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FIXED POINT RESULTS FOR A GENERALIZED θ-REICH TYPE CONTRACTION IN A GENERALIZED b_2-METRIC SPACE

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Abstract. In this paper, we introduce a new type of contractive maps referred to as θ-Reich-type contractions and prove fixed point theorems for such maps on a setting of a generalized b_2-metric space.

Keywords: Reich-type contraction; b_2-metric; fixed point.

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1. Introduction

In 1968, Kannan provided a generalization of the Banach contraction principle in that a contraction mapping with a fixed point need not be continuous, [1]. Kannan's theorem characterizes the completenesss of the metric space and was proved by Subrahmanyam in 1975, [2]. In [3], the authors presented results dealing with fixed point for maps that are not continuous on a metric space and addressed the aspect of convergence which improves the classical results.

Reich's fixed point theorem is a generalization of Banach's fixed point theorem,

Theorem 1.1. [4] *Let (X,d) be a complete metric space. Suppose that the self-map $T : X \to X$ satisfies the following:*

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$$(1) \qquad d(Tx,Ty) \leq \alpha_1 d(x,Tx) + \alpha_2 d(y,Ty) + \alpha_3 d(x,y)$$

for $x,y \in X$ where $\alpha_1 + \alpha_2 + \alpha_3 < 1$. Then T admits a unique fixed point.

with $\alpha_1 = \alpha_2 = 0$ while $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $\alpha_3 = 0$ yields Kannan's fixed point theorem, [4]. Several generalizations of the Banach contraction principle were derived by either changing the contraction conditions or by changing the space. Samet et.al., in [6], introduced a new type of θ-contractive maps and established a new fixed point theorem for such maps on the setting of a generalized metric space. In this paper, we introduce a new type of θ-contractive maps that display a sublinearity nature on a generalized b_2-metric space. Before we proceed, we provide a generalization of the concept of a b_2-metric space. The authors in a similar way provided generalizations of the concept of a metric in [7, 8, 9].

Definition 1.2. [5] *Let X be a non-empty set and $d : X \times X \times X \to [0,\infty)$ be a map satisfying the following properties:*

(i) $d(x,y,z) = 0$ for $x,y,z \in X$, if at least two of the three points are the same.

(ii) For $x,y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

(iii) symmetry property: for $x,y,z \in X$,

$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).$$

(iv) rectangle inequality:

$$d(x,y,z) \leq d(x,y,t) + d(y,z,t) + d(z,x,t)$$

for $x,y,z,t \in X$.

Then d is a 2-metric and (X,d) is a 2-metric space.

Definition 1.3. *Let X be a non-empty set and $d : X \times X \times X \to [0,\infty)$ be a map satisfying the following properties:*

(i) $d(x,y,z) = 0$ for $x,y,z \in X$, if at least two of the three points are the same.

(ii) For $x,y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

(iii) symmetry property: for $x,y,z \in X$,

$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).$$

(iv) modified rectangle inequality:there exists $\alpha,\beta,\gamma \geq 1$ such that

$$d(x,y,z) \leq \alpha d(x,y,t) + \beta d(y,z,t) + \gamma d(z,x,t)]$$

for $x,y,z,t \in X$.

Then d is a generalized b_2-metric and (X,d) is a generalized b_2- metric space.

If $\alpha = \beta = \gamma = 1$ then the generalized b_2-metric is a 2-metric. If $\alpha = \beta = \gamma = s > 1$ then the generalized b_2-metric is a b_2-metric.

Example 1.4. *Let $X = (0,1)$ and define*

$$d(x,y,z) = \begin{cases} 0, & \text{if at least two of the three points are the same} \\ e^{|x-y|^{\xi}+|y-z|^{\xi}+|z-x|^{\xi}}, & \text{otherwise}. \end{cases} \tag{2}$$

for $x,y,z \in X$ and $\xi \geq 1$. Properties $i)-iii)$ of definition 1.3, can be easily verified. We shall verify property $iv)$:

For $x,y,z \in X$ and using Jensens' inequality, we get

$$\begin{aligned} d(x,y,z) &= e^{|x-y|^{\xi}+|y-z|^{\xi}+|z-x|^{\xi}} \\ &= e^{\frac{1}{2}|x-y|^{\xi}+\frac{1}{3}|y-z|^{\xi}+\frac{1}{6}|z-x|^{\xi}} e^{\frac{1}{2}|x-y|^{\xi}+\frac{2}{3}|y-z|^{\xi}+\frac{5}{6}|z-x|^{\xi}} \\ &\leq e^{2} e^{\frac{1}{2}|x-y|^{\xi}+\frac{1}{3}|y-z|^{\xi}+\frac{1}{6}|z-x|^{\xi}} \\ &\leq e^{2}\left[\frac{1}{2}e^{|x-y|^{\xi}} + \frac{1}{3}e^{|y-z|^{\xi}} + \frac{1}{6}e^{|z-x|^{\xi}}\right] \\ &\leq e^{2}\left[\frac{1}{2}e^{|x-y|^{\xi}+|y-t|^{\xi}+|t-x|^{\xi}} + \frac{1}{3}e^{|z-y|^{\xi}+|y-t|^{\xi}+|t-z|^{\xi}} + \frac{1}{6}e^{|z-x|^{\xi}+|x-t|^{\xi}+|t-z|^{\xi}}\right] \\ &= \alpha d(x,y,t) + \beta d(z,y,t) + \gamma d(z,x,t) \end{aligned}$$

where $\alpha = \frac{e^2}{2} \geq 1$, $\beta = \frac{e^2}{3} \geq 1$ and $\gamma = \frac{e^2}{6} \geq 1$.

