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b-METRIC SPACES AND THE RELATED APPROXIMATE BEST PROXIMITY PAIR RESULTS USING CONTRACTION MAPPINGS

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Abstract: The aim of this paper is to prove some new approximate best proximity pair theorems on b-metric spaces using contraction mappings, including P-Bianchini contraction, $P-B$ contraction, etc. A few examples are provided to exemplify our findings. Finally, we discuss some applications that are related to the main results.

Keywords: b-metric space; best approximate pair; $P-B$ contraction; P-Bianchini contraction; diameter best approximate pair.

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1. Introduction

Fixed point theory and operator theory are presently essential in many mathematics-related fields and applications, particularly in the fields of finance, astrophysics, dynamical systems, the logic of decisions, and parameter estimation. In 1922, Banach [6] proposed the renowned Banach contraction principle. After that, various authors extended these principle and gave many results using contraction mappings on metric spaces (see also, [12], [13], [16], [17], [18], [19], [25] & [34]). After that, many researchers found new approximate fixed point theorems on

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metric spaces that do not require completeness in both contraction and rational type contraction mappings (refer, [8], [9], [10], [14], [23], [27], [28], [29] & [30]). On the other hand, the best proximity point theory also has the same importance as fixed point theory. In the absence of exact best proximity points, approximate best proximity points may be used because the best proximity point results have overly strict limitations. There seem to be numerous problems in branches of mathematics that can be handled using the concept of best proximity pair theory. Nonetheless, experience demonstrates that for many instances, an approximate computation is more than acceptable; hence, having the best proximity pair is not always necessary, but having an almost-best proximity pair is essential. Another type of growing challenge that leads to this approximate occurs when the requirements that must be enforced to ensure the presence of the best proximity pairings for the major challenge at hand are much more stringent. In [20], the authors achieved some results on the optimum proximity pairs. In the same way, the authors Antony Eldred. A., et al [15], proved many results on proximity pairs. One can also refer to many results about proximity point of the pairs and their theorems in [7], [26], [31], [32], [33]. Moreover, B-contraction and Bianchini contraction definitions are located in [11] & [21], and using these, we define $P-B$ contraction and P-Bianchini contraction.

Meanwhile, the author Backhtin [1], demonstrated the notion of b-metric space in 1989. Especially the fixed point theorems in b-metric space was developed by the authors ([2], [4], [5]). Particularly, in 1993, Czerwik [3] introduced the notion of b-metric space with a view of generalizing the Banach contraction mapping theorem. After that, a lot of authors have worked in this directions and have presented some nice results related to the fixed point theory. Also, an extension of the Banach fixed point theorem in b-metric spaces to address various difficulties of the convergence of the measurable functions with regards to measure.

The rest of the paper is laid out as follows: Section 1 is a general introductory part. In Section 2, we recall the basics from the previous literature. In Section 3, we present the main results, which include the approximate best proximity pair results on b-metric spaces using various contraction mappings such as $P-B$ contraction, P-Bianchini contraction and so on. Mainly, we discuss the diameter of an approximate best proximity point for the pair (W,V) by using various contraction mappings based on the results of [22] and [24]. In Section 4, we present some application related to our main findings in the area of differential equations. Finally, in section 5, we reach a conclusion.

2. Preliminaries

In this section, some definitions and lemmas from earlier research are recalled. These are then employed throughout the remainder of the main findings of this manuscript.

Definition 2.1. [2] *Let M be a non-empty set and $b \geq 1$ be a given real number. A function $d : M \times M \longrightarrow \mathbb{R}_+$ is called a b-metric provided that for all $p, q, r \in M$ satisfies the following conditions.*

(i) $d(p,q) = 0$ *iff* $p = q$;

(ii) $d(p,q) = d(q,p)$;

(iii) $d(p,q) \leq b[d(p,r) + (r,q)]$

The pair (M,d) is called a b-metric space. Immediately from the notion of b-metric space we have the result every metric space is a b-metric space with $b = 1$. But the converse does not hold.

