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APPROXIMATING ZEROS OF MONOTONE OPERATORS IN BANACH SPACES

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Abstract. In this study, we present a novel iterative algorithm that aims to identify the zeros of the sum of two monotone operators. Our algorithm demonstrates strong convergence in a Banach space. Additionally, we provide a few examples of the practical implications of our findings. Finally, we discuss numerical examples and demonstrate the convergence behavior of the proposed algorithm.

Keywords: monotone mapping; strong convergence; zeros; Banach space.

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1. Introduction

The following inclusion problem is an important problem in nonlinear analysis.

$$\text{find } u \in K \text{ such that } 0 \in (R+S)(u), \tag{1.1}$$

where K is a Banach space and $R,S : K \to K$ are self-mappings. The solution set of the aforementioned problem is represented by the notation by $(R+S)^{-1}(u) = \{u \in K : 0 \in (R+S)(u)\}$.

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This type of inclusion problems arise in convex minimization, variational inequalities, split feasibility and elsewhere with a number of applications to image and signal processing, machine learning and statistical regression [12, 19, 6, 5]. Rockafellar [14] and Brézis and Lions [2] investigated convergence of the resolvent iteration for a monotone operator. Passty [10] extended the resolvent iteration by involving more than one maximal monotone operator and introduced an important algorithm in a Hilbert space. He proved that the generated sequence converges weakly to a zero of the sum of two maximal monotone operators. This method converges if the inverse of the forward mapping is strongly monotone. To weaken or get rid of this restrictions on mappings, Tseng [20] presented a modified forward backward splitting method for maximal monotone mappings and obtained some weak convergence results in Hilbert spaces. In 1996, Alber [1] presented some strong results to find zeros of monotone mappings in Banach spaces. Subsequently, many researchers obtained certain results to find zeros for monotone operators [13, 15, 9, 24]. Recently, Chidume [3] used normalized duality mapping and presented a new algorithm in uniformly convex and uniformly smooth Banach spaces. He obtained some strong convergence results in under certain conditions.

This paper introduces a novel iterative algorithm designed for the purpose of locating zeros of the sum of two monotone operators in Banach spaces. Additionally, we provide a few instances where our findings can be applied. In conclusion, an example is presented to analyse the convergence characteristics of the proposed algorithm when various initial guesses and coefficients are employed.

2. Preliminaries

Definition 2.1. [8]. A Banach space K is said to be uniformly convex if for each $\varepsilon \in (0,2]$ $\exists\ \delta > 0$ such that $\left\|\frac{u+v}{2}\right\| \leq 1-\delta$ for all $u,v \in K$ with $\|u\| = \|v\| = 1$ and $\|u-v\| > \varepsilon$. The Banach space K is said to be strictly convex if

$$\left\|\frac{u+v}{2}\right\| < 1,$$

whenever $u,v \in K$ with $\|u\| = \|v\| = 1$, $u \neq v$.

Definition 2.2. [15]. The normalized duality mapping J from K into K^* is defined as

$$J(u) = \{u^* \in K^* : \langle u, u^* \rangle = \|u\|^2 = \|u^*\|^2\}, \ \forall\, u \in K.$$

Definition 2.3. [4]. The modulus of smoothness of a Banach space K is defined by

$$\rho_K(t) = \sup\left\{ \frac{\|u+tv\| + \|u-tv\|}{2} - 1 : \|u\| = \|v\| = 1 \right\}, t \geq 0.$$

The Banach space K is said to be uniformly smooth if $\lim\limits_{t\to 0} \frac{\rho_K(t)}{t} = 0$. The Banach space K is uniformly smooth if and only if J is single valued and uniformly continuous on each bounded subset of K.

Let K be a smooth Banach space and consider the following function studied in Alber [1] and Kamimura and Takahashi [7] $\Psi : K \times K \to \mathbb{R}$ as

$$\Psi(u,v) = \|u\|^2 - 2\langle u, J(v)\rangle + \|v\|^2, \text{ for each } u,v \in K.$$

Using the definition of the function Ψ we can also have

$$(\|u\| - \|v\|)^2 \leq \Psi(u,v) \leq (\|u\| + \|v\|)^2, \text{ for each } u,v \in K.$$

We also know that [7]

$$\Psi(u,v) = \Psi(u,w) + \Psi(w,v) - 2\langle u-w, J(w) - J(v)\rangle, \text{ for each } u,v \in K.$$

Suppose $\Gamma : K \times K^* \to \mathbb{R}$ be mapping defined by

$$\Gamma(u,u^*) = \|u\|^2 - 2\langle u, u^*\rangle + \|u^*\|^2, \text{ for each } u \in K, u^* \in K^*.$$

for each $u \in K$, $u^* \in K^*$. Then

$$\Gamma(u,u^*) = \Psi(u, J^{-1}(u^*)), \text{ for each } u \in K, u^* \in K^*.$$

Definition 2.4. A mapping $S : K \to K^*$ is said to be monotone if for each $u,v \in K$

$$\langle u-v, S(u) - S(v)\rangle \geq 0.$$

The set of zeros of the mapping S is defined by

$$S^{-1}(0) = \{u \in K : 0 \in S(u)\}.$$

Definition 2.5. [1]. A continuous strictly increasing function $\Delta : (0,\infty) \to (0,\infty)$ is said to be modulus of continuity if $\Delta(u) \to 0$ as $u \to 0$.

A function is said to be uniformly continuous iff it has modulus of continuity [1]. So, we can say that J^{-1} has modulus of continuity if the Banach space K is strictly convex and reflexive. We denote the modulus of J^{-1} by Δ satisfying

$$\|J^{-1}(u) - J^{-1}(v)\| \leq \Delta(\|u-v\|).$$

Lemma 2.6. [15]. *Let K be a smooth Banach space and $J : K \to K^*$ a duality mapping on K. Then $\langle u-v, J(u)-J(v)\rangle \geq 0$ for all $u,v \in K$. Further, if K is strictly convex and $\langle u-v, J(u)-J(v)\rangle = 0$, then $u = v$.*

Lemma 2.7. [1]. *Let K be a reflexive strictly convex smooth Banach space and K^* be it's dual. Then*

$$\Gamma(u,u^*) + 2\langle J^{-1}(u^* - u, v^*)\rangle \leq \Gamma(u, u^* + v^*)$$

for all $u \in K$ and $u^*, v^* \in K^*$.

Lemma 2.8. [21, 23]. *Let K be a 2-uniformly smooth Banach space. Then there exists a constant $k > 0$ such that*

$$\|u-v\|^2 \leq \|u\|^2 - 2\langle v, J(u)\rangle + k\|v\|^2$$

for each $u,v \in K$.

Lemma 2.9. [7]. *Let K be a smooth and uniformly convex Banach space and let $\{u_n\}$ and $\{v_n\}$ be sequences in K such that either $\{u_n\}$ or $\{v_n\}$ is bounded. If $\lim_{n\to\infty} \Psi(u_n, v_n) = 0$, then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.*

Lemma 2.10. [16]. *Assume $\{\tau_n\}$ be a sequence of non negative real numbers satisfying*

$$\tau_{n+1} \leq \tau_n + \xi_n,$$

for all $n \geq 0$. If $\sum_{n=1}^{\infty} |\xi_n| < \infty$, then $\lim_{n\to\infty} \tau_n$ exists.

Lemma 2.11. [22, 11]. *Assume $\{\tau_n\}$ be a sequence of non negative real numbers satisfying*

$$\tau_{n+1} \leq (1-\sigma_n)\tau_n + \xi_n + \eta_n,$$

for all $n \geq 0$, where $\{\sigma_n\}$ is a subsequence in $(0,1)$, $\{\xi_n\}$ and $\{\eta_n\}$ are sequences in $\mathbb{R}$. Suppose that:

(1) $\sum_{n=1}^{\infty} \sigma_n = \infty$,
(2) $\sum_{n=1}^{\infty} |\xi_n| < \infty$ or $\limsup_{n\to\infty} \frac{\xi_n}{\sigma_n} \leq 0$,
(3) $\sum_{n=1}^{\infty} \eta_n < \infty$.

Then, $\lim_{n\to\infty} \tau_n = 0$.

Lemma 2.12. [17] *The function $\Psi : K \times K \to \mathbb{R}$ satisfies the following inequality*

$$\Psi(u,(1-t)v) = (1-t)\Psi(u,v) + t\|u\|^2 - t(1-t)\|v\|^2$$

for all $t \in (0,1)$ and $u, v \in K$.