It follows that d is a generalized b_2-metric and not a b_2-metric.

Definition 1.5. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a generalized b_2-metric space (X,d).*

a) the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to $x\in X$ iff for all $\xi\in X$,

$$\lim_{n\to\infty} d(x_n,x,\xi)=0.$$

b) the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X iff for all $\xi\in X$,

$$\lim_{n,m\to\infty} d(x_n,x_m,\xi)=0.$$

2. Main Result

Definition 2.1. *Let (X,d) be a generalized b_2-metric space and a mapping $T:X\to X$ is a θ-Reich-type contraction if $x,y,z\in X$,*

$$\begin{aligned}&\theta(d(Tx,Ty,z))\\ &\le \alpha_1\theta(d(x,y,z))+\alpha_2\theta(d(x,Tx,z))+\alpha_3\theta(d(y,Ty,z))\end{aligned}$$

where $\alpha_1+\alpha_2+\alpha_3<1$,

$\theta:[0,\infty)\to[0,\infty)$ is a function satisfying the following conditions:

(i) θ is continuous and non-decreasing.

(ii) for each sequence $\{t_n\}\subset(0,\infty)$,

$$\lim_{n\to\infty}\theta(t_n)=0 \iff \lim_{n\to\infty} t_n=0.$$

(iii) there exists $k\in(0,1)$ and $l\in[0,\infty)$ such that $\lim_{t\to 0^+}\frac{\theta(t)}{t^k}=l$.

Theorem 2.2. *Let (X,d) be a complete generalized b_2-metric space and if a mapping $T:X\to X$ is a θ-Reich-type contraction then T has a unique fixed point in X.*

Proof. Let $x_0\in X$ be arbitrary. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$, where $x_n=Tx_{n-1}=T^nx_0$ is a Cauchy sequence in X. If $x_{n-1}=x_n$ for some $n\in\mathbb{N}$ then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. To prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence we suppose that $x_{n-1}\neq x_n$ for $n\in\mathbb{N}$ and let $x=x_{n-1}$ and $y=x_n$ in the assumption, then we get

$$\begin{aligned}&\theta(d(x_n,x_{n+1},z))\\ &=\theta(d(Tx_{n-1},Tx_n,z))\end{aligned}$$

$$\leq \alpha_1\theta(d(x_{n-1},x_n,z))+\alpha_2\theta(d(x_n,x_{n-1},z))+\alpha_3\theta(d(x_n,x_{n+1},z)).$$

It follows that

$$(3)\qquad \begin{aligned}(1-\alpha_3)\theta(d(x_n,x_{n+1},z)) &\leq (\alpha_1+\alpha_2)\theta(d(x_n,x_{n-1},z))\\ \theta(d(x_n,x_{n+1},z)) &\leq \left(\frac{\alpha_1+\alpha_2}{1-\alpha_3}\right)\theta(d(x_n,x_{n-1},z))\\ \theta(d(x_n,x_{n+1},z)) &\leq \theta(d(x_n,x_{n-1},z)),\end{aligned}$$

since $\alpha_1+\alpha_2+\alpha_3<1$. Since θ is an increasing function it follows that $d(x_n,x_{n+1}) \leq d(x_n,x_{n-1},z)$, thus $\{d(x_n,x_{n+1},z)\}_{n\in\mathbb{N}}$ is a decreasing sequence. Next, we show that $d(x_n,x_{n+1},z)\to 0$ as $n\to\infty$. Recursively using (3), we get

$$\begin{aligned}&\theta\left(d(x_{n+1},x_n,z)\right)\\ &\leq \left(\frac{\alpha_1+\alpha_2}{1-\alpha_3}\right)\theta\left(d(x_{n-1},x_n,z)\right)\\ &\leq \left(\frac{\alpha_1+\alpha_2}{1-\alpha_3}\right)^2\theta\left(d(x_{n-1},x_{n-2},z)\right)\\ &\ \vdots\\ (4)\qquad &\leq \left(\frac{\alpha_1+\alpha_2}{1-\alpha_3}\right)^n\theta(d(x_0,x_1,z)),\end{aligned}$$

since $\left(\frac{\alpha_1+\alpha_2}{1-\alpha_3}\right)<1$, it follows that as $n\to\infty$, we get $\theta\left(d(x_{n+1},x_n,z)\right)\to 0$ thus $d(x_{n+1},x_n,z)\to 0$. We now show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. It follows from the property of θ that from $k\in(0,1)$ and $l\in(0,\infty)$ that

$$(5)\qquad \lim_{n\to\infty}\frac{\theta(d(x_n,x_{n+1},z))}{[d(x_n,x_{n+1},z)]^k}=l.$$

