(f\mathscr{A}_{x_1},f\mathscr{A}_{x_2})+\tilde{d_{c^*}}(f\mathscr{V}_{x_1},f\mathscr{V}_{x_2})\right)\\
&\preceq \frac{2}{\sqrt{5}}\max\left\{\tilde{d_{c^*}}(f\mathscr{A}_{x_1},f\mathscr{A}_{x_2}),\tilde{d_{c^*}}(f\mathscr{V}_{x_1},f\mathscr{V}_{x_2})\right\}\\
&\preceq \varphi\left(\tilde{\kappa}^{\star}\mathbb{K}(\mathscr{A}_{x_1},\mathscr{A}_{x_2},\mathscr{V}_{x_1},\mathscr{V}_{x_2})\tilde{\kappa}\right).
\end{aligned}
$$

Here $\tilde{\kappa}=\begin{bmatrix}\frac{2}{\sqrt{5}} & 0\\ 0 & \frac{2}{\sqrt{5}}\end{bmatrix}$ with $||\tilde{\kappa}||=\frac{2}{\sqrt{5}}<1$ Therefore, all the conditions of Theorem 3.2 satisfied and $(\frac{1}{3},\frac{1}{3})$ is coupled fixed point of S and f.

3.1. Applications to Homotopy.

In this section, we study the existence of an unique solution to Homotopy theory.

Theorem 3.1.1: Let $(\tilde{\mathscr{B}},\tilde{C},\tilde{d_{c^*}})$ be complete C^*-algebra valued fuzzy soft metric space, Δ and $\overline{\Delta}$ be an open and closed subset of $\tilde{\mathscr{B}}$ such that $\Delta\subseteq\overline{\Delta}$. Suppose $\mathscr{H}:\overline{\Delta}^2\times[0,1]\to\tilde{\mathscr{B}}$ be an operator with following conditions are satisfying,

$\tau_0)$ $\wp_x\neq\mathscr{H}(\wp_x,\varpi_x,s)$, $\varpi_x\neq\mathscr{H}(\varpi_x,\wp_x,s)$, for each $\wp_x,\varpi_x\in\partial\Delta$ and $s\in[0,1]$ (Here $\partial\Delta$ is boundary of Δ in $\mathscr{B}$);

$\tau_1)$ for all $\wp_x,\varpi_x,æ_x,ß_x\in\overline{\Delta}$, $s\in[0,1]$ and $\varphi\in\mathscr{I}$ and $\tilde{\kappa}\in\tilde{C}$ with $||\sqrt{2}\tilde{\kappa}||<1$ such that

$$\tilde{d_{c^*}}\left(\mathscr{H}(\wp_x,\varpi_x,s),\mathscr{H}(æ_x,ß_x,s)\right)\preceq\varphi\left(\tilde{\kappa}\tilde{d_{c^*}}(\wp_x,æ_x)\tilde{\kappa}^*+\tilde{\kappa}\tilde{d_{c^*}}(\varpi_x,ß_x)\tilde{\kappa}^*\right).$$

$\tau_2)$ $\exists\,\tilde{M}\in\tilde{C_+}\ni\tilde{d_{c^*}}(\mathscr{H}(\wp_x,\varpi_x,s),\mathscr{H}(\wp_x,\varpi_x,t))\preceq||\tilde{M}|||s-t|$

for every $\wp_x,\varpi_x\in\overline{\Delta}$and $s,t\in[0,1]$.

Then $\mathscr{H}(.,0)$ has a coupled fixed point $\iff$ $\mathscr{H}(.,1)$ has a coupled fixed point.