3. Algorithm and Convergence Result

In this section, we introduce a new algorithm to solve the inclusion problem (1.1). Throughout $\{\zeta_n\}$ is a real sequence in $(0,1)$ satisfying the following conditions:

(1) $\sum_{n=1}^{\infty} \zeta_n \Delta(\zeta_n C)C < \infty$, $C = \sup\{\|R(u)\| : u \in K\}$.
(2) $\sum_{n=1}^{\infty} \zeta_n \Delta(\zeta_n D)D < \infty$, $D = \sup\{\|S(u)\| : u \in K\}$.
(3) $\lim_{n\to\infty} \frac{\zeta_n}{\zeta_{n+1}} = d$ (constant).
(4) $\lim_{n\to\infty} \zeta_n = 0$, $\sum_{n=1}^{\infty} \zeta_n = \infty$.
(5) For any constant $k > 0$, $\sum_{n=1}^{\infty} k\zeta_n^2 C^2 < \infty$.

Algorithm 1. *Let $u_0 \in K$ be given. Define the sequence $\{u_n\}$ as follows:*

$$v_{n+1} = J^{-1}(J(u_n) - \zeta_n R(u_n)),$$

$$w_{n+1} = J^{-1}(J(v_{n+1}) - \zeta_{n+1} S(v_{n+1})),$$

$$u_{n+1} = J^{-1}(J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1})))$$

for all $n \in \mathbb{N} \cup \{0\}$.

Theorem 3.1. *Suppose K be a strictly convex and 2-uniformly smooth real Banach space with the dual K^*. Let $R,S : K \to K^*$ be monotone and bounded mappings with $(R+S)^{-1}(0) \neq \emptyset$. Let us consider the parameter $\{\zeta_n\}$ satisfies the conditions $(1)-(5)$, then the sequence $\{u_n\}$ defined by Algorithm (1), converges strongly to u^*, a solution of $(R+S)(u) = 0$ if $u^* \in R^{-1}(0)$ and $u^* \in S^{-1}(0)$.*

Proof. Since the mappings $R,S : K \to K^*$ are bounded, suppose

$$C = \sup\{\|S(u)\| : u \in K\}, \text{ and } D = \sup\{\|R(u)\| : u \in K\}.$$

First, we prove that the sequence $\{u_n\}$ is bounded. Since $(R+S)^{-1}(0) \neq \emptyset$, let $u^* \in (R+S)^{-1}(0)$, such that $u^* \in R^{-1}(0)$ and $u^* \in S^{-1}(0)$

$$\begin{aligned}
\Psi(u^*, u_{n+1}) &= \Psi(u^*, J^{-1}(J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1})))) \\
&= \Gamma(u^*, J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1}))) \\
&= \|u^*\|^2 - 2\langle u^*, J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1}))\rangle \\
&\quad + \|J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1}))\|^2 \\
&\leq \|u^*\|^2 - 2\langle u^*, J(w_{n+1})\rangle + 2\zeta_{n+1}\langle u^*, S(w_{n+1}) - S(v_{n+1})\rangle \\
&\quad + \|J(w_{n+1})\|^2 - 2\zeta_{n+1}\langle w_{n+1}, S(w_{n+1}) - S(v_{n+1})\rangle + k\|\zeta_{n+1}(S(w_{n+1}) - S(v_{n+1}))\|^2 \\
&= \|u^*\|^2 - 2\langle u^*, J(w_{n+1})\rangle + \|w_{n+1}\|^2 - 2\zeta_{n+1}\langle w_{n+1} - u^*, S(w_{n+1}) - S(v_{n+1})\rangle \\
&\quad + k\zeta_{n+1}^2\|(S(w_{n+1}) - S(v_{n+1}))\|^2 \\
&= \Psi(u^*, w_{n+1}) - 2\zeta_{n+1}\langle w_{n+1} - u^*, S(w_{n+1}) - S(v_{n+1})\rangle \\
&\quad + k\zeta_{n+1}^2\|(S(w_{n+1}) - S(v_{n+1}))\|^2. \qquad (3.1)
\end{aligned}$$

Now we show that $2\zeta_{n+1}\langle w_{n+1} - u^*, S(w_{n+1}) - S(v_{n+1})\rangle \geq 0$,

$$\begin{aligned}
2\zeta_{n+1}\langle w_{n+1} - u^*, S(w_{n+1}) - S(v_{n+1})\rangle &= 2\zeta_{n+1}\langle w_{n+1} - u^*, S(w_{n+1}) - S(v_{n+1}) - S(u^*)\rangle \\
&= 2\zeta_{n+1}\langle w_{n+1} - u^*, S(w_{n+1}) - S(u^*)\rangle \\
&\quad - 2\zeta_{n+1}\langle w_{n+1} - u^*, S(v_{n+1})\rangle.
\end{aligned}$$

Since S is monotone operator so $\langle u_n - u^*, S(u_n) - S(u^*)\rangle \geq 0$ for any sequence $\{u_n\}$ without loss of generality we can assume $2\zeta_{n+1}\langle w_{n+1} - u^*, S(w_{n+1}) - S(u^*)\rangle = Q_1$ and $2\zeta_{n+1}\langle v_{n+1} - u^*, S(v_{n+1}) - S(u^*)\rangle = Q_2$. Let $Q = \min\{Q_1, Q_2\}$.

Now,

$$2\zeta_{n+1}\langle w_{n+1} - u^*, S(w_{n+1}) - S(v_{n+1})\rangle \geq Q - 2\zeta_{n+1}\langle w_{n+1} - u^*, S(v_{n+1})\rangle. \tag{3.2}$$

$$\begin{aligned}
-2\zeta_{n+1}\langle w_{n+1} - u^*, S(v_{n+1})\rangle &= -2\zeta_{n+1}\langle J^{-1}(J(v_{n+1}) - \zeta_{n+1}S(v_{n+1})) - u^*, S(v_{n+1})\rangle \\
&= -2\zeta_{n+1}\langle v_{n+1} - u^*, S(v_{n+1}) - S(u^*)\rangle \\
&+ 2\zeta_{n+1}\langle \zeta_{n+1}J^{-1}(S(v_{n+1})), S(v_{n+1})\rangle \\
&\geq -Q + 2\zeta_{n+1}^2\langle J^{-1}(S(v_{n+1}))), S(v_{n+1})\rangle \\
&= -Q + 2\zeta_{n+1}^2\|J^{-1}(S(v_{n+1}))\| \cdot \|S(v_{n+1})\|.
\end{aligned}$$

$$Q - 2\zeta_{n+1}\langle w_{n+1} - u^*, S(v_{n+1})\rangle \geq 2\zeta_{n+1}^2\|J^{-1}(S(v_{n+1}))\| \cdot \|S(v_{n+1})\|. \tag{3.3}$$

With the help of equations (3.2) and (3.3) we get

$$2\zeta_{n+1}\langle w_{n+1} - u^*, S(w_{n+1}) - S(v_{n+1})\rangle \geq 2\zeta_{n+1}^2\|J^{-1}(S(v_{n+1}))\| \cdot \|S(v_{n+1})\| \geq 0.$$

Now (3.1) becomes

$$\begin{aligned}
\Psi(u^*, u_{n+1}) &\leq \Psi(u^*, w_{n+1}) + k\zeta_{n+1}^2\|S(w_{n+1}) - S(v_{n+1})\|^2 \\
&\leq \Psi(u^*, w_{n+1}) + k\zeta_{n+1}^2 C^2. \qquad (3.4)
\end{aligned}$$