For $0<\lambda<l$, by the definition of a limit there exists $n_1\in\mathbb{N}$ such that

$$\begin{aligned}\lambda &< \frac{\theta(d(x_n,x_{n+1},z))}{[d(x_n,x_{n+1},z)]^k}\\ \lambda[d(x_n,x_{n+1},z)]^k &\leq \theta(d(x_n,x_{n+1},z))\end{aligned}$$

From inequality (4), we get

$$n[d(x_n,x_{n+1},z)]^k < n\lambda^{-1}(\theta(d(x_n,x_{n+1},z)))$$
$$\leq n\lambda^{-1}\left(\frac{\alpha_1+\alpha_2}{1-\alpha_3}\right)^n \theta(d(x_0,x_1,z))$$

for all $n > n_1$ which yields that

$$\lim_{n\to\infty} n[d(x_n,x_{n+1},z)]^k = 0.$$

Hence there exists $n_2 \in \mathbb{N}$ such that

$$n[d(x_n,x_{n+1},z)]^k \leq 1$$

which implies that

$$d(x_n,x_{n+1},z) \leq \frac{1}{n^{\frac{1}{k}}} \tag{6}$$

for all $n > n_2$. For $m \in \mathbb{N}$, using (6) we obtain

$$\begin{aligned}
&d(x_n,x_{n+m},z)\\
&\leq \alpha d(x_n,x_{n+m},x_{n+1}) + \beta d(x_{n+m},z,x_{n+1}) + \gamma d(z,x_n,x_{n+1})\\
&\leq \max\{\alpha,\beta,\gamma\}\,(d(x_n,x_{n+m},x_{n+1}) + d(x_{n+m},z,x_{n+1}) + d(z,x_n,x_{n+1}))\\
&\leq \max\{\alpha,\beta,\gamma\}\left(\frac{2}{n^{\frac{1}{k}}} + d(x_{n+m},z,x_{n+1})\right)\\
&\leq \max\{\alpha,\beta,\gamma\}\left(\frac{2}{n^{\frac{1}{k}}} + \alpha d(x_{n+m},z,x_{n+2}) + \beta d(z,x_{n+1},x_{n+2}) + \gamma d(x_{n+1},x_{n+m},x_{n+2})\right)\\
&\leq \max\{\alpha,\beta,\gamma\}\left(\max\{\alpha,\beta,\gamma\}\frac{2}{n^{\frac{1}{k}}} + \max\{\alpha,\beta,\gamma\}\left(\frac{2}{(n+1)^{\frac{1}{k}}} + d(x_{n+m},z,x_{n+2})\right)\right)\\
&= (\max\{\alpha,\beta,\gamma\})^2\left(\frac{2}{n^{\frac{1}{k}}} + \frac{2}{(n+1)^{\frac{1}{k}}} + d(x_{n+m},z,x_{n+2})\right)\\
&\leq (\max\{\alpha,\beta,\gamma\})^{m+1}\left(\frac{2}{n^{\frac{1}{k}}} + \frac{2}{(n+1)^{\frac{1}{k}}} + \cdots + \frac{2}{(n+m)^{\frac{1}{k}}}\right)\\
&= (\max\{\alpha,\beta,\gamma\})^{m+1}\sum_{j=n}^{n+m}\frac{2}{j^{\frac{1}{k}}}\\
&\leq (\max\{\alpha,\beta,\gamma\})^{m+1}\,2\sum_{j=n}^{\infty}\frac{1}{j^{\frac{1}{k}}}.
\end{aligned}$$

Based on the convergence of the series $\sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{k}}}$, since $0 < \frac{1}{j} < 1$, we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since (X,d) is a complete generalized b_2-metric space there exist $u \in X$ such that $u = \lim_{n\to\infty} x_n$. We show that u is a fixed of T,

$$\begin{aligned}
&\theta(d(x_{n+1}, Tu, z)) \\
&= \theta(d(Tx_n, Tu, z)) \\
&\leq \alpha_1\theta(d(x_n, u, z)) + \alpha_2\theta(d(x_n, Tx_n, z)) + \alpha_3\theta(d(u, Tu, z)) \\
&\leq \alpha_1\theta(d(x_n, u, z)) + \alpha_2\theta(d(x_n, x_{n+1}, z)) + \alpha_3\theta(d(u, Tu, z)).
\end{aligned}$$

Taking the limits on both sides, we get

$$\theta(d(u, Tu, z)) \leq \alpha_3\theta(d(u, Tu, z)), \tag{7}$$

which is a contradication, unless $d(u,Tu,z) = 0$, $u = Tu$. To prove the uniqueness, we assume that $Tu'' = u''$:

$$\begin{aligned}
&\theta(d(u', u'', z)) = \theta(d(Tu', Tu'', z)) \\
&\leq \alpha_1\theta(d(u', u'', z)) + \alpha_2\theta(d(u', Tu', z)) + \alpha_3\theta(d(u'', Tu'', z))
\end{aligned}$$

It follows that

$$\theta(d(u', u'', z)) \leq \alpha_1\theta(d(u', u'', z))$$

which is a contradiction unless $d(u', u'', z) = 0$, i.e., $u' = u''$. □

Example 2.3. *Let* $X = \left[0, \frac{1+\sqrt{5}}{2}\right]$ *and define*

$$d(x,y,z) = \begin{cases} 0, & \text{if at least two of the three are the same.} \\ \mu e^{|x-y|+|y-z|+|z-x|}, & \text{otherwise.} \end{cases} \tag{8}$$

where $0 < \mu < \alpha_1 < 1$.