Proof Let the set

$$\mathscr{B}=\left\{\ s\in[0,1]:\mathscr{H}(\wp_x,\varpi_x,s)=\wp_x,\mathscr{H}(\varpi_x,\wp_x,s)=\varpi_x\text{ for some }\wp_x,\varpi_x\in\Delta\ \right\}.$$

Suppose that $\mathscr{H}(.,0)$ has a coupled fixed point in Δ^2, we have that $(0,0)\in\mathscr{B}^2$. So that $\mathscr{B}$ is non-empty set. Now we show that $\mathscr{B}$ is both closed and open in $[0,1]$ and hence by the connectedness $\mathscr{B}=[0,1]$. As a result, $\mathscr{H}(.,1)$ has a coupled fixed point in Δ^2. First we show that $\mathscr{B}$ closed in $[0,1]$. To see this, Let $\left\{s_{x_p}\right\}_{p=1}^{\infty}\subseteq\mathscr{B}$ with $s_{x_p}\to s_{x'}\in[0,1]$ as $p\to\infty$. We must

show that $s_{x'} \in \mathscr{B}$. Since $s_{x_p} \in \mathscr{B}$
for $p = 0,1,2,3,\cdots$, there exists sequences $\left\{\wp_{x_p}\right\}, \left\{\varpi_{x_p}\right\} \subseteq \Delta$ with
$\wp_{x_{p+1}} = \mathscr{H}(\wp_{x_p}, \varpi_{x_p}, s_{x_p})$, $\varpi_{x_{p+1}} = \mathscr{H}(\varpi_{x_p}, \wp_{x_p}, s_{x_p})$.
Consider

$$\begin{aligned}
\tilde{d_{c^*}}(\wp_{x_p}, \wp_{x_{p+1}}) &= \tilde{d_{c^*}}\left(\mathscr{H}(\wp_{x_{p-1}}, \varpi_{x_{p-1}}, s_{x_{p-1}}), \mathscr{H}(\wp_{x_p}, \varpi_{x_p}, s_{x_p})\right) \\
&\preceq \begin{array}{c} \tilde{d_{c^*}}\left(\mathscr{H}(\wp_{x_{p-1}}, \varpi_{x_{p-1}}, s_{x_{p-1}}), \mathscr{H}(\wp_{x_p}, \varpi_{x_p}, s_{x_{p-1}}) \right) \\ +\tilde{d_{c^*}}\left(\mathscr{H}(\wp_{x_p}, \varpi_{x_p}, s_{x_{p-1}}), \mathscr{H}(\wp_{x_p}, \varpi_{x_p}, s_{x_p}) \right) \end{array} \\
&\preceq \tilde{d_{c^*}}\left(\mathscr{H}(\wp_{x_{p-1}}, \varpi_{x_{p-1}}, s_{x_{p-1}}), \mathscr{H}(\wp_{x_p}, \varpi_{x_p}, s_{x_{p-1}})\right) + ||\tilde{M}|||s_{x_{p-1}} - s_{x_p}|.
\end{aligned}$$

Letting $p \to \infty$, we get

$$\begin{aligned}
\lim_{p\to\infty} \tilde{d_{c^*}}(\wp_{x_p}, \wp_{x_{p+1}}) &\preceq \lim_{p\to\infty} \tilde{d_{c^*}}\left(\mathscr{H}(\wp_{x_{p-1}}, \varpi_{x_{p-1}}, s_{x_{p-1}}), \mathscr{H}(\wp_{x_p}, \varpi_{x_p}, s_{x_{p-1}})\right) + 0 \\
&\preceq \lim_{p\to\infty} \varphi\left(\tilde{\kappa}\tilde{d_{c^*}}(\wp_{x_{p-1}}, \wp_{x_p})\tilde{\kappa}^* + \tilde{\kappa}\tilde{d_{c^*}}(\varpi_{x_{p-1}}, \varpi_{x_p})\tilde{\kappa}^*\right) \\
&\prec \lim_{p\to\infty} \left(\tilde{\kappa}\tilde{d_{c^*}}(\wp_{x_{p-1}}, \wp_{x_p})\tilde{\kappa}^* + \tilde{\kappa}\tilde{d_{c^*}}(\varpi_{x_{p-1}}, \varpi_{x_p})\tilde{\kappa}^*\right) \\
&\prec \lim_{p\to\infty} \tilde{\kappa}\left(\tilde{d_{c^*}}(\wp_{x_{p-1}}, \wp_{x_p}) + \tilde{d_{c^*}}(\varpi_{x_{p-1}}, \varpi_{x_p})\right)\tilde{\kappa}^*. \quad (9)
\end{aligned}$$