Example 2.2. [2] *Let* $M = \{0,1,2\}$ *and* $d(2,0) = d(0,2) = m \geq 2$
$d(0,1) = d(1,2) = d(1,0) = d(2,1) = 1$ *and*
$d(0,0) = d(1,1) = d(2,2) = 0$
Then, $d(p,q) \leq \frac{m}{2}[d(p,r) + d(r,q)]$ *for all* $p, q, r \in M$.
if $m > 2$ *then the triangle inequality does not hold.*

Definition 2.3. [22],[24] *Let W and V be two nonempty subsets of a metric space M and $B : W \cup V \to W \cup V$ such that $B(W) \subseteq V$ and $B(V) \subseteq W$. Then w is said to be an approximate best proximity point of the pair (W,V), if*

$$d_b(w, Bw) \leq d_b(W,V) + \varepsilon.$$

Remark 2.4. [22],[24] *Let*

$$P_{B\varepsilon}(W,V) = \{w \in (W,V) : d_b(w,Bw) < d_b(W,V) + \varepsilon, \text{ for some } \varepsilon > 0\}$$

be denotes the set of all approximate best proximity pairs of pair (W,V) for a given $\varepsilon > 0$. Also the pair (W,V) is said to be an approximate best proximity pair property, if

$$d_b(w,Bw) \leq d_b(W,V) \neq 0.$$

Example 2.5. *Let us take $M = \mathbb{R}^2$ and $W = \{(w,v) \in M : (w-v)^2 + v^2 \leq 1\}$ and $V = \{(w,v) \in M : (w+v)^2 + v^2 \leq 1\}$ with $B(w,v) = (-w,v)$ for $(w,v) \in M$. Then*

$$d_b((w,v),B(w,v)) \leq d_b(W,V) + \varepsilon \text{ for some } \varepsilon > 0.$$

Hence, $P_{B\varepsilon}(W,V) \neq \emptyset$.

Theorem 2.6. [22],[24] *Let W and V be two nonempty subsets of a metric space M. Suppose that the mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ and*

$$\lim_{n\to\infty} d_b(B^n W, B^{n+1} W) = d_b(W,V), \text{ for some} w \in (W \cup V).$$

Then the pair (W,V) is called an approximate best proximity pair.

Definition 2.7. [22],[24] *Let $B : W \cup V \to W \cup V$ be a continuous map such that $B(W) \subseteq V, B(V) \subseteq W$ and $\varepsilon > 0$. Then, we define the diameter $Dtr(P_{B\varepsilon}(W,V))$, i.e.,*

$$Dtr(P_{B\varepsilon}(W,V)) = sup\{d_b(w,v) : w,v \in P_{B\varepsilon}(W,V)\}.$$

Theorem 2.8. [22],[24] *Let W and V be two non-empty subsets of a metric space M. Suppose that a mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V, B(V) \subseteq W$ is a $P-\alpha$ contraction and $\varepsilon > 0$. Suppose that:*

- *(i) $P_{B\varepsilon}(W,V) \neq \emptyset$;*
- *(ii) for every $\varphi > 0$, there exists $\psi(\varphi) > 0$ such that $d_b(w,v) - d_b(Bw,Bv) \leq \varphi$ implies that $d_b(w,v) \leq \psi(\varphi)$, for every $w,v \in P_{B\varepsilon}(W,V) \neq \emptyset$.*

Then, $Dtr(P_{B\varepsilon}(W,V)) \leq \psi(2d_b(W,V) + \varepsilon)$.

Definition 2.9. *A mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a P-Chatterjea contraction operator if there exists $b_1 \in (0, \frac{1}{2})$ such that*

$$d_b(Bw,Bv) \leq b_1[d_b(w,Bv) + d_b(v,Bw)], \text{ for all } w,v \in W \cup V. \tag{2.1}$$

Definition 2.10. *A mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a $P-B$ contraction operator if there exists $b_1, b_2, b_3 \in (0,1)$ with $2b_1 + b_2 + 2b_3 < 1$ such that*

$$\begin{aligned} d_b(Bw,Bv) &\leq b_1[d_b(w,Bw) + d_b(v,Bv)] + b_2[d_b(w,v)] \\ &\quad + b_3[d_b(w,Bv) + d_b(v,Bw)], \text{ for all } w,v \in W \cup V. \end{aligned} \tag{2.2}$$

Definition 2.11. *A mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a P-Bianchini contraction operator if there exists $b_1 \in (0,1)$ such that*

$$\begin{aligned} &d_b(Bw,Bv) \leq b_1 B_{ia}(w,v), \\ &\text{where } B_{ia}(w,v) = max\{d_b(w,Bw), d_b(v,Bv)\}, \text{ for all } w,v \in W \cup V. \end{aligned} \tag{2.3}$$