Then d is a generalized b_2-metric. Let $T : X \to X$ and define

$$Tx = \sqrt{x+1}$$

and $\theta(t) = t$. *Then for* $x, y \in X$, *from the Mean Value Theorem, we get*

$$\begin{aligned} \left|\sqrt{x+1} - \sqrt{y+1}\right| &\leq \left|\frac{1}{\sqrt{\xi+1}}\right| |x-y| \\ &\leq |x-y| \end{aligned}$$

since $0 < \xi < \frac{1+\sqrt{5}}{2}$.
For $x, z \in X$: *we obtain* $\left|z - \sqrt{x+1}\right| \leq |z-x|$ *since*

$$\sqrt{x+1} \geq x$$

for $0 < x < \frac{1+\sqrt{5}}{2}$.
Since the exponential function is increasing, we obtain that

$$\begin{aligned} &\theta(d(Tx,Ty,z)) \\ &= \mu e^{|\sqrt{x+1}-\sqrt{y+1}|+|\sqrt{y+1}-z|+|z-\sqrt{x+1}|} \\ &\leq \mu e^{|x-y|+|y-z|+|z-x|} \\ &\leq \alpha_1 \theta(d(x,y,z)) \end{aligned}$$

with $\mu < \alpha_1 < 1$ *and* $\alpha_2 = \alpha_3 = 0$. *By applying theorem 2.2, it follows that* T *has a fixed point in* X.

The following theorem is a result of [11], in which they introduced the notion of a θ-contractions and Suzuki contractions, but in this case the underlying space is a generalized b_2-metric space.

Theorem 2.4. *Let* (X,d) *be a complete generalized* b_2*-metric space and a mapping* $T : X \to X$ *satisfying: for all* $x, y, z \in X$,

$$\begin{aligned} &\theta(d(Tx,Ty,z)) \\ &\leq \left[\theta\left(\max\{d(x,y,z), d(x,Tx,z), d(y,Ty,z)\}\right)\right]^r \end{aligned}$$

where $0 < r < 1$, $\theta : (0,\infty) \to (1,\infty)$ *is a function satisfying the following conditions:*
(i) θ *is continuous and non-decreasing.*

(ii) for each sequence $\{t_n\} \subset (0,\infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \iff \lim_{n\to\infty} t_n = 0.$$

(iii) there exists $k \in (0,1)$ *and* $l \in (0,\infty)$ *such that* $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^k} = l$.
Then T *has a unique fixed point in* X.

Proof. Let $x_0 \in X$ be arbitrary. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$, where $x_n = Tx_{n-1} = T^n x_0$ is a Cauchy sequence in X. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$ then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. To prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence we suppose that $x_{n-1} \neq x_n$ for $n \in \mathbb{N}$ and let $x = x_{n-1}$ and $y = x_n$ in the assumption, then we get

$$\begin{aligned}
&\theta(d(x_n,x_{n+1},z))\\
&= \theta(d(Tx_{n-1},Tx_n,z))\\
&\leq [\max\{d(x_n,x_{n-1},z), d(x_n,x_{n+1},z)\}]^r.
\end{aligned}$$

If $\max\{d(x_n,x_{n-1},z), d(x_n,x_{n+1},z)\} = d(x_n,x_{n+1},z)$ then

$$\theta(d(x_n,x_{n+1},z)) \leq [\theta(d(x_n,x_{n+1},z))]^r, \tag{9}$$

which leads to a contradiction, since $0 < r < 1$. It follows that

$$\theta(d(x_n,x_{n+1},z)) \leq [\theta(d(x_n,x_{n-1},z))]^r < \theta(d(x_n,x_{n-1},z)) \tag{10}$$

Since θ is an increasing function it follows that $d(x_n,x_{n+1}) \leq d(x_n,x_{n-1},z)$ thus $\{d(x_n,x_{n+1},z)\}_{n\in\mathbb{N}}$ is a decreasing sequence. Next, we show that $d(x_n,x_{n+1},z) \to 0$ as $n \to \infty$. Recursively, using (10), we get

$$\begin{aligned}
&\theta\left(d(x_{n+1},x_n,z)\right)\\
&\leq \left[\theta\left(d(x_{n-1},x_n,z)\right)\right]^r\\
&\leq \left[\theta\left(d(x_{n-1},x_{n-2},z)\right)\right]^{r^2}\\
&\vdots\\
&\leq \left[\theta(d(x_0,x_1,z))\right]^{r^n}, \qquad (11)
\end{aligned}$$

since $r<1$, it follows that as $n\to\infty$, we get $\theta\left(d(x_{n+1},x_n,z)\right)\to 1$ thus $d(x_{n+1},x_n,z)\to 0$. We now show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. It follows from the property of θ that from $k\in(0,1)$ and $l\in(0,\infty)$ that

$$\lim_{n\to\infty}\frac{\theta(d(x_n,x_{n+1},z))-1}{[d(x_n,x_{n+1},z)]^k}=l. \tag{12}$$