Similarly,

$$\lim_{p\to\infty} \tilde{d_{c^*}}(\varpi_{x_p}, \varpi_{x_{p+1}}) \prec \lim_{p\to\infty} \tilde{\kappa}\left(\tilde{d_{c^*}}(\wp_{x_{p-1}}, \wp_{x_p}) + \tilde{d_{c^*}}(\varpi_{x_{p-1}}, \varpi_{x_p})\right)\tilde{\kappa}^* \quad (10)$$

and now from (9) and (10), we have

$$\begin{aligned}
\tilde{\delta}_p &= \lim_{p\to\infty} \left(\tilde{d_{c^*}}(\varpi_{x_p}, \varpi_{x_{p+1}}) + \tilde{d_{c^*}}(\wp_{x_p}, \wp_{x_{p+1}})\right) \\
&\prec \lim_{p\to\infty} (\sqrt{2}\tilde{\kappa})\left(\tilde{d_{c^*}}(\wp_{x_{p-1}}, \wp_{x_p}) + \tilde{d_{c^*}}(\varpi_{x_{p-1}}, \varpi_{x_p})\right)(\sqrt{2}\tilde{\kappa})^* \\
&\prec \lim_{p\to\infty} (\sqrt{2}\tilde{\kappa})\tilde{\delta}_{p-1}(\sqrt{2}\tilde{\kappa})^*
\end{aligned}$$

which, together with the property: if $\tilde{a}, \tilde{b} \in \tilde{C_+}$ and $\tilde{a} \preceq \tilde{b}$ implies $\tilde{x}^\star \tilde{a}\tilde{x} \preceq \tilde{x}^\star \tilde{b}\tilde{x}$, yields that for each $p \in \mathbb{N} \cup \{0\}$.

$$\tilde{0}_{\tilde{C}} \preceq \tilde{\delta}_p \prec \lim_{p\to\infty} (\sqrt{2}\tilde{\kappa})\tilde{\delta}_{p-1}(\sqrt{2}\tilde{\kappa})^* \prec \cdots \prec \lim_{p\to\infty} (\sqrt{2}\tilde{\kappa})^p \tilde{\delta}_0[(\sqrt{2}\tilde{\kappa})^*]^p$$

since $||\sqrt{2}\tilde{\kappa}|| < 1$ it follows that

$$\lim_{p\to\infty} \tilde{d_{c^*}}(\wp_{x_p},\wp_{x_{p+1}}) = \tilde{0}_{\tilde{C}} \text{ and } \lim_{p\to\infty} \tilde{d_{c^*}}(\varpi_{x_p},\varpi_{x_{p+1}}) = \tilde{0}_{\tilde{C}}.$$

Now for $p \in \mathbb{N}$ and any $n \in \mathbb{N}$,

$$\tilde{d_{c^*}}\left(\wp_{x_{p+n}},\wp_{x_p}\right) \preceq \tilde{d_{c^*}}\left(\wp_{x_{p+n}},\wp_{x_{p+n-1}}\right) + \tilde{d_{c^*}}\left(\wp_{x_{p+n-1}},\wp_{x_{p+n-2}}\right) + \ldots + \tilde{d_{c^*}}\left(\wp_{x_{p+1}},\wp_{x_p}\right)$$

$$\tilde{d_{c^*}}\left(\varpi_{x_{p+n}},\varpi_{x_p}\right) \preceq \tilde{d_{c^*}}\left(\varpi_{x_{p+n}},\varpi_{x_{p+n-1}}\right) + \tilde{d_{c^*}}\left(\varpi_{x_{p+n-1}},\varpi_{x_{p+n-2}}\right) + \ldots + \tilde{d_{c^*}}\left(\varpi_{x_{p+1}},\varpi_{x_p}\right).$$