Now,

$$\begin{aligned}
\Psi(u^*, w_{n+1}) &= \Psi(u^*, J^{-1}(J(v_{n+1}) - \zeta_{n+1}S(v_{n+1}))) \\
&= \Gamma(u^*, J(v_{n+1}) - \zeta_{n+1}S(v_{n+1})) \\
&\leq \Gamma(u^*, J(v_{n+1})) - 2\langle J^{-1}(J(v_{n+1}) - \zeta_{n+1}S(v_{n+1})) - u^*, \zeta_{n+1}S(v_{n+1})\rangle \\
&= \Gamma(u^*, J(v_{n+1})) - 2\zeta_{n+1}\langle v_{n+1} - u^*, S(v_{n+1}) - S(u^*)\rangle \\
&+ 2\zeta_{n+1}\langle J^{-1}(J(v_{n+1})) - J^{-1}(J(v_{n+1}) - \zeta_{n+1}S(v_{n+1})), S(v_{n+1})\rangle.
\end{aligned}$$

$$\begin{aligned}
\Psi(u^*,w_{n+1}) &\leq \Gamma(u^*,J(v_{n+1})) \\
&\quad + 2\zeta_{n+1}\|J^{-1}(J(v_{n+1})) - J^{-1}(J(v_{n+1}) - \zeta_{n+1}S(v_{n+1})\| \cdot \|S(v_{n+1})\| \\
&\leq \Psi(u^*,v_{n+1}) + 2\zeta_{n+1}\Delta(\|\zeta_{n+1}S(v_{n+1})\|)C \\
&\leq \Psi(u^*,v_{n+1}) + 2\zeta_{n+1}\Delta(\|\zeta_{n+1}C\|)C \qquad (3.5)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Psi(u^*,v_{n+1}) &= \Psi(u^*,J^{-1}(J(u_n) - \zeta_n R(u_n))) \\
&= \Gamma(u^*,J(u_n) - \zeta_n R(u_n)) \\
&\leq \Gamma(u^*,J(u_n)) - 2\langle J^{-1}(J(u_n) - \zeta_n R(u_n)) - u^*, \zeta_n R(u_n)\rangle \\
&= \Gamma(u^*,J(u_n)) - 2\zeta_n\langle u_n - u^*, R(u_n) - R(u^*)\rangle \\
&\quad + 2\zeta_n\langle J^{-1}(J(u_n)) - J^{-1}(J(u_n) - \zeta_n R(u_n)), R(u_n)\rangle.
\end{aligned}$$

$$\begin{aligned}
\Psi(u^*,v_{n+1}) &\leq \Gamma(u^*,J(u_n)) \\
&\quad + 2\zeta_n\|J^{-1}(J(u_n)) - J^{-1}(J(u_n) - \zeta_n R(u_n)\| \cdot \|R(u_n)\| \\
&\leq \Psi(u^*,u_n) + 2\zeta_n\Delta(\|\zeta_n R(u_n)\|)D \\
&\leq \Psi(u^*,u_n) + 2\zeta_n\Delta(\|\zeta_n D\|)D \qquad (3.6)
\end{aligned}$$

Now using (3.5) and (3.6) in (3.4), we get

$$\Psi(u^*,u_{n+1}) \leq \Psi(u^*,u_n) + 2\zeta_{n+1}\Delta(\|\zeta_{n+1}C\|)C + 2\zeta_n\Delta(\|\zeta_n D\|)D + k\zeta_{n+1}^2 C^2.$$

Applying Lemma 2.10 and conditions $(1),(2)$ and (4), we get $\lim_{n\to\infty}\Psi(u^*,u_n)$ exists. Hence the sequence $\{\Psi(u^*,u_n)\}$ is bounded, which implies that the sequence $\{u_n\}$ is also bounded. Next we show that the sequence $\{u_n\}$ converges strongly to u^*. Since the space K is uniformly smooth, denote $C_1 = \sup\|J(w_n)\|$ without loss of generality. Then

$$\begin{aligned}
\Psi(u^*,u_{n+1}) &= \Psi(u^*,J^{-1}(J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1})))) \\
&= \Gamma(u^*,J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1})))
\end{aligned}$$

$$
\begin{aligned}
&= \Gamma(u^*, (1-\zeta_{n+1})J(w_{n+1}) + \zeta_{n+1}J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1})))\\
&\leq \Gamma(u^*, (1-\zeta_{n+1})J(w_{n+1}))\\
&+ 2\langle w_{n+1} - u^*, \zeta_{n+1}J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1}))\rangle\\
&+ 2\langle J^{-1}(J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1}))) - w_{n+1}, \zeta_{n+1}J(w_{n+1})\\
&- \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1}))\rangle.
\end{aligned}
$$

$$
\begin{aligned}
\Psi(u^*, u_{n+1}) &\leq \Gamma(u^*, (1-\zeta_{n+1})J(w_{n+1})) + 2\langle w_{n+1} - u^*, \zeta_{n+1}J(w_{n+1})\rangle\\
&- 2\zeta_{n+1}\langle w_{n+1} - u^*, S(w_{n+1}) - S(v_{n+1}) - S(u^*)\rangle\\
&+ 2\|J^{-1}(J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1}))) - J^{-1}(J(w_{n+1}))\| \times\\
&\times \|\zeta_{n+1}J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1}))\|\\
&\leq (1-\zeta_{n+1})\Gamma(u^*, J(w_{n+1})) + 2\zeta_{n+1}\langle u^*, J(u^*) - J(w_{n+1})\rangle\\
&+ 2\Delta(\|(J(w_{n+1}) - \zeta_{n+1}(S(w_{n+1}) - S(v_{n+1}))) - J(w_{n+1})\|) \times\\
&\times (\zeta_{n+1}\|J(w_{n+1})\| + \zeta_{n+1}\|S(w_{n+1}) - S(v_{n+1})\|)\\
&\leq (1-\zeta_{n+1})\Psi(u^*, w_{n+1}) + 2\zeta_{n+1}\langle u^*, J(u^*) - J(w_{n+1})\rangle\\
&+ 2\zeta_{n+1}\Delta(\zeta_{n+1}C)(C_1 + C) \qquad (3.7)
\end{aligned}
$$

Substituting inequalities (3.5) and (3.6) in (3.7), we get

$$
\begin{aligned}
\Psi(u^*, u_{n+1}) &\leq (1-\zeta_{n+1})(\Psi(u^*, u_n) + 2\zeta_{n+1}\Delta(\|\zeta_{n+1}C\|)C + 2\zeta_n\Delta(\|\zeta_n D\|)D\\
&+ 2\zeta_{n+1}\langle u^*, J(u^*) - J(w_{n+1})\rangle + 2\zeta_{n+1}\Delta(\zeta_{n+1}C)(C_1 + C)\\
&\leq (1-\zeta_{n+1})\Psi(u^*, u_n) + 2\zeta_{n+1}\Delta(\|\zeta_{n+1}C\|)C + 2\zeta_n\Delta(\|\zeta_n D\|)D\\
&+ 2\zeta_{n+1}\langle u^*, J(u^*) - J(w_{n+1})\rangle + 2\zeta_{n+1}\Delta(\zeta_{n+1}C)(C_1 + C). \qquad (3.8)
\end{aligned}
$$

Now we prove that the sequence $\{\Psi(u^*, u_n)\}$ converges to zero. From (3.8), we get

$$
\begin{aligned}
\Psi(u^*, u_{n+1}) &\leq \Psi(u^*, u_n) + 2\zeta_{n+1}\Delta(\|\zeta_{n+1}C\|)C + 2\zeta_n\Delta(\|\zeta_n D\|)D\\
&+ 2\zeta_{n+1}\langle u^*, J(u^*) - J(w_{n+1})\rangle + 2\zeta_{n+1}\Delta(\zeta_{n+1}C)(C_1 + C)
\end{aligned}
$$

$$2\zeta_{n+1}\langle u^*, J(w_{n+1}) - J(u^*)\rangle \leq \Psi(u^*, u_n) - \Psi(u^*, u_{n+1}) + 2\zeta_{n+1}\Delta(\|\zeta_{n+1}C\|)C$$
$$+ 2\zeta_n\Delta(\|\zeta_n D\|)D + 2\zeta_{n+1}\Delta(\zeta_{n+1}C)(C_1 + C). \tag{3.9}$$

Summing the terms, we get

$$2\sum_{n=1}^{\infty}\zeta_{n+1}\langle u^*, J(w_{n+1}) - J(u^*)\rangle \leq \sum_{n=1}^{\infty}(\Psi(u^*, u_n) - \Psi(u^*, u_{n+1})) + 2\sum_{n=1}^{\infty}\zeta_{n+1}\Delta(\|\zeta_{n+1}C\|)C$$
$$+ 2\sum_{n=1}^{\infty}\zeta_n\Delta(\|\zeta_n D\|)D + 2\sum_{n=1}^{\infty}\zeta_{n+1}\Delta(\zeta_{n+1}C)(C_1 + C).$$

Using conditions (1), (2) and (4), we get $\lim_{n\to\infty}\langle u^*, J(w_{n+1}) - J(u^*)\rangle = 0$. Applying lemma 2.11 in (3.8), we get $\lim_{n\to\infty}\Psi(u^*, u_n) = 0$. Now by Lemma 2.9, we get $\lim_{n\to\infty}\|u_n - u^*\| = 0$. This implies that the sequence $\{u_n\}$ converges strongly to $u^* \in (R+S)^{-1}(0)$. It completes the proof. □

4. Applications

In this section, we present couple of applications of our result.