For $0<\lambda<l$, by the definition of a limit there exists $n_1\in\mathbb{N}$ such that

$$\lambda<\frac{\theta(d(x_n,x_{n+1},z))-1}{[d(x_n,x_{n+1},z)]^k}$$

$$\lambda[d(x_n,x_{n+1},z)]^k\le\theta(d(x_n,x_{n+1},z)). \tag{13}$$

From inequality (11), we get

$$\begin{aligned} n[d(x_n,x_{n+1},z)]^k &< n\lambda^{-1}(\theta(d(x_n,x_{n+1},z)))-1\\ &\le n\lambda^{-1}\left[\theta(d(x_0,x_1,z))\right]^{r^n}-1 \end{aligned} \tag{14}$$

for all $n>n_1$ which yields that

$$\lim_{n\to\infty}n[d(x_n,x_{n+1},z)]^k=0. \tag{15}$$

Hence, there exists $n_2\in\mathbb{N}$ such that

$$n[d(x_n,x_{n+1},z)]^k\le 1, \tag{16}$$

which implies that

$$d(x_n,x_{n+1},z)\le\frac{1}{\frac{1}{n^k}}, \tag{17}$$

for all $n>n_2$. For $m\in\mathbb{N}$, we obtain

$$\begin{aligned} &d(x_n,x_{n+m},z)\\ &\le\alpha d(x_n,x_{n+m},x_{n+1})+\beta d(x_{n+m},z,x_{n+1})+\gamma d(z,x_n,x_{n+1})\\ &\le\max\{\alpha,\beta,\gamma\}\left(d(x_n,x_{n+m},x_{n+1})+d(x_{n+m},z,x_{n+1})+d(z,x_n,x_{n+1})\right)\\ &\le\max\{\alpha,\beta,\gamma\}\left(\frac{2}{n^{\frac{1}{k}}}+d(x_{n+m},z,x_{n+1})\right)\\ &\le\max\{\alpha,\beta,\gamma\}\left(\frac{2}{n^{\frac{1}{k}}}+\alpha d(x_{n+m},z,x_{n+2})+\beta d(z,x_{n+1},x_{n+2})+\gamma d(x_{n+1},x_{n+m},x_{n+2})\right) \end{aligned}$$

$$\leq \max\{\alpha,\beta,\gamma\}\left(\max\{\alpha,\beta,\gamma\}\frac{2}{n^{\frac{1}{k}}}+\max\{\alpha,\beta,\gamma\}\left(\frac{2}{(n+1)^{\frac{1}{k}}}+d(x_{n+m},z,x_{n+2})\right)\right)$$
$$=(\max\{\alpha,\beta,\gamma\})^2\left(\frac{2}{n^{\frac{1}{k}}}+\frac{2}{(n+1)^{\frac{1}{k}}}+d(x_{n+m},z,x_{n+2})\right)$$
$$\leq(\max\{\alpha,\beta,\gamma\})^{m+1}\left(\frac{2}{n^{\frac{1}{k}}}+\frac{2}{(n+1)^{\frac{1}{k}}}+\cdots+\frac{2}{(n+m)^{\frac{1}{k}}}\right)$$
$$=(\max\{\alpha,\beta,\gamma\})^{m+1}\sum_{j=n}^{n+m}\frac{2}{j^{\frac{1}{k}}}$$
$$\leq(\max\{\alpha,\beta,\gamma\})^{m+1}2\sum_{j=n}^{\infty}\frac{1}{j^{\frac{1}{k}}}.$$

Based on the convergence of the series $\sum_{j=n}^{\infty}\frac{1}{j^{\frac{1}{k}}}$, since $0<\frac{1}{j}<1$, we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since (X,d) is a complete generalized b_2-metric space there exist $u\in X$ such that $u=\lim_{n\to\infty}x_n$. We show that u is a fixed of T,

$$\theta(d(x_{n+1},Tu,z))$$
$$=\theta(d(Tx_n,Tu,z))$$
$$\leq[\theta\max\{(d(x_n,u,z),d(x_n,Tx_n,z),d(u,Tu,z)\}]^r$$

Taking the limits on both side we get

$$\theta(d(u,Tu,z))\leq[\theta(d(u,Tu,z))]^r \tag{18}$$

which is a contradication, unless $d(u,Tu,z)=0$, $u=Tu$. To prove the uniqueness, we assume that $Tu''=u''\neq u'=Tu'$:

$$\theta(d(u',u'',z))=\theta(d(Tu',Tu'',z))$$
$$\leq\left[\theta\left(\max\left\{d(u',u'',z),d(u',Tu',z),d(u'',Tu'',z)\right\}\right)\right]^r \tag{19}$$

It follows that

$$\theta(d(u',u'',z))\leq\left[\max\left\{\theta(d(u',u'',z))\right\}\right]^r,$$

which is a contradiction, since $0<r<1$, thus $u'=u''$. □

The following theorem provides a θ-type contraction of the principle result found in [12].

