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COMMON FIXED POINT RESULTS FOR FOUR MAPS SATISFYING CONTRACTIVE CONDITION IN MULTIPLICATIVE B-METRIC SPACES

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Abstract. In this paper, we discuss the unique common fixed point of two pair of weakly compatible mappings on a complete multiplicative b-metric space, which satisfies the following inequality:

$$d(Sx,Ty) \leq [k\{max\{d(Ax,By),d(Ax,Sx),d(By,Ty),d(Sx,By),d(Ax,Ty)\}\}]^{\lambda},$$

where A and S are weakly compatible, B and T also are weakly compatible. Our results improve and generalize the results of X. He et al. [3].

Keywords: multiplicative metric space; common fixed point; compatible mappings; weakly compatible mappings.
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1. INTRODUCTION

The study for the fixed point of contractive mappings is a famous topic in metric spaces. fixed point theory is, in fact, a simple, powerful, and useful tool for research area. In addition to an acceptable contraction condition, the metrical common fixed point theorems usually include

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constraints on commutativity, continuity, completeness, and appropriate containment of ranges of detailed maps. Since Banach [1] proved the Banach contraction principle in 1922.

Bashirov [2] introduced the usefullness of multiplicative calculus with some interesting applications. With the help of multiplicative absolute value function, they defined the multiplicative distance between two non-negative real numbers as well as between two positive square matrices. In 1976, Jungck [4] introduced the notion of commuting maps to prove the existence of a common fixed point theorems on a metric space

In 2012, Ozavsar et al.[6] investigate the multiplicative metric space by remarking its topological properties and introduced the concept of multiplicative contraction mapping and some fixed-point theorem of multiplicative, contraction mappings on multiplicative metric space. They recently proved a common fixed-point theorem for four self-mappings in multiplicative metric spaces.

We present some definition and result in common fixed-point theorem for compatible mappings in complete multiplicative b-metric space. For, we have introduced the notion of b-metric in multiplicative metric space.

2. PRELIMINARIES

Definition 2.1. [3] Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to R^+$ satisfying the following conditions:

(i) $d(x,y) \geq 1$ for all $x, y \in X$ and $d(x,y) = 1$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(iii) $d(x,y) \leq d(x,z)d(z,y)$ for all $x, y \in X$,

(multiplicative triangle inequality).

We use the following definition for our main result:

Definition 2.2. Let X be a nonempty set. A multiplicative b-metric is a mapping $d : X \times X \to R^+$ satisfying the following conditions:

$[B1]$ $d(x,y) \geq 1$ for all $x, y \in X$ and $d(x,y) = 1$ if and only if $x = y$;

$[B2]$ $d(x,y) = (y,x)$ for all $x, y \in X$;

$[B3]$ $d(x,y) \leq b.d(x,z).d(z,y)$ for all $x,y,z \in X$ (multiplicative triangle inequality), where $b \geq 1$.

Definition 2.3. [3] Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x) = \{y \mid d(x,y) < \varepsilon\}$, $\varepsilon > 1$, there exists a natural number N such that $n \geq N$, then $x_n \in B(x)$. The sequence $\{x_n\}$is said to be multiplicative converging to x, denoted by $x_n \to x$ $(n \to \infty)$.

Definition 2.4. [3] Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence is called a multiplicative Cauchy sequence if it holds that for all $\varepsilon > 1$, there exists $N \in \mathbf{N}$ such that $d(x_n,x_m) < \varepsilon$ for all $m,n > N$.

Definition 2.5. [3] We call a multiplicative metric space complete if every multiplicative Cauchy sequence in it is multiplicative convergence to $x \in X$.

Definition 2.6. [3] Suppose that S,T are two self-mappings of a multiplicative metric space (X,d); S,T are called commutative mappings if it holds that for all $x \in X$, $STx = TSx$.

Definition 2.7. [3] Suppose that S,T are two self-mappings of a multiplicative metric space (X,d); S,T are called weak commutative mappings if it holds that for all $x \in X$, $d(STx,TSx) \leq d(Sx,Tx)$.

Definition 2.8. [3] Let (X,d) be a multiplicative metric space. A mapping $f : X \to X$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0,1)$ such that $d(f(x_1), f(x_2)) \leq d(x_1,x_2)^\lambda$ for all $x,y \in X$.