Consequently,

$$\begin{aligned}
\tilde{d_{c^*}}\left(\wp_{x_{p+n}},\wp_{x_p}\right) + \tilde{d_{c^*}}\left(\varpi_{x_{p+n}},\varpi_{x_p}\right) &\preceq \tilde{\delta}_{p+n-1} + \tilde{\delta}_{p+n-2} + \ldots + \tilde{\delta}_p \\
&\preceq \sum_{m=p}^{p+n-1} (\sqrt{2}\tilde{\kappa})^m \tilde{\delta}_0 [(\sqrt{2}\tilde{\kappa})^\star]^m
\end{aligned}$$

and then

$$\begin{aligned}
||\tilde{d_{c^*}}\left(\wp_{x_{p+n}},\wp_{x_p}\right) + \tilde{d_{c^*}}\left(\varpi_{x_{p+n}},\varpi_{x_p}\right)|| &\leq \sum_{m=p}^{p+n-1} ||\sqrt{2}\tilde{\kappa}||^{2m}\tilde{\delta}_0 \\
&\leq \sum_{m=p}^{\infty} ||\sqrt{2}\tilde{\kappa}||^{2m}\tilde{\delta}_0 \leq \frac{||\sqrt{2}\tilde{\kappa}||^{2m}}{1-||\sqrt{2}\tilde{\kappa}||^2}\tilde{\delta}_0.
\end{aligned}$$

Since $||\tilde{\kappa}|| < \frac{1}{\sqrt{2}}$, we have

$$||\tilde{d_{c^*}}\left(\wp_{x_{p+n}},\wp_{x_p}\right) + \tilde{d_{c^*}}\left(\varpi_{x_{p+n}},\varpi_{x_p}\right)|| \leq \frac{||\sqrt{2}\tilde{\kappa}||^{2m}}{1-||\sqrt{2}\tilde{\kappa}||^2}\tilde{\delta}_0 \to 0.$$

Hence $\left\{\wp_{x_p}\right\}$ and $\left\{\varpi_{x_p}\right\}$ are Cauchy sequence in C^*-algebra valued fuzzy soft metric spaces $(\tilde{\mathscr{B}},\tilde{C},\tilde{d_{c^*}})$ and by the completeness of $(\tilde{\mathscr{B}},\tilde{C},\tilde{d_{c^*}})$, there exist $\tilde{u}',\tilde{v}' \in \mathscr{B}$ with

$$\lim_{p\to\infty}\wp_{x_{p+1}} = u_{x'} = \lim_{p\to\infty}\wp_{x_p} \quad \lim_{p\to\infty}\varpi_{x_{p+1}} = v_{x'} = \lim_{p\to\infty}\varpi_{x_p}$$

we have

$$\begin{aligned}
\tilde{d_{c^*}}(u_{x'},\mathscr{H}(u_{x'},v_{x'},s_{x'})) &= \lim_{p\to\infty}\tilde{d_{c^*}}\left(\mathscr{H}(\wp_{x_p},\varpi_{x_p},s_{x'}),\mathscr{H}(u_{x'},v_{x'},s_{x'})\right) \\
&\preceq \lim_{n\to\infty}\varphi\left(\tilde{\kappa}\tilde{d_{c^*}}(\wp_{x_p},u_{x'})\tilde{\kappa}^\star + \tilde{\kappa}\tilde{d_{c^*}}(\varpi_{x_p},v_{x'})\tilde{\kappa}^\star\right) \\
&\prec \lim_{n\to\infty}\left(\tilde{\kappa}\tilde{d_{c^*}}(\wp_{x_p},u_{x'})\tilde{\kappa}^\star + \tilde{\kappa}\tilde{d_{c^*}}(\varpi_{x_p},v_{x'})\tilde{\kappa}^\star\right) \\
&= \tilde{0}_{\tilde{C}}.
\end{aligned}$$