Theorem 2.5. *Let (X,d) be a complete generalized b_2-metric space and a mapping $T: X \to X$ satisfying:*

$$\begin{aligned}&\theta(d(Tx,Ty,z))\\ &\leq \alpha_1\theta(d(x,y,z))+\alpha_2\theta(d(x,Tx,z))+\alpha_3\theta(d(y,Ty,z))+\alpha_4\theta(d(x,Ty,z))\\ &+\alpha_5\theta(d(y,Tx,z))\end{aligned} \tag{20}$$

for all $x,y,z \in X$, where $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 < 1$, and
$\theta : [0,\infty) \to [0,\infty)$ is a function satisfying the following conditions:
(i) θ is continuous and non-decreasing.
(ii) for each sequence $\{t_n\} \subset (0,\infty)$,

$$\lim_{n\to\infty}\theta(t_n)=0 \iff \lim_{n\to\infty}t_n=0.$$

(iii) there exists $k \in (0,1)$ and $l \in (0,\infty)$ such that $\lim_{t\to 0^+}\frac{\theta(t)}{t^k}=l$.
Then T has a unique fixed point in X.

Proof. For $x,y,z \in X$,

$$\begin{aligned}&\theta(d(Tx,Ty,z))\\ &\leq \alpha_1\theta(d(x,y,z))+\alpha_2\theta(d(x,Tx,z))+\alpha_3\theta(d(y,Ty,z))+\alpha_4\theta(d(x,Ty,z))\\ &+\alpha_5\theta(d(y,Tx,z)).\end{aligned} \tag{21}$$

It follows that

$$\begin{aligned}&\theta(d(Ty,Tx,z))\\ &\leq \alpha_1\theta(d(y,x,z))+\alpha_2\theta(d(y,Ty,z))+\alpha_3\theta(d(x,Tx,z))+\alpha_4\theta(d(y,Tx,z))\\ &+\alpha_5\theta(d(x,Ty,z)).\end{aligned} \tag{22}$$

Adding (21) and (22) and by the symmetry of the metric, we get

$$\begin{aligned}&\theta(d(Tx,Ty,z))\\ &\leq \alpha_1\theta(d(x,y,z))+\frac{\alpha_2+\alpha_3}{2}\left[\theta(d(x,Tx,z))+\theta(d(y,Ty,z))\right]\end{aligned}$$

$$+\frac{\alpha_4+\alpha_5}{2}\left[\theta(d(x,Ty,z))+\theta(d(y,Tx,z))\right]. \tag{23}$$

Taking $y = Tx$ in (23), we get

$$\begin{aligned}
&\theta(d(Tx,T^2x,z))\\
&\le \alpha_1\theta(d(x,Tx,z))+\frac{\alpha_2+\alpha_3}{2}\left[\theta(d(x,Tx,z))+\theta(d(Tx,T^2x,z))\right]\\
&+\frac{\alpha_4+\alpha_5}{2}\left[\theta(d(x,T^2x,z))+\theta(d(Tx,Tx,z))\right].
\end{aligned} \tag{24}$$

Replacing z by Tx, (24) reduces to

$$\begin{aligned}
&\left(1-\frac{\alpha_2+\alpha_3}{2}\right)\theta(d(Tx,T^2x,z))\\
&\le\left(\alpha_1+\frac{\alpha_2+\alpha_3}{2}\right)\left[\theta(d(x,Tx,z))\right]\\
&+\frac{\alpha_4+\alpha_5}{2}\left[\theta(d(x,T^2x,z))\right].
\end{aligned} \tag{25}$$

It follows that

$$\begin{aligned}
&\theta(d(Tx,T^2x,z))\\
&\le\left(\frac{2\alpha_1+\alpha_2+\alpha_3}{2-\alpha_2-\alpha_3}\right)\left[\theta(d(x,Tx,z))\right]\\
&+\frac{\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3}\left[\theta(d(x,T^2x,z))\right]
\end{aligned} \tag{26}$$

Using the modified triangle inequality, we obtain

$$\theta(d(T^2x,x,z))\le\alpha\theta(d(T^2x,x,t))+\beta\theta(d(x,z,t))+\gamma\theta(d(z,T^2x,t)) \tag{27}$$

Taking $t = Tx$, inequality (27) becomes

$$\theta(d(T^2x,x,z))\le\alpha\theta(d(T^2x,x,Tx))+\beta\theta(d(x,z,Tx))+\gamma\theta(d(z,T^2x,Tx)). \tag{28}$$

Rearrange terms, and using inequality (26),we get

$$\begin{aligned}
&\theta(d(T^2x,x,z))-\alpha\theta(d(T^2x,x,Tx))-\beta\theta(d(x,z,Tx))\\
&\le\gamma\theta(d(z,T^2x,Tx))\\
&\le\gamma\left(\frac{2\alpha_1+\alpha_2+\alpha_3}{2-\alpha_2-\alpha_3}\right)\left[\theta(d(x,Tx,z))\right]
\end{aligned}$$

$$+\gamma\left(\frac{\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3}\right)\left[\theta(d(x,T^2x,z))\right]. \tag{29}$$