Definition 2.12. *A mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a P-Hardy and Rogers contraction operator if there exists $b_1, b_2, b_3, b_4, b_5 \in (0,1)$ with $b_1 + b_2 + b_3 + b_4 + b_5 < 1$ such that*

$$\begin{aligned} d_b(Bw,Bv) &\leq b_1 d_b(w,v) + b_2 d_b(w,Bw) + b_3 d_b(v,Bv) \\ &\quad + b_4 d_b(w,Bv) + b_5 d_b(v,Bw), \text{ for all } w,v \in W \cup V. \end{aligned} \tag{2.4}$$

3. Main Results

In this section, we prove some approximate best proximity pair theorems on b-metric spaces using various contraction mappings including P-chatterjea contraction, $P-B$ contraction, P-Hardy Rogers contraction and etc. The proof of these theorems is split into two parts. The first one deals with qualitative results, and the other one deals with quantitative results; both are related to the approximate best proximity points for the pairs (V,W) on b-metric spaces.

Theorem 3.1. *Let W and V be two non-empty subsets of a b-metric space M with the coefficient $s \geq 1$. Suppose that a mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a $P-B$ Chatterjea contraction mapping with $sb_1(s+1) < 1$ then for all $\varepsilon > 0$,*

(i) $P_{B\varepsilon}(W,V) \neq \emptyset$; *and*
(ii) $\Delta(P_{B\varepsilon}(W,V)) \leq \dfrac{2b_1 s d_b(W,V) + 2\varepsilon(b_1 s + 1)}{1 - 2b_1 s}$.

Proof. Let $\varepsilon > 0$ and $w \in W \cup V$. Consider,

$$\begin{aligned} d_b(B^n w, B^{n+1} w) &= d_b(B(B^{n-1} w), B(B^n w)) \\ &\leq b_1[d_b(B^{n-1} w, B^{n+1} w) + d_b(B^n w, B^n w)] \\ &= sb_1 d_b(B^{n-1} w, B^n w) + sb_1 d_b(B^n w, B^{n+1} w) \end{aligned}$$

That is,

$$\begin{aligned} d_b(B^n w, B^{n+1} w) &\leq \left(\frac{sb_1}{1-sb_1}\right) d_b(B^{n-1} w, B^n w) \\ &= \lambda d_b(B^{n-1} w, B^n w), \text{ where } \lambda = \frac{sb_1}{1-sb_1} \\ &\leq \lambda^2 d_b(B^{n-2} w, B^{n-1} w) \\ &\vdots \\ &\leq \lambda^n d_b(w, Bw) \end{aligned}$$

But $b_1 \in (0, \frac{1}{2}]$ implies that $\lambda \in (0,1)$. Therefore,

$$\lim_{n\to\infty} d_b(B^n w, B^{n+1} w) = 0, \text{ for all } w \in W \cup V.$$

Hence, by Theorem 2.6, it follows that

$$P_{B\varepsilon}(W,V) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

Clearly, condition (i) is proved. For proving condition (ii), take $\varphi > 0$ and $w, v \in P_{B\varepsilon}(W,V)$. Also, by Theorem 2.6, $d_b(w,v) - d_b(Bw,Bv) \leq \varphi$. Then $d_b(w,v) \leq d_b(Bw,Bv) + \varphi$. Since $w, v \in P_{B\varepsilon}(W,V)$ implies that $d_b(w,Bw) \leq d_b(W,V) + \varepsilon_1$ and $d_b(v,Bv) \leq d_b(W,V) + \varepsilon_2$. Choose $\varepsilon = max\{\varepsilon_1, \varepsilon_2\}$. Now,

$$\begin{aligned} d_b(w,v) &\leq d_b(Bw,Bv) + \varphi \\ &= b_1 d_b(w,Bv) + b_1 d_b(v,Bw) + \varphi \\ &= b_1 s d_b(w,v) + b_1 s d_b(v,Bv) + b_1 s d_b(v,w) + b_1 s d_b(w,Bw) + \varphi \\ &= 2b_1 s d_b(w,v) + 2b_1 s d_b(W,V) + 2b_1 s\varepsilon + \varphi \\ &= \frac{2b_1 s d_b(W,V) + 2b_1 s\varepsilon + \varphi}{1-2b_1 s} \end{aligned}$$

$$= \psi(\varphi)$$

Thus, for every $\varphi > 0$, there exists $\psi(\varphi) > 0$ such that $d_b(w,v) - d_b(Bw,Bv) \leq \varphi$ implies $d_b(w,v) = \psi(\varphi)$. Then the Theorem 2.8 gives,

$$\Delta(P_{B\varepsilon}(W,V)) \leq \psi(2\varepsilon), \text{ for all } \varepsilon > 0.$$