4.1. Convex minimization problem.

Let $S, R : K \to K^*$ be mappings. Then the convex minimization problem is defined as

$$\min\{S(u) + R(u) : u \in K\}.$$

Let $g, h : K \to \mathbb{R}$ be convex, differential and bounded functions. Assume that ∇g and ∇h are the gradient of functions g and h, respectively. Now, we present first application of our result.

Theorem 4.1. *Suppose K be a strictly convex and 2-uniformly smooth real Banach space with the dual K^*. Let $\nabla g, \nabla h : K \to K^*$ be monotone and bounded mappings with $(\nabla g + \nabla h)^{-1}(0) \neq \emptyset$. Let us consider parameter $\{\zeta_n\}$ satisfies the above $(1)-(5)$ conditions, then sequence $\{u_n\}$ defined by following Algorithm:*

$$v_{n+1} = J^{-1}(J(u_n) - \zeta_n\nabla g(u_n)), \ \ \forall n \geq 0,$$
$$w_{n+1} = J^{-1}(J(v_{n+1}) - \zeta_{n+1}\nabla h(v_{n+1})), \ \ \forall n \geq 0,$$
$$u_{n+1} = J^{-1}(J(w_{n+1}) - \zeta_{n+1}(\nabla h(w_{n+1}) - \nabla h(v_{n+1}))), \ \ \forall n \geq 0.$$

converges strongly to some $u^* \in (\nabla g + \nabla h)^{-1}(0)$, if $u^* \in (\nabla h)^{-1}(0)$ and $u^* \in (\nabla g)^{-1}(0)$ and u^* is a solution of above minimization problem.

Proof. Taking $S = \nabla g$ and $R = \nabla h$ in Theorem 3.1, one can easily get the desired result. $\square$

4.2. ℓ_1 regularization problem. ℓ_1 regularization problem is also known as the LASSO problem. The LASSO problem is considered as a image or signal recovery problem. The image or signal processing problem is described as estimating the original and clean image or signal, using known information from the polluted image or signal. The following linear inverse problem is commonly used to express this type of problem:

$$Y = Ax + w,$$

where w is a noise function, x is the unknown image or signal to recover, A is the degradation function and Y is contaminated image. We use the Basis Pursuit denoising method, which uses the ℓ_1 norm as a sparsity imposing penalty, to recover an approximation of the image or signal x

$$\min_x \left\{ \frac{1}{2}\|Y - A(x)\|_2^2 + \lambda \|x\|_1 \right\}.$$

Suppose H_1, H_2 be two Hilbert spaces $A : H_1 \to H_2$ bounded monotone mapping and $A^* : H_2 \to H_1$ is adjoint of mapping A. Let $P(x), Q(x) : H_1 \to H_2$ such that $P(x) = \frac{1}{2}\|Y - A(x)\|_2^2$ and $Q(x) = \|x\|_1$. Then $\nabla P(x) = A^*(Ax - Y)$ and proximal mapping of $Q(x)$ is $(I + \lambda \partial Q)^{-1}$, where ∂Q is the subdifferential of function $Q(x)$ defined by $\partial Q(x) = \{s \in H_1 : Q(y) \geq Q(x) + \langle s, y - x\rangle, \forall y \in H_1\}$. If we set $G = A^*(I + \lambda \partial Q)^{-1}A$ and $T = \nabla P$, where ∇P is the gradient of $P(x)$. Since the mapping $P(x)$ is differentiable with L-Lipschitz continuous gradient $L = \|A^*A\|$ and A is bounded hence the map $T = \nabla P(x)$ is monotone and bounded. If we follow [18, Theorem 4.5] we can easily get that $G = A^*(I + \lambda \partial Q)^{-1}A$ is $\frac{1}{\|A\|^2}$-inverse strongly monotone. Thus G is bounded and monotone mapping. The following theorem is second application of our result.

Theorem 4.2. *Suppose H_1, H_2 be two Hilbert spaces. Let $G, T : H_1 \to H_2$ be monotone and bounded mappings with $(G + T)^{-1}(0) \neq \emptyset$. Let us consider parameter $\{\zeta_n\}$ satisfies the above $(1) - (5)$ conditions, then sequence $\{u_n\}$ defined by following Algorithm:*

$$v_{n+1} = J^{-1}(J(u_n) - \zeta_n(A^*(I + \lambda \partial Q)^{-1}A)(u_n)), \;\; \forall n \geq 0,$$

$$w_{n+1} = J^{-1}(J(v_{n+1}) - \zeta_{n+1}(\nabla P(v_{n+1}))), \;\; \forall n \geq 0,$$

$$u_{n+1} = J^{-1}(J(w_{n+1}) - \zeta_{n+1}(\nabla P(w_{n+1}) - \nabla P(v_{n+1}))), \;\; \forall n \geq 0.$$

converges strongly to $u^* \in (G+T)^{-1}(0)$ *if* $u^* \in G^{-1}(0)$ *and* $u^* \in T^{-1}(0)$.

Proof. Taking $R = G$ and $S = T$ in Theorem 3.1, one can easily get the desired result. □

5. EXAMPLES

In this section, we present some examples and convergence behaviour of our algorithm 1 for different initial guesses and parameters numerically.

Example 5.1. Let $H = L^2([0,2\pi])$, norm $\|u\| = \left(\int_0^{2\pi} |u(t)|^2 dt\right)^{1/2}$, and inner product $\langle u, v\rangle = \int_0^{2\pi} u(t)v(t)dt$, for all $u, v \in H$. Let us consider the closed ball centred at $\sin \in H$ with radius 4.

$$B = \{u \in H : \|u - \sin\|_2^2 \leq 16\} = \left\{ u \in H : \int_0^{2\pi} |u(t) - \sin(t)|^2 dt \leq 16 \right\}.$$

Let the mapping $S : H \to H$ such that $(Su)(t) = \dfrac{u(t)}{2}$ for all $u \in H$. Now, define the mapping $R : H \to H$ such that $R(u) = \nabla\left(\frac{1}{2}\|S(u) - P_B S(u)\|^2\right) = S^*(I - P_B)S(u)$. Where, ∇ is the gradient, S^* is the adjoint mapping of S and P_B is the metric projection onto B for all $u \in H$ defined as:

$$P_B(u) = \begin{cases} \sin + \dfrac{4}{\sqrt{\int_0^{2\pi} |u(t)-\sin(t)|^2 dt}}(u - \sin), & \int_0^{2\pi} |u(t) - \sin(t)|^2 > 16, \\ u, & \int_0^{2\pi} |u(t) - \sin(t)|^2 \leq 16. \end{cases}$$

Here the mappings R, S are monotone and bounded and $0 \in (R+S)^{-1}(0)$ with $0 \in R^{-1}(0)$ and $0 \in S^{-1}(0)$. Hence, all the conditions of Theorem 3.1 are satisfying, so the sequence $\{u_n\}$ generated by Algorithm 1 converges to $0 \in (R+S)^{-1}(0)$.

Example 5.2. Let $X = \mathbb{R}$, $K = \left[\frac{-1}{2}, 1\right]$ be a subset of X and $R, S : K \to K$ be mappings defined as

$$R(u) = \tanh(u), \quad S(u) = \frac{u}{1+u}$$

for all $u \in K$. The mappings R, S are monotone, bounded and $0 \in (R+S)^{-1}(0)$ with $0 \in R^{-1}(0)$ and $0 \in S^{-1}(0)$.

Now we present the convergence behaviour of our algorithm for different choices of ζ_n and initial guesses.