It follows that

$$\left(1-\gamma\left(\frac{\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3}\right)\right)\left[\theta(d(x,T^2x,z))\right]-\alpha\theta(d(T^2x,x,Tx))$$
$$\leq\left(\gamma\left(\frac{2\alpha_1+\alpha_2+\alpha_3}{2-\alpha_2-\alpha_3}\right)+\beta\right)\left[\theta(d(x,Tx,z))\right].$$

(30)

Replacing $z=Tx$ in (30),

$$\left(1-\gamma\left(\frac{\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3}\right)-\alpha\right)\left[\theta(d(T^2x,x,Tx))\right]$$
$$\leq\left(\gamma\left(\frac{2\alpha_1+\alpha_2+\alpha_3}{2-\alpha_2-\alpha_3}\right)+\beta\right)\left[\theta(d(x,Tx,Tx))\right]$$

(31)

It follows that

$$\left(1-\gamma\left(\frac{\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3}\right)-\alpha\right)\left[\theta(d(T^2x,x,z))\right]$$
$$\leq\left(\gamma\left(\frac{2\alpha_1+\alpha_2+\alpha_3}{2-\alpha_2-\alpha_3}\right)+\beta\right)\left[\theta(d(x,Tx,z))\right].$$

(32)

Substituting (32) into (26), we get

$$\theta(d(Tx,T^2x,z))$$
$$\leq\left(\frac{2\alpha_1+\alpha_2+\alpha_3}{2-\alpha_2-\alpha_3}+\frac{\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3}\left(\frac{\left(\gamma\left(\frac{2\alpha_1+\alpha_2+\alpha_3}{2-\alpha_2-\alpha_3}\right)+\beta\right)}{\left(1-\gamma\left(\frac{\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3}\right)-\alpha\right)}\right)\right)\theta(d(x,Tx,z))$$
$$\leq\mu\theta(d(x,Tx,z)), \tag{33}$$

where $\mu=\left(\frac{2\alpha_1+\alpha_2+\alpha_3}{2-\alpha_2-\alpha_3}+\frac{\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3}\left(\frac{\left(\gamma\left(\frac{2\alpha_1+\alpha_2+\alpha_3}{2-\alpha_2-\alpha_3}\right)+\beta\right)}{\left(1-\gamma\left(\frac{\alpha_4+\alpha_5}{2-\alpha_2-\alpha_3}\right)-\alpha\right)}\right)\right)<1$. Let $x_0\in X$ be arbitrary. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$, where $x_n=Tx_{n-1}=T^nx_0$ is a Cauchy sequence in X. If $x_{n-1}=x_n$ for some $n\in\mathbb{N}$ then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. To prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence we suppose that $x_{n-1}\neq x_n$ for $n\in\mathbb{N}$ and let $x=x_{n-1}$ in (33), then we get

$$\theta(d(x_n,x_{n+1},z)) = \theta(d(Tx_{n-1},T(Tx_{n-1}),z))$$
$$\leq \mu\theta(d(x_{n-1},Tx_{n-1},z)) = \mu\theta(d(x_{n-1},,x_n,z))$$

since $\mu < 1$ and θ is an increasing function it follows that $d(x_n,x_{n+1},z) \leq d(x_n,x_{n-1},z)$ thus $\{d(x_n,x_{n+1},z)\}_{n\in\mathbb{N}}$ is a decreasing sequence. Next we show that $d(x_n,x_{n+1},z) \to 0$ as $n \to \infty$. Recursively, using (33), we get

$$\begin{aligned}
&\theta\left(d(x_{n+1},x_n,z)\right)\\
&\leq \mu\theta\left(d(x_{n-1},x_n,z)\right)\\
&\leq (\mu)^2\theta\left(d(x_{n-1},x_{n-2},z)\right)\\
&\vdots\\
&\leq (\mu)^n\theta(d(x_0,x_1,z)),
\end{aligned} \tag{34}$$

since $\mu < 1$, it follows that as $n \to \infty$, we get $\theta\left(d(x_{n+1},x_n,z)\right) \to 0$ thus $d(x_{n+1},x_n,z) \to 0$. We now show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. It follows from the property of θ that from $k \in (0,1)$ and $l \in (0,\infty)$ that

$$\lim_{n\to\infty} \frac{\theta(d(x_n,x_{n+1},z))}{[d(x_n,x_{n+1},z)]^k} = l. \tag{35}$$