Which means exactly that

$$\Delta(P_{B\varepsilon}(W,V)) \leq \frac{2b_1 s d_b(W,V) + 2b_1 s\varepsilon + 2\varepsilon}{1 - 2b_1 s}$$

Hence,

$$\Delta(P_{B\varepsilon}(W,V)) \leq \frac{2b_1 s d_b(W,V) + 2\varepsilon(b_1 s + 1)}{1 - 2b_1 s}, \text{ for all } \varepsilon > 0.$$

□

Theorem 3.2. *Let W and V be two non-empty subsets of a b-metric space M with the coefficient $s \geq 1$. Suppose that a mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a $P - B$ contraction mapping with $b_1(s+1) + b_2 s + b_3 s(s+1) < 1$ then for all $\varepsilon > 0$,*

(i) $P_{B\varepsilon}(W,V) \neq \emptyset$*; and*
(ii) $\Delta(P_{B\varepsilon}(W,V)) \leq \dfrac{2(b_1 + sb_3)d(W,V) + 2\varepsilon(b_1 + sb_3 + 1)}{1 - b_2 - 2sb_3}$.

Proof. Let $\varepsilon > 0$ and $w \in W \cup V$. Consider,

$$\begin{aligned}
d_b(B^n w, B^{n+1} w) &= d_b(B(B^{n-1} w), B(B^n w)) \\
&\leq b_1[d_b(B^{n-1} w, B^n w) + d_b(B^n w, B^{n+1} w)] + b_2[d_b(B^{n-1} w, B^n w)] \\
&\quad + b_3[d_b(B^{n-1} w, B^{n+1} w) + d_b(B^n w, B^n w)] \\
&= b_1 d_b(B^{n-1} w, B^n w) + b_1 d_b(B^n w, B^{n+1} w) + b_2 d_b(B^{n-1} w, B^n w) \\
&\quad + sb_3 d_b(B^{n-1} w, B^n w) + sb_3 d_b(B^n w, B^{n+1} w)
\end{aligned}$$

That is,

$$\begin{aligned}
d_b(B^n w, B^{n+1} w) &= \left(\frac{b_1 + b_2 + sb_3}{1 - b_1 - sb_3}\right) d_b(B^{n-1} w, B^n w) \\
&= \lambda d_b(B^{n-1} w, B^n w), \text{ where } \lambda = \frac{b_1 + b_2 + sb_3}{1 - b_1 - sb_3}
\end{aligned}$$

$$\leq \lambda^2 d_b(B^{n-2}w, B^{n-1}w)$$

$$\vdots$$

$$\leq \lambda^n d_b(w, Bw)$$

But b_1, b_2 and $b_3 \in (0,1)$ with $2b_1 + b_2 + 2b_3 < 1$ implies that $\lambda \in (0,1)$. Therefore,

$$\lim_{n\to\infty} d_b(B^n w, B^{n+1}w) = 0, \text{ for all } w \in W \cup V.$$

Hence, by Theorem 2.6, it follows that

$$P_{B\varepsilon}(W,V) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

Clearly, condition (i) is proved. For proving condition (ii), take $\varphi > 0$ and $w, v \in P_{B\varepsilon}(W,V)$. Also, by Theorem 2.8, $d_b(w,v) - d_b(Bw,Bv) \leq \varphi$. Then $d_b(w,v) \leq d_b(Bw,Bv) + \varphi$. Since $w, v \in P_{B\varepsilon}(W,V)$ implies that $d_b(w,Bw) \leq d_b(W,V) + \varepsilon_1$ and $d_b(v,Bv) \leq d_b(W,V) + \varepsilon_2$. Choose $\varepsilon = max\{\varepsilon_1, \varepsilon_2\}$. Now,