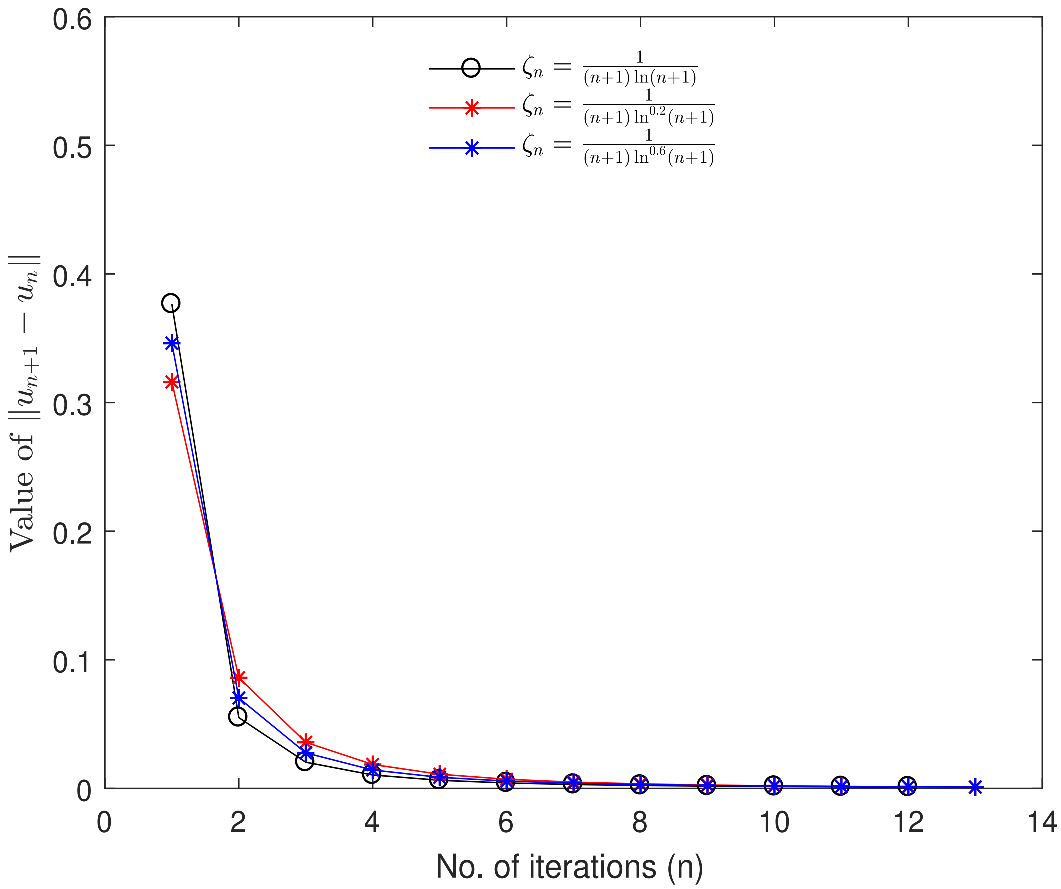

FIGURE 1. Convergence behaviour for fixed $u_0 = 0.5$, different values of $\zeta_n = \frac{1}{(n+1)\ln(n+1)}, \zeta_n = \frac{1}{(n+1)\ln^{0.2}(n+1)}, \zeta_n = \frac{1}{(n+1)\ln^{0.6}(n+1)}$.

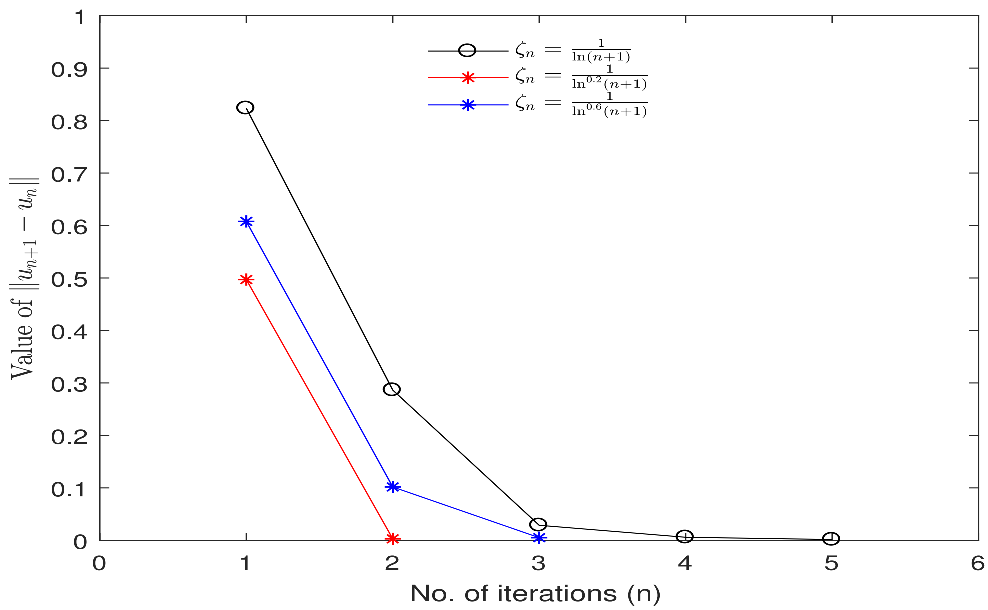

FIGURE 2. Convergence behaviour for fixed $u_0 = 0.5$, different values of $\zeta_n = \frac{1}{\ln(n+1)}, \zeta_n = \frac{1}{\ln^{0.2}(n+1)}, \zeta_n = \frac{1}{\ln^{0.6}(n+1)}$.

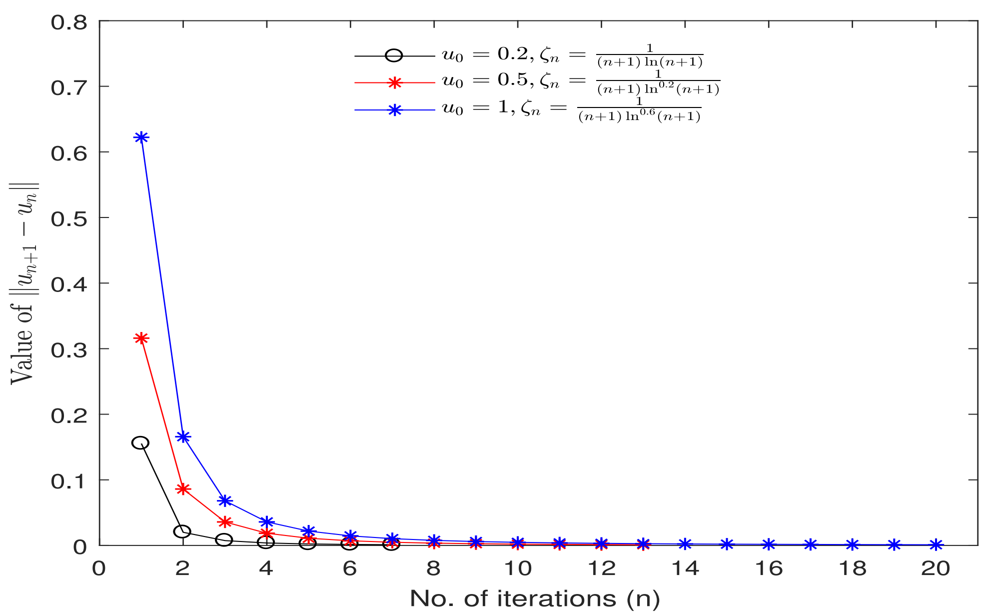

FIGURE 3. Convergence behaviour for different initial values $u_0 = 0.2, u_0 = 0.5, u_0 = 1$, different values of $\zeta_n = \frac{1}{(n+1)\ln(n+1)}, \zeta_n = \frac{1}{(n+1)\ln^{0.2}(n+1)}, \zeta_n = \frac{1}{(n+1)\ln^{0.6}(n+1)}$.

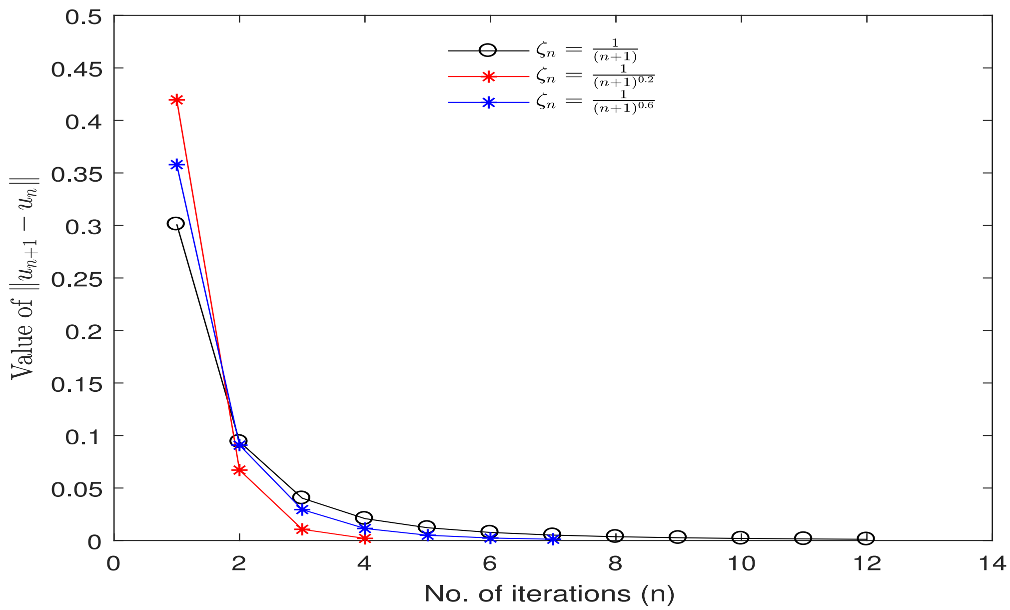

FIGURE 4. Convergence behaviour for fixed $u_0 = 0.5$, different values of $\zeta_n = \frac{1}{n+1}, \zeta_n = \frac{1}{(n+1)^{0.2}}, \zeta_n = \frac{1}{(n+1)^{0.6}}$.