It follows that $\mathscr{H}(u_{x'}, v_{x'}, s_{x'}) = u_{x'}$. Similarly, we can prove $\mathscr{H}(v_{x'}, u_{x'}, s_{x'}) = v_{x'}$. Thus $s_{x'} \in \mathscr{B}$. Hence $\mathscr{B}$ is closed in $[0,1]$. Let $s_{x_0} \in \mathscr{B}$, then there exist $\wp_{x_0}, \varpi_{x_0} \in \Delta$ with $\wp_{x_0} = \mathscr{H}(\wp_{x_0}, \varpi_{x_0}, s_{x_0})$, $\varpi_{x_0} = \mathscr{H}(\varpi_{x_0}, \wp_{x_0}, s_{x_0})$. Since Δ is open, then there exist $\tilde{r} > 0$ such that $B_{d_{c^*}}(\wp_{x_0}, \tilde{r}) \subseteq \Delta$.

Choose $s_{x'} \in (s_{x_0} - \varepsilon, s_{x_0} + \varepsilon)$ such that $|s_{x'} - s_{x_0}| \leq \frac{1}{||\tilde{M}^p||} < \frac{\varepsilon}{2}$, then for $\wp_{x'} \in \overline{B_{\tilde{d_{c^*}}}(\wp_{x_0}, \tilde{r})} = \left\{\wp_{x'} \in \mathscr{B}/\tilde{d_{c^*}}(\wp_{x'}, \wp_{x_0}) \leq \frac{\tilde{r}}{2} + \tilde{d_{c^*}}(\wp_{x_0}, \wp_{x_0})\right\}$ and $\varpi_{x'} \in \overline{B_{\tilde{d_{c^*}}}(\varpi_{x_0}, \tilde{r})} = \left\{\varpi_{x'} \in \mathscr{B}/\tilde{d_{c^*}}(\varpi_{x'}, \varpi_{x_0}) \leq \frac{\tilde{r}}{2} + \tilde{d_{c^*}}(\varpi_{x_0}, \varpi_{x_0})\right\}$. Now we have

$$\begin{aligned}
&\tilde{d_{c^*}}(\mathscr{H}(\wp_{x'}, \varpi_{x'}, s_{x'}), \wp_{x_0}) = \tilde{d_{c^*}}(\mathscr{H}(\wp_{x'}, \varpi_{x'}, s_{x'}), \mathscr{H}_b(\wp_{x_0}, \varpi_{x_0}, s_{x_0})) \\
\preceq\ & \tilde{d_{c^*}}(\mathscr{H}(\wp_{x'}, \varpi_{x'}, s_{x'}), \mathscr{H}(\wp_{x'}, \varpi_{x'}, s_{x_0})) \\
& + \tilde{d_{c^*}}(\mathscr{H}(\wp_{x'}, \varpi_{x'}, s_{x_0}), \mathscr{H}(\wp_{x_0}, \varpi_{x_0}, s_{x_0})) \\
\preceq\ & ||\tilde{M}|||s_{x'} - s_{x_0}| + \tilde{d_{c^*}}(\mathscr{H}(\wp_{x'}, \varpi_{x'}, s_{x_0}), \mathscr{H}(\wp_{x_0}, \varpi_{x_0}, s_{x_0})) \\
\preceq\ & \frac{1}{||\tilde{M}^{p-1}||} + \tilde{d_{c^*}}(\mathscr{H}(\wp_{x'}, \varpi_{x'}, s_{x_0}), \mathscr{H}(\wp_{x_0}, \varpi_{x_0}, s_{x_0})).
\end{aligned}$$

Letting $p \to \infty$, we obtain

$$\begin{aligned}
\tilde{d_{c^*}}(\mathscr{H}(\wp_{x'}, \varpi_{x'}, s_{x'}), \wp_{x_0}) &\preceq \tilde{d_{c^*}}(\mathscr{H}(\wp_{x'}, \varpi_{x'}, s_{x_0}), \mathscr{H}(\wp_{x_0}, \varpi_{x_0}, s_{x_0})) \\
&\preceq \varphi\left(\tilde{\kappa}\tilde{d_{c^*}}(\wp_{x'}, \wp_{x_0})\tilde{\kappa}^* + \tilde{\kappa}\tilde{d_{c^*}}(\varpi_{x'}, \varpi_{x_0})\tilde{\kappa}^*\right) \\
&\prec \tilde{\kappa}\tilde{d_{c^*}}(\wp_{x'}, \wp_{x_0})\tilde{\kappa}^* + \tilde{\kappa}\tilde{d_{c^*}}(\varpi_{x'}, \varpi_{x_0})\tilde{\kappa}^* \qquad (11)
\end{aligned}$$