$$\begin{aligned}
d_b(w,v) &\leq d_b(Bw,Bv) + \varphi \\
&\leq b_1[d_b(W,V) + \varepsilon + d_b(W,V) + \varepsilon] + b_2[d_b(w,v)] \\
&\quad + b_3[sd_b(w,v) + sd_b(v,Bv) + sd_b(v,w) + sd_b(w,Bw)] + \varphi \\
&= b_1[2d_b(W,V) + 2\varepsilon] + b_2 d_b(w,v) + b_3[2sd_b(w,v) + 2sd_b(W,V) + 2s\varepsilon] + \varphi \\
&= \frac{2(b_1 + sb_3)d_b(W,V) + 2\varepsilon(b_1 + sb_3) + \varphi}{1 - b_2 - 2sb_3} \\
&= \psi(\varphi)
\end{aligned}$$

Thus, for every $\varphi > 0$, there exists $\psi(\varphi) > 0$ such that $d_b(w,v) - d_b(Bw,Bv) \leq \varphi$ implies $d_b(w,v) = \psi(\varphi)$. Then the Theorem 2.8 gives,

$$\Delta(P_{B\varepsilon}(W,V)) \leq \psi(2\varepsilon), \text{ for all } \varepsilon > 0.$$

Which means exactly that

$$\Delta(P_{B\varepsilon}(W,V)) \leq \frac{2(b_1 + sb_3)d_b(W,V) + 2\varepsilon(b_1 + sb_3) + 2\varepsilon}{1 - b_2 - 2sb_3}$$

Hence,

$$\Delta(P_{B\varepsilon}(W,V)) \leq \frac{2(b_1+sb_3)d_b(W,V)+2\varepsilon(b_1+sb_3+1)}{1-b_2-2sb_3}, \text{ for all } \varepsilon > 0.$$

□

Theorem 3.3. *Let W and V be two non-empty subsets of a b-metric space M with the coefficient $s \geq 1$. Suppose that a mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a P-Bianchini contraction mapping with $sb_1 < 1$ then for every $\varepsilon > 0$,*

(i) $P_{B\varepsilon}(W,V) \neq \emptyset$; *and*

(ii) $\Delta(P_{B\varepsilon}(W,V)) \leq b_1 d_b(W,V) + \varepsilon(b_1+2)$.

Proof. Let $\varepsilon > 0$ and $w \in W \cup V$. Consider,

Case 1. *Suppose $B_{ia}(w,v) = d_b(w,Bw)$. Then the Definition 2.11 becomes*

$$d_b(Bw,Bv) \leq b_1 d_b(w,Bw)$$

Substitute $v = Bw$ we get,

$$d_b(Bw,B^2w) \leq b_1 d_b(w,Bw)$$

Again substituting $w = Bw$ implies,

$$\begin{aligned} d_b(B^2w,B^3w) &\leq b_1 d_b(Bw,B^2w) \\ &\leq (b_1)^2 d_b(w,Bw) \end{aligned}$$

Continuing this process we have,

$$d_b(B^n w,B^{n+1}w) \leq (b_1)^n d_b(w,Bw)$$

Case 2. *Suppose $B_{ia}(w,v) = d_b(v,Bv)$. Then the Definition 2.11 becomes*

$$d_b(Bw,Bv) \leq b_1 d_b(v,Bv)$$

Substitute $v = Bw$, we get

$$d_b(Bw,B^2w) \leq b_1 d_b(w,B^2w)$$

This is impossible because $b_1 \in (0,1)$. Therefore, **Case 2** does not exist. Now using **Case 1** and Theorem 2.6, we have

$$\lim_{n\to\infty} d_b(B^n w, B^{n+1} w) = 0, \text{ for all } w \in W \cup V.$$

And it follows that

$$P_{B\varepsilon}(W,V) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

Clearly, condition (i) is proved. For proving condition (ii), take $\varphi > 0$ and $w,v \in P_{B\varepsilon}(W,V)$. Also, by Theorem 2.8, $d_b(w,v) - d_b(Bw,Bv) \leq \varphi$. Then $d_b(w,v) \leq d_b(Bw,Bv) + \varphi$. Since $w,v \in P_{B\varepsilon}(W,V)$ implies that $d_b(w,Bw) \leq d_b(W,V) + \varepsilon$. Now,

$$\begin{aligned}
d_b(w,v) &\leq d_b(Bw,Bv) + \varphi \\
&\leq b_1 d_b(w,Bw) + \varphi \\
&\leq b_1(d_b(W,V) + \varepsilon) + \varphi \\
&= \psi(\varphi)
\end{aligned}$$