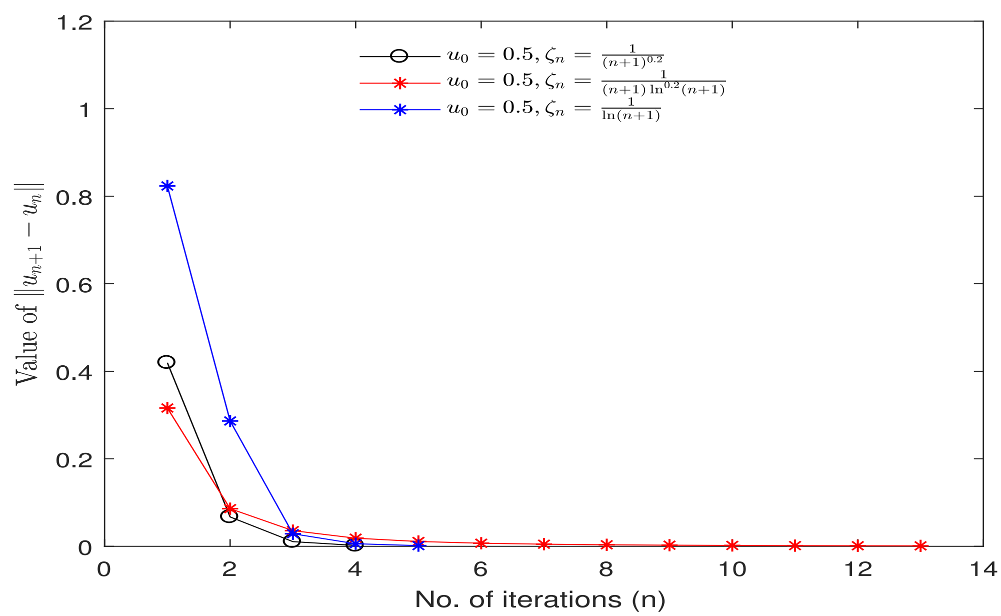

FIGURE 5. Convergence behaviour for fixed initial value $u_0 = 0.5$ and different $\zeta_n = \frac{1}{(n+1)^{0.2}}, \zeta_n = \frac{1}{(n+1)\ln(n+1)^{0.2}}, \zeta_n = \frac{1}{\ln(n+1)}$.

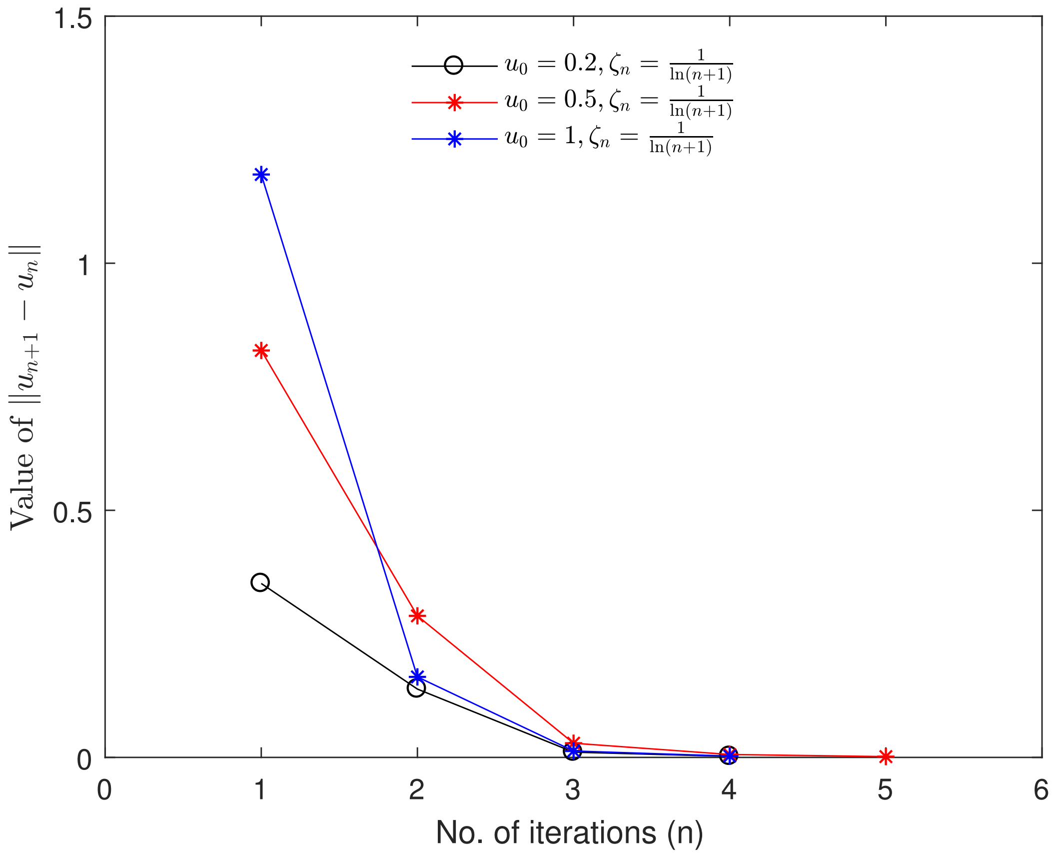

FIGURE 6. Convergence behaviour for different initial values $u_0 = 0.2, u_0 = 0.5, u_0 = 1$,for fixed $\zeta_n = \frac{1}{\ln(n+1)}$.

Example 5.3. Let $X = \mathbb{R}$ be equipped with the standard norm and $K = [-1,1]$ a subset of X. Suppose $R, S : K \to K$ are mappings defined by

$$R(u) = 1+u, \quad S(u) = \frac{1+u}{3}$$

for all $u \in K$. It is easy to see that the mappings R, and S are monotone, bounded and satisfy all the conditions of Theorem 3.1. Also, $-1 \in (R+S)^{-1}(0)$ with $-1 \in R^{-1}(0)$ and $-1 \in S^{-1}(0)$.

Now we present the convergence behaviour of our algorithm.

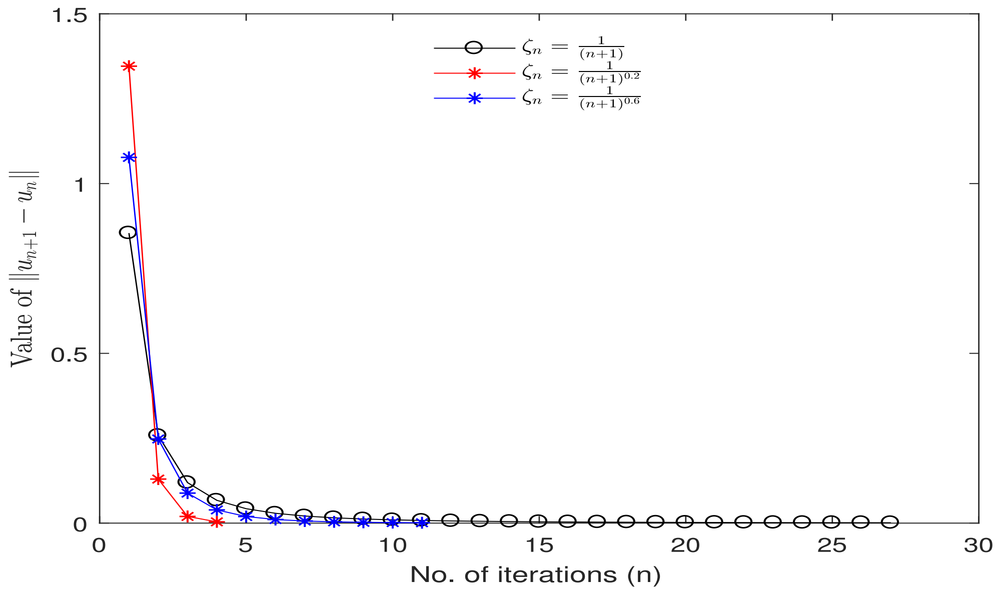

FIGURE 7. Convergence behaviour for fixed $u_0 = 0.5$, different values of $\zeta_n = \frac{1}{n+1}, \zeta_n = \frac{1}{(n+1)^{0.2}}, \zeta_n = \frac{1}{(n+1)^{0.6}}$.