For $0 < \lambda < l$, by the definition of a limit there exists $n_1 \in \mathbb{N}$ such that

$$\lambda < \frac{\theta(d(x_n,x_{n+1},z))}{[d(x_n,x_{n+1},z)]^k}$$

$$\lambda[d(x_n,x_{n+1},z)]^k \leq \theta(d(x_n,x_{n+1},z)). \tag{36}$$

From inequality (34), we get

$$\begin{aligned}
n[d(x_n,x_{n+1},z)]^k &< n\lambda^{-1}(\theta(d(x_n,x_{n+1},z)))\\
&\leq n\lambda^{-1}(\mu)^n\theta(d(x_0,x_1,z))
\end{aligned} \tag{37}$$

for all $n > n_1$ which yields that

$$\lim_{n\to\infty} n[d(x_n,x_{n+1},z)]^k = 0. \tag{38}$$

Hence, there exists $n_2 \in \mathbb{N}$ such that

$$n[d(x_n, x_{n+1}, z)]^k \leq 1, \tag{39}$$

which implies that

$$d(x_n, x_{n+1}, z) \leq \frac{1}{\frac{1}{n^k}} \tag{40}$$

for all $n > n_2$. For $m \in \mathbb{N}$, we obtain

$$\begin{aligned}
& d(x_n, x_{n+m}, z) \\
& \leq \alpha d(x_n, x_{n+m}, x_{n+1}) + \beta d(x_{n+m}, z, x_{n+1}) + \gamma d(z, x_n, x_{n+1}) \\
& \leq \max\{\alpha, \beta, \gamma\} \left(d(x_n, x_{n+m}, x_{n+1}) + d(x_{n+m}, z, x_{n+1}) + d(z, x_n, x_{n+1}) \right) \\
& \leq \max\{\alpha, \beta, \gamma\} \left(\frac{2}{n^{\frac{1}{k}}} + d(x_{n+m}, z, x_{n+1}) \right) \\
& \leq \max\{\alpha, \beta, \gamma\} \left(\frac{2}{n^{\frac{1}{k}}} + \alpha d(x_{n+m}, z, x_{n+2}) + \beta d(z, x_{n+1}, x_{n+2}) + \gamma d(x_{n+1}, x_{n+m}, x_{n+2}) \right) \\
& \leq \max\{\alpha, \beta, \gamma\} \left(\max\{\alpha, \beta, \gamma\} \frac{2}{n^{\frac{1}{k}}} + \max\{\alpha, \beta, \gamma\} \left(\frac{2}{(n+1)^{\frac{1}{k}}} + d(x_{n+m}, z, x_{n+2}) \right) \right) \\
& = (\max\{\alpha, \beta, \gamma\})^2 \left(\frac{2}{n^{\frac{1}{k}}} + \frac{2}{(n+1)^{\frac{1}{k}}} + d(x_{n+m}, z, x_{n+2}) \right) \\
& \leq (\max\{\alpha, \beta, \gamma\})^{m+1} \left(\frac{2}{n^{\frac{1}{k}}} + \frac{2}{(n+1)^{\frac{1}{k}}} + \cdots + \frac{2}{(n+m)^{\frac{1}{k}}} \right) \\
& = (\max\{\alpha, \beta, \gamma\})^{m+1} \sum_{j=n}^{n+m} \frac{2}{j^{\frac{1}{k}}} \\
& \leq (\max\{\alpha, \beta, \gamma\})^{m+1} 2 \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{k}}}.
\end{aligned}$$

Based on the convergence of the series $\sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{k}}}$, since $0 < \frac{1}{j} < 1$, we conclude that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X. Since (X, d) is a complete generalized b_2-metric space there exist $u \in X$ such that $u = \lim_{n \to \infty} x_n$. We show that u is a fixed of T,

$$\begin{aligned}
& \theta(d(x_{n+1}, Tu, z)) \\
& = \theta(d(Tx_n, Tu, z))
\end{aligned}$$

$$\leq \alpha_1\theta(d(x,y,z))+\alpha_2\theta(d(x,Tx,z))+\alpha_3\theta(d(y,Ty,z))+\alpha_4\theta(d(x,Ty,z))$$
$$+\alpha_5\theta(d(y,Tx,z)).$$

Taking the limits on both sides, we get

$$\theta(d(u,Tu,z)) \leq (\alpha_3+\alpha_4)\theta(d(u,Tu,z)), \tag{41}$$

which is a contradication, since $\alpha_3+\alpha_4<1$, unless $d(u,Tu,z)=0$, $u=Tu$. To prove the uniqueness, we assume that $Tu''=u''$. Then using inequality (20)

$$\theta(d(u',u'',z)) = \theta(d(Tu',Tu'',z))$$
$$\leq \alpha_1\theta(d(u',u'',z))+\alpha_2\theta(d(u',Tu',z))+\alpha_3\theta(d(u'',Tu'',z))$$
$$+\alpha_4\theta(d(u',Tu'',z))+\alpha_5\theta(d(u'',Tu',z))$$
$$=\alpha_1\theta(d(u',u'',z))+\alpha_4\theta(d(u',Tu'',z))+\alpha_5\theta(d(u'',Tu',z))$$
$$=\alpha_1\theta(d(u',u'',z))+\alpha_4\theta(d(Tu',T^2u'',z))+\alpha_5\theta(d(Tu'',T^2u',z))$$
$$\leq \alpha_1\theta(d(u',u'',z))+\alpha_4\mu\theta(d(u',Tu',z))+\alpha_5\theta(d(u'',Tu'',z)).$$

It follows that

$$\theta(d(u',u'',z)) \leq \alpha_1\theta(d(u',u'',z)),$$

which is a contradiction, since $\alpha_1<1$, unless $d(u',u'',z)=0$, i.e., $u'=u''$. □

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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