Thus, for every $\varphi > 0$, there exists $\psi(\varphi) > 0$ such that $d_b(w,v) - d_b(Bw,Bv) \leq \varphi$ implies $d_b(w,v) = \psi(\varphi)$. Then the Theorem 2.8 gives,

$$\Delta(P_{B\varepsilon}(W,V)) \leq \psi(2\varepsilon), \text{ for all } \varepsilon > 0.$$

This means exactly

$$\Delta(P_{B\varepsilon}(W,V)) \leq b_1 d_b(W,V) + \varepsilon(b_1 + 2), \text{ for all } \varepsilon > 0.$$

□

Corollary 3.4. *Let W and V be two non-empty subsets of a b-metric space M with the coefficient $s \geq 1$. Suppose that a mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ and defined by $d_b(Bw,Bv) \leq b_1 d_b(w,Bw)$ operator then for every $\varepsilon > 0, P_{B\varepsilon}(W,V) \neq \emptyset$ and $\Delta(P_{B\varepsilon}(W,V)) \leq b_1 d_b(W,V) + \varepsilon(b_1 + 2)$.*

Proof. Proof is trivial when one can follow the above Theorem 3.3. □

Theorem 3.5. *Let W and V be two non-empty subsets of a b-metric space M with the coefficient $s \geq 1$. Suppose that a mapping $B : W \cup V \to W \cup V$ satisfying $B(W) \subseteq V$ and $B(V) \subseteq W$ is a P-Hardy Rogers contraction mapping with $s(b_1 + b_2) + b_3 + sb_4(s+1) < 1$ then for every $\varepsilon > 0$,*

(i) $P_{B\varepsilon}(W,V) \neq \emptyset$; *and*
(ii) $\Delta(P_{B\varepsilon}(W,V)) \leq \dfrac{(b_2 + b_3 + sb_4 + sb_5)d_b(W,V) + (b_2 + b_3 + sb_4 + sb_5 + 2)\varepsilon}{1 - b_1 - sb_4 - sb_5}$, $\forall\, \varepsilon > 0$.

Proof. Let $\varepsilon > 0$ and $w \in W \cup V$. Consider,

$$\begin{aligned}
d_b(B^n w, B^{n+1} w) &= d_b(B(B^{n-1}w), B(B^n w)) \\
&\leq b_1 d_b(B^{n-1}w, B^n w) + b_2 d_b(B^{n-1}w, B^n w) + b_3 d_b(B^n w, B^{n+1} w) \\
&\quad + b_4 d_b(B^{n-1}w, B^{n+1}w) + b_5 d_b(B^n w, B^n w) \\
&= b_1 d_b(B^{n-1}w, B^n w) + b_2 d_b(B^{n-1}w, B^n w) + b_3 d_b(B^n w, B^{n+1} w) \\
&\quad + b_4 d_b(B^{n-1}w, B^n w) + b_4 d_b(B^n w, B^{n+1} w) \\
&= \left(\frac{b_1 + b_2 + sb_4}{1 - b_3 - sb_4}\right) d_b(B^{n-1}w, B^n w) \\
&= \lambda d_b(B^{n-1}w, B^n w), \text{ where } \lambda = \left(\frac{b_1 + b_2 + sb_4}{1 - b_3 - sb_4}\right)
\end{aligned}$$

But b_1, b_2, b_3, b_4 and $b_5 \in (0,1)$ implies that $\lambda \in (0,1)$. Therefore,

$$\lim_{n\to\infty} d_b(B^n w, B^{n+1} w) = 0, \text{ for all } w \in W \cup V.$$