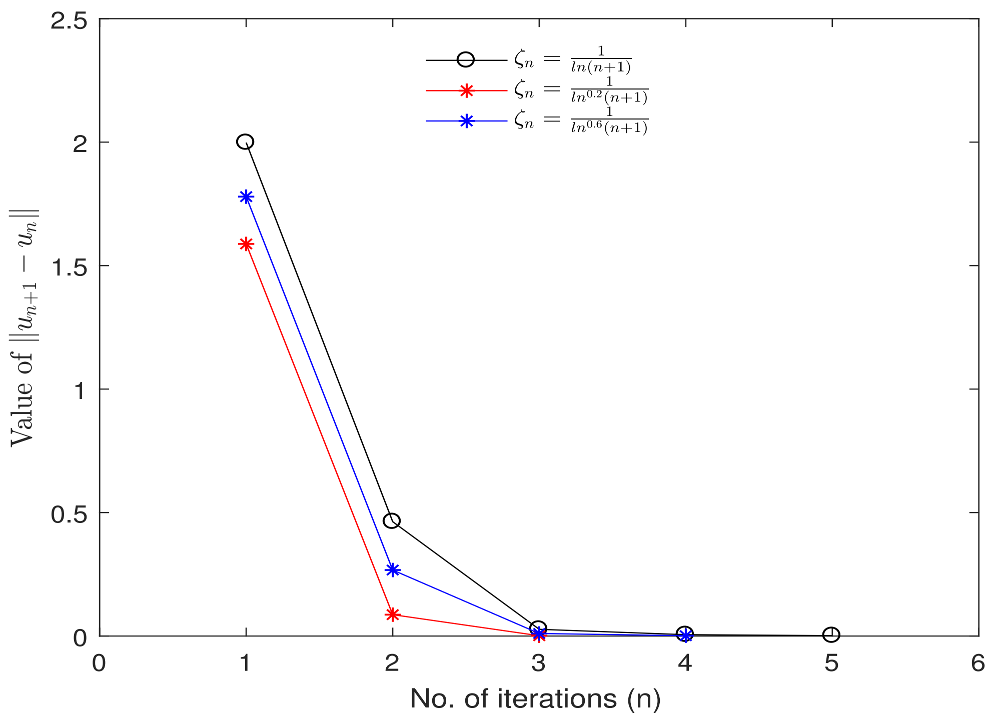

FIGURE 8. Convergence behaviour for fixed $u_0 = 0.5$, different values of $\zeta_n = \frac{1}{\ln(n+1)}, \zeta_n = \frac{1}{\ln^{0.2}(n+1)}, \zeta_n = \frac{1}{\ln^{0.6}(n+1)}$.

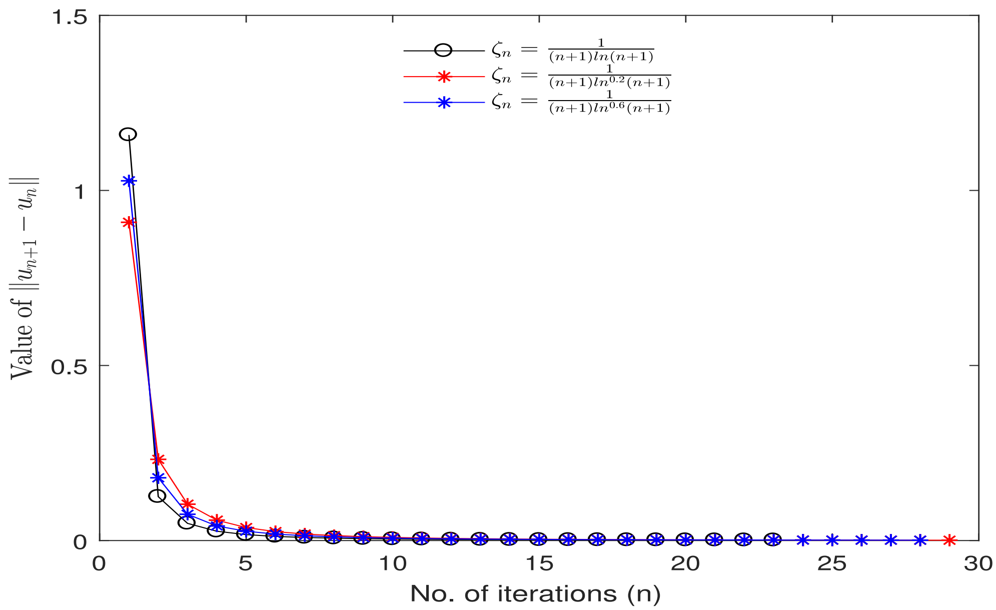

FIGURE 9. Convergence behaviour for fixed $u_0 = 0.5$, different values of $\zeta_n = \frac{1}{(n+1)\ln(n+1)}, \zeta_n = \frac{1}{(n+1)\ln^{0.2}(n+1)}, \zeta_n = \frac{1}{(n+1)\ln^{0.6}(n+1)}$.

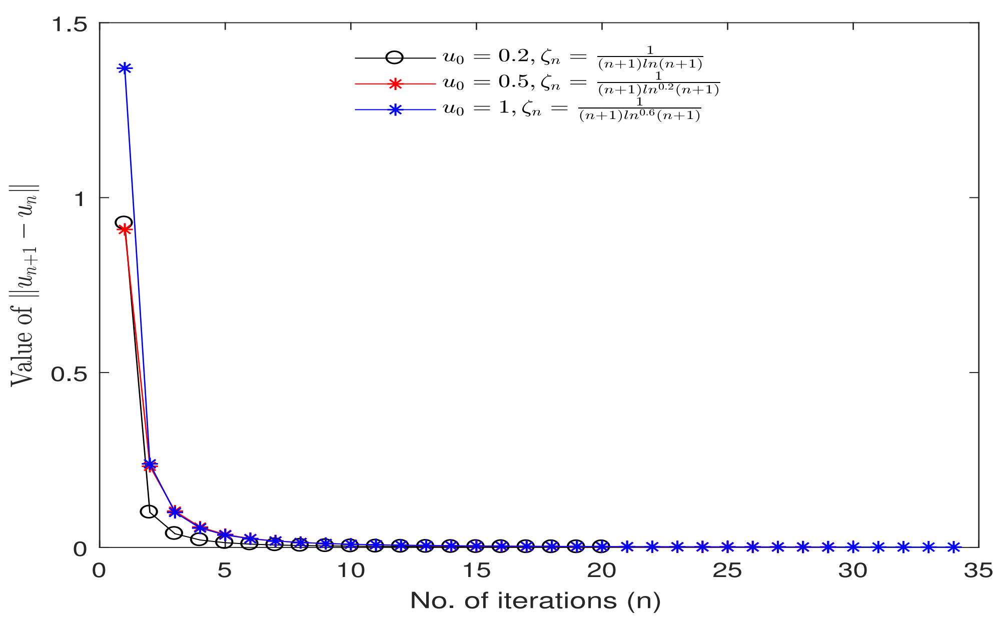

FIGURE 10. Convergence behaviour for different initial values $u_0 = 0.2, u_0 = 0.5, u_0 = 1$, different values of $\zeta_n = \frac{1}{(n+1)\ln(n+1)}, \zeta_n = \frac{1}{(n+1)\ln^{0.2}(n+1)}, \zeta_n = \frac{1}{(n+1)\ln^{0.6}(n+1)}$.