and

$$\tilde{d_{c^*}}(\mathscr{H}(\varpi_{x'}, \wp_{x'}, s_{x'}), \varpi_{x_0}) \prec \tilde{\kappa}\tilde{d_{c^*}}(\wp_{x'}, \wp_{x_0})\tilde{\kappa}^* + \tilde{\kappa}\tilde{d_{c^*}}(\varpi_{x'}, \varpi_{x_0})\tilde{\kappa}^* \qquad (12)$$

and now from (11) and (12), we have

$$\tilde{d_{c^*}}(\mathscr{H}(\wp_{x'}, \varpi_{x'}, s_{x'}), \wp_{x_0}) + \tilde{d_{c^*}}(\mathscr{H}(\varpi_{x'}, \wp_{x'}, s_{x'}), \varpi_{x_0}) \prec (\sqrt{2}\tilde{\kappa})\left(\tilde{d_{c^*}}(\wp_{x'}, \wp_{x_0}) + \tilde{d_{c^*}}(\wp_{x'}, \wp_{x_0})\right)(\sqrt{2}\tilde{\kappa})^*$$

then

$$\begin{aligned}
||\tilde{d_{c^*}}(\mathscr{H}(\wp_{x'}, \varpi_{x'}, s_{x'}), \wp_{x_0}) + \tilde{d_{c^*}}(\mathscr{H}(\varpi_{x'}, \wp_{x'}, s_{x'}), \varpi_{x_0})|| &< ||\sqrt{2}\tilde{\kappa}||^2||\tilde{d_{c^*}}(\wp_{x'}, \wp_{x_0}) + \tilde{d_{c^*}}(\wp_{x'}, \wp_{x_0})|| \\
&\leq \tilde{r} + ||\tilde{d_{c^*}}(\wp_{x_0}, \wp_{x_0})|| + ||\tilde{d_{c^*}}(\wp_{x_0}, \wp_{x_0})||.
\end{aligned}$$

Thus for each fixed $s_{x'} \in (s_{x_0}-\varepsilon, s_{x_0}+\varepsilon)$, $\mathscr{H}(.,s_{x'}) : \overline{B_{\tilde{d_{c^*}}}(\wp_{x_0},\tilde{r})} \to \overline{B_{\tilde{d_{c^*}}}(\wp_{x_0},\tilde{r})}$, $\mathscr{H}(.,s_{x'}) : \overline{B_{\tilde{d_{c^*}}}(\varpi_{x_0},\tilde{r})} \to \overline{B_{\tilde{d_{c^*}}}(\varpi_{x_0},\tilde{r})}$. Then all conditions of Theorem 3.1.1 are satisfied. Thus we conclude that $\mathscr{H}(.,s_{x'})$ has a coupled fixed point in $\overline{\Delta}^2$. But this must be in Δ^2. Since (τ_0) holds. Thus, $s_{x'} \in \mathscr{B}$ for any $s_{x'} \in (s_{x_0}-\varepsilon, s_{x_0}+\varepsilon)$. Hence $(s_{x_0}-\varepsilon, s_{x_0}+\varepsilon) \subseteq \mathscr{B}$. Clearly $\mathscr{B}$ is open in [0, 1]. For the reverse implication, we use the same strategy.

4. Conclusions

In the setting up of C^*-algebra valued fuzzy soft metric spaces, this work concludes several applications to homotopy theory via coupled fixed point theorems for (α,φ)-$\mathbb{K}$-contraction type mappings.

Conflict of Interests

The authors declare that there is no conflict of interests.

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