Hence, by Theorem 2.6, it follows that

$$P_{B\varepsilon}(W,V) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

Clearly, condition (i) is proved. For proving condition (ii), take $\varphi > 0$ and $w, v \in P_{B\varepsilon}(W,V)$. Also, by Theorem 2.8, $d_b(w,v) - d_b(Bw, Bv) \leq \varphi$. Then $d_b(w,v) \leq d_b(Bw, Bv) + \varphi$. Since $w, v \in P_{B\varepsilon}(W,V)$ implies that $d_b(w, Bw) \leq d_b(W,V) + \varepsilon_1$ and $d_b(v, Bv) \leq d_b(W,V) + \varepsilon_2$. Choose $\varepsilon = max\{\varepsilon_1, \varepsilon_2\}$. Now,

$$\begin{aligned}
d_b(w,v) &\leq d_b(Bw, Bv) + \varphi \\
&\leq b_1 d_b(w,v) + b_2[d_b(W,V) + \varepsilon] + b_3[d_b(W,V) + \varepsilon] + b_4 s d_b(w,v) \\
&\quad + b_4 s[d_b(W,V) + \varepsilon] + b_5 s d_b(w,v) + sb_5[d_b(W,V) + \varepsilon] + \varphi
\end{aligned}$$

$$= (b_1 + sb_4 + sb_5)d_b(w,v) + (b_2 + b_3 + sb_4 + sb_5)[d_b(W,V) + \varepsilon] + \varphi$$
$$= \frac{(b_2 + b_3 + sb_4 + sb_5)[d_b(W,V) + \varepsilon] + \varphi}{1 - (b_1 + sb_4 + sb_5)}$$
$$= \psi(\varphi)$$

Thus, for every $\varphi > 0$, there exists $\psi(\varphi) > 0$ such that $d_b(w,v) - d_b(Bw,Bv) \leq \varphi$ implies $d_b(w,v) = \psi(\varphi)$. Then the Theorem 2.8 gives,

$$\Delta(P_{B\varepsilon}(W,V)) \leq \psi(2\varepsilon), \text{ for all } \varepsilon > 0.$$

Which means exactly that

$$\Delta(P_{B\varepsilon}(W,V)) \leq \frac{(b_2 + b_3 + sb_4 + sb_5)d_b(W,V) + (b_1 + sb_3) + 2\varepsilon}{1 - b_2 - 2sb_3}$$

Hence,

$$\Delta(P_{B\varepsilon}(W,V)) \leq \frac{(b_2 + b_3 + sb_4 + sb_5)d_b(W,V) + (b_2 + b_3 + sb_4 + sb_5 + 2)\varepsilon}{1 - b_1 - sb_4 - sb_5}, \text{ for all } \varepsilon > 0.$$

□

Remark 3.6. *(1) In Definition 2.10, substitute $b_2 = \alpha$ and $b_1 = b_3 = 0$, then it becomes $P - \alpha$ contraction operator and for every $\varepsilon > 0, P_{B\varepsilon}(W,V) \neq \emptyset$ and*

$$\Delta(P_{B\varepsilon}(W,V)) \leq \frac{2(\varepsilon + d_b(W,V))}{b_2}.$$

(2) In Definition 2.10, substitute $b_2 = b_3 = 0$, then it becomes P-Kannan operator and for every $\varepsilon > 0, P_{B\varepsilon}(W,V) \neq \emptyset$ and

$$\Delta(P_{B\varepsilon}(W,V)) \leq 2\varepsilon(1 + b_1) + 2b_1 d_b(W,V).$$

(3) In Definition 2.12, substitute $b_4 = b_5 = 0$, then it becomes P-Reich operator and for every $\varepsilon > 0, P_{B\varepsilon}(W,V) \neq \emptyset$ and

$$\Delta(P_{B\varepsilon}(W,V)) \leq \frac{(b_2 + b_3)d_b(W,V) + (b_2 + b_3 + 2)\varepsilon}{1 - b_1}.$$

(4) In Definition 2.12, substitute $b_4 = b_5$, then it becomes P-Ciric operator and for every $\varepsilon > 0, P_{B\varepsilon}(W,V) \neq \emptyset$ and

$$\Delta(P_{B\varepsilon}(W,V)) \leq \frac{(b_2 + b_3 + 2sb_4)d_b(W,V)) + (b_2 + b_3 + 2sb_4 + 2)\varepsilon}{1 - b_1 - 2sb_4}.$$

(5) In P-Mohseni-saheli operator, for every $\varepsilon > 0, P_{B\varepsilon}(W,V) \neq \emptyset$ *and*

$$\Delta(P_{B\varepsilon}(W,V)) \leq \frac{2b_1 s d_b(W,V)) + (b_1 s + 1)2\varepsilon}{1 - b_1 - sb_1}.$$

4. APPLICATIONS

Approximate best proximity point theory has many applications in mathematical fields especially in differential equations. The following examples shows that it has the applications in Green's functions in differential equations.