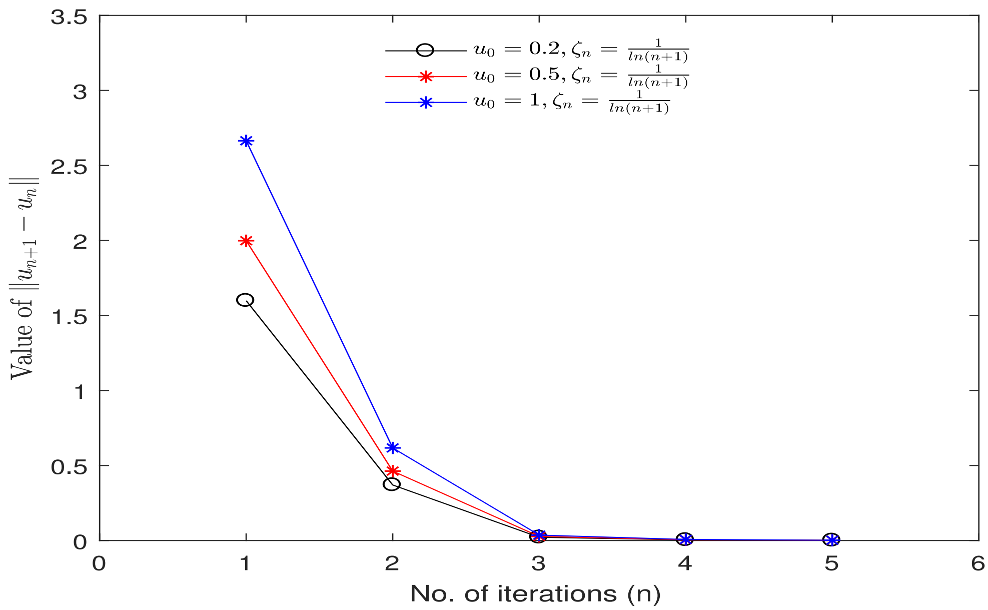

FIGURE 11. Convergence behaviour for different initial values $u_0 = 0.2, u_0 = 0.5, u_0 = 1$,for fixed $\zeta_n = \frac{1}{\ln(n+1)}$.

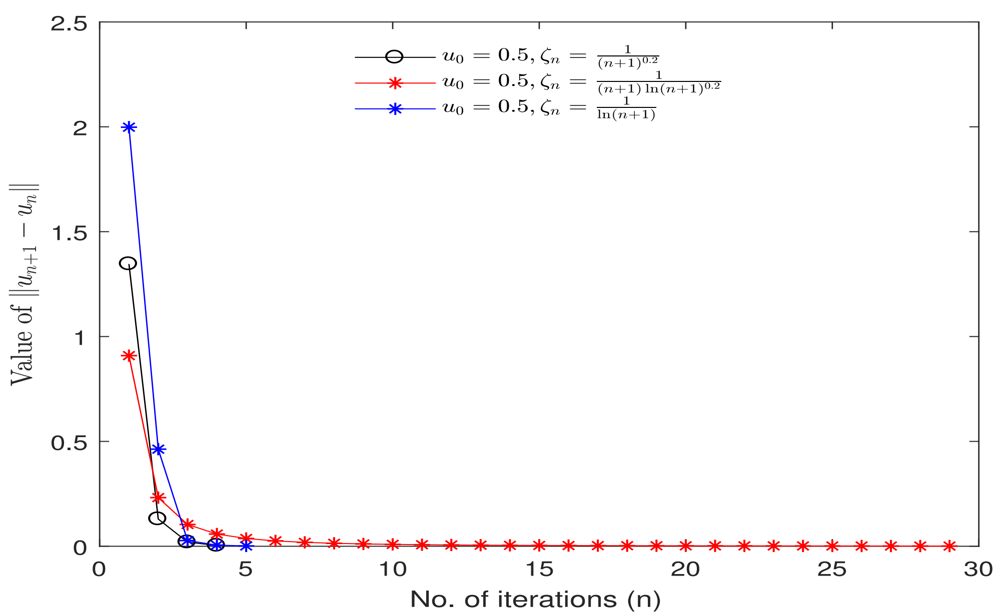

FIGURE 12. Convergence behaviour for fixed initial value $u_0 = 0.5$ and different $\zeta_n = \frac{1}{(n+1)^{0.2}}, \zeta_n = \frac{1}{(n+1)\ln(n+1)^{0.2}}, \zeta_n = \frac{1}{\ln(n+1)}$.

TABLE 1. Influence of coefficient ζ_n with initial guess $u_0 = 0.5$.

Number of iterations	For $\zeta_n = \frac{1}{(n+1)^{0.2}}$, $\|u_{n+1} - u_n\|$
1	1.345821263653190
2	0.129726066522860
3	0.019649562225108
4	0.003723403437351
Number of iterations	$\zeta_n = \frac{1}{(n+1)\,\mathrm{In}(n+1)^{0.2}}$, $\|u_{n+1} - u_n\|$
1	0.909030839333177
2	0.231951092478638
3	0.103865821269175
4	0.058284004399984
5	0.037078643969271
6	0.025566077343709
7	0.018646888927333
8	0.014176337396629
9	0.011126869427510
10	0.008957043717802
$\cdots$	$\cdots\ \cdots\ \cdots\ \cdots$
25	0.001356516081114
26	0.001251521785546
27	0.001158208962482
28	0.001074913065072
29	0.001000255015068
Number of iterations	$\zeta_n = \frac{1}{\mathrm{In}(n+1)}$, $\|u_{n+1} - u_n\|$
1	1.998274212300339
2	0.463001623732462
3	0.027238860107510
4	0.005491197630068
5	0.001589144513118

Now we compare our algorithm 1 with the algorithms presented in [17, 3]. For comparison, we choose initial guess $u_0 = 0.5$, $\zeta_n = \frac{1}{(n+1)\ln(n+1)^{0.8}}$. The stopping criteria is $\|u_{n+1} - u_n\| < 10^{-3}$. It shows that our algorithm 1 converges faster than the others.

TABLE 2. Comparison with initial guess $u_0 = 0.5$ and $\zeta_n = \frac{1}{(n+1)\ln(n+1)^{0.8}}$.

Algorithm	**Number of iterations**
Our algorithm	26
Yan Tang	59
Chidume	66

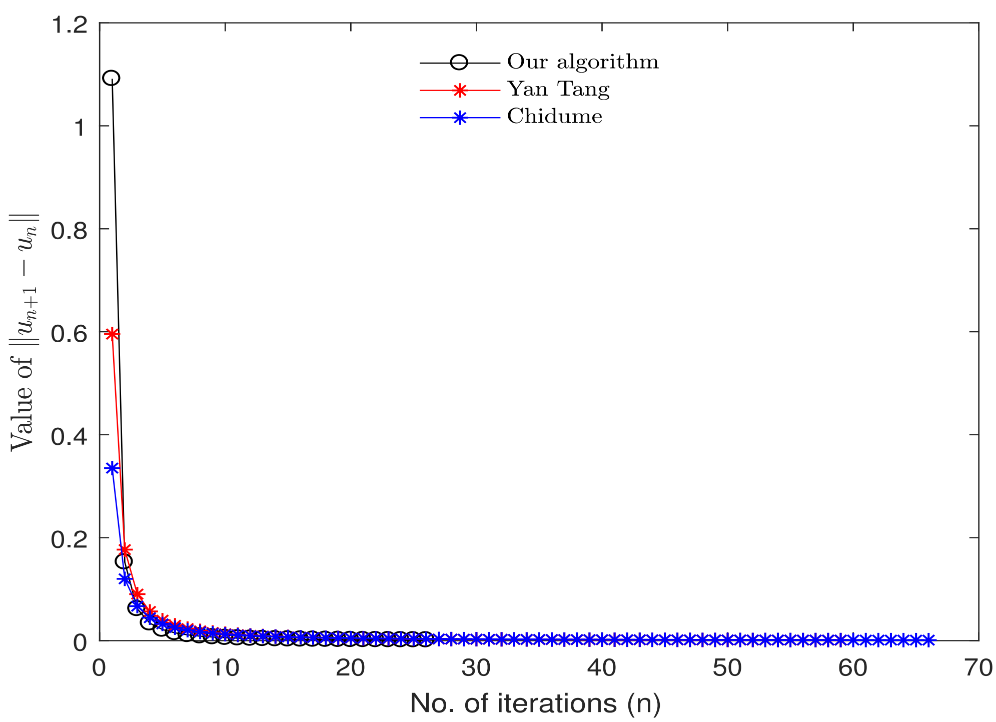

FIGURE 13. Comparison with initial guess $u_0 = 0.5$ and $\zeta_n = \frac{1}{(n+1)\ln(n+1)^{0.8}}$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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