Example 4.1. *Consider* $z''(w) = \frac{3v^2(w)}{2}, 0 \leq v \leq 1$ *subect to* $z(0) = 4, z(1) = 1$. *Exact solution is* $z(w) = \frac{4}{(1+w)^2}$. *Consider a mapping* $W : [0,1] \to [0,1]$ *by*

$$W(z) = z(v) + \int_0^1 G(v,w)[z''(w) - \phi(w,z(w))]dw \tag{4.1}$$

Consider, $z''(v) = 0$ *which implies*

$$z(v) = c_1 v + c_2 \tag{4.2}$$

By initial condition we have $c_2 = 4$ *and* $c_1 = -3$. *Then (4.2) becomes* $z(v) = -3w_1 + 4$.

$$\begin{aligned} W(z) &= -3v + 4 + \int_0^1 G(v,w)[z''(w) - \phi(w,z(w))]dw \\ &= -3v + 4 + \int_0^1 G(v,w)z''(w)dw - \int_0^1 G(v,w)\phi(w,z(w))dw \\ &= -3v + 4 + \int_0^1 G(v,w)\frac{3}{2}z^2(w)dw \end{aligned}$$

Consider,

$$\begin{aligned} |W(z_1) - W(z_2)| &= \left| -\int_0^1 G(v,w)\frac{3}{2}z_2^2(w)dw + \int_0^1 G(v,w)\frac{3}{2}z_2^2(w)dw \right| \\ &= \frac{3}{2}\left| \int_0^1 G(v,w)[z_2^2(w) - z_1^2(w)]dw \right| \\ &\leq \frac{3}{2}\left(\int_0^1 |G(v,w)|^2 dw \right)^{\frac{1}{2}} \left[\int_0^1 |z_2^2(w) - z_1^2(w)|^2 dw \right]^{\frac{1}{2}} \\ &\leq \frac{3}{2}\left(\int_0^w w^2(1-v)^2 dv + \int_v^1 v^2(1-w)^2 dw \right)^{\frac{1}{2}} \left[\int_0^1 |z_2^2(w) - z_1^2(w)|^2 dw \right]^{\frac{1}{2}} \\ &\leq \frac{3}{2}\left\{ \frac{(1-v)^2 v^3}{3} + \frac{v^2(1-v)^3}{3} \right\}^{\frac{1}{2}} \left[\int_0^1 |z_2^2(w) - z_1^2(w)|^2 dw \right]^{\frac{1}{2}} \end{aligned}$$

$$\leq \frac{3}{2}\left\{\frac{(1-v)^2}{3}[v^3+v^2(1-v)]\right\}^{\frac{1}{2}}\left[\int_0^1 |z_2^2(w)-z_1^2(w)|^2 dw\right]^{\frac{1}{2}}$$

$$\leq \frac{3}{2}\left\{\frac{(1-v)^2v^2}{3}\right\}^{\frac{1}{2}}\left[\int_0^1 |z_2^2(w)-z_1^2(w)|^2 dw\right]^{\frac{1}{2}}$$

$$\leq \frac{3}{8\sqrt{3}}\left[\int_0^1 |z_2^2(w)-z_1^2(w)|^2 dw\right]^{\frac{1}{2}}$$

$$\leq \frac{\sqrt{3}}{8}\left[\int_0^1 |z_2^2(w)-z_1^2(w)|^2 dw\right]^{\frac{1}{2}}$$

$$\leq \frac{\sqrt{3}}{8}\sup_{[0,1]}|z_2(w)-z_1(w)|$$

$$\leq \sup_{[0,1]}|z_2(w)-z_1(w)|$$

Hence, W is contraction, it has approximate best proximity point.

5. Conclusion

This paper has introduced some new approximations for best proximity pairs theorems that are applicable to contraction operators in a b-metric space. This paper finds out that as the parameter ε approaches zero, the results bring about a set of strict restrictions on the estimated diameters approximate best proximity points. Finding approximate best proximity point of the pairs may be even more critical than locating exact ones; it is just as essential, if not more so.

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Author Contributions

All authors contributed equally, read and approved the final manuscript.

Conflict of Interests

The authors declare that there is no conflict of interests.

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