Definition 2.9. [3] Suppose that f and g are two self-maps of a multiplicative metric space (X,d). The pair (fg) are called weakly compatible mappings if $fx = gx$, $x \in X$ implies $fg_x = gf_x$. That is, $d(fx,gx) = 1 \Rightarrow d(fgx,gfx) = 1$.

Proposition 2.10. [5] Let S and A be compatible mappings of a multiplicative metric space (X,d) into itself. If for some $t \in X$, then $SAt = SSt = AAT = ASt$.

Proposition 2.11. [5] Let S and A be compatible mappings of a multiplicative metric space (X,d) into itself. Suppose that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$

for some $t \in X$.

Then we have

1. $\lim_{n\to\infty} ASx_n = St$ if S is continuous at t;

2. $\lim_{n\to\infty} SAx_n = At$ if A is continuous at t;

3. $SAt = ASt$ and $St = At$ if S and A is continuous at t.

Proposition 2.12. [6] Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\} \to x$ $(n \to \infty)$ if and only if $d(x_n,x) \to 1$ $(n \to \infty)$.

Proposition 2.13. [6] Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n,x_m) \to 1$ $(n,m \to \infty)$.

Proposition 2.14. [6] Let (X,d_x) be a multiplicative metric space, $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \to x, y_n \to y$ $(n \to \infty)$, $x,y \in X$. Then $d(x_n,y_n) \to d(x,y)$ $(n \to \infty)$.

Bashirov [2] proved the result i 2008 [see Theorem 2.1]. In 2012, Ozavsar [6] proved the multiplicative contraction mapping [see Theorem 2.2] and in 2014, X. He. [3] Proved the fixed point result using weakly commuting in mappings [see Theorem 2.3].

3. Main Results

In this section, we prove some common fixed point results for generalized contaction mappings satisfying compatible conditions:

Theorem 3.1. *Let S,T,A and B be self-mappings of a complete multiplicative b-metric space X; which satisfy the following conditions:*

(i) $SX \subset BX, TX \subset AX$;

(ii) A and S are weakly compatible, B and T also are weakly compatible;

(iii) One of S, T , A and B is continuous;

(iv) $d(Sx,Ty) \leq [k\{max\{d(Ax,By),d(Ax,Sx),d(By,Ty),d(Sx,By),d(Ax,Ty)\}\}]^{\lambda}$

Then S, T, A and B have a unique common fixed point

where $b \geq 1$ such that $\lim_{m,n\to\infty}(kb)^{\frac{h}{1-h}^{(m-n)}} = 1$.

Proof. Since $SX \subset BX,\ and\ T(X) \subset AX$, for an arbitrary chosen point x_0 in X we obtain x_1 in X. For this $x_1 \in X$, we may obtain $x_2 \in X$; etc. Continuing in this way we obtain a sequence $\{y_n\} \in X$,

$\exists x_2 \in X$ such that $Tx_1 = Ax_2 = y_1, \dots$;

$\exists x_{2n+1} \in X$ such that $Bx_{2n+1} = y_{2n}$,

$\exists x_{2n+2} \in X$ such that $Tx_{2n+1} = Ax_{2n+2} = y_{2n+1}, \dots$; $\forall n = 0, 1, 2....\infty$.

define a sequence $\{y_n\} \in X$.

In order to show $\{y_n\}$ Cauchy sequence, let us put x_{2n} for x, and x_{2n+1} for y in condition (iv), and using (1) we have

$$
\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq [k\ (max\{d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Sx_{2n}, Bx_{2n+1}), \\
&\quad d(Ax_{2n}, Tx_{2n+1})\}]^{\lambda} \\
&= [k\,(max\,\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n+1})\})]^{\lambda} \\
&\leq [k\,(max\,\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\
&\quad 1, d(y_{2n-1}, y_{2n})\ .\ d(y_{2n}, y_{2n+1})\})]^{\lambda} \\
&\leq [k\,(max\,\{bd(y_{2n-1}, y_{2n})\ .\ d(y_{2n}, y_{2n+1}), bd(y_{2n-1}, y_{2n})\ .d(y_{2n}, y_{2n+1}), \\
&\quad bd(y_{2n-1}, y_{2n})\ .\ d(y_{2n}, y_{2n+1}), 1, bd(y_{2n-1}, y_{2n})\ .\ d(y_{2n}, y_{2n+1})\})]^{\lambda} \\
&\quad (using\ B3, as\ d(x,y) \leq bd(x,z).d(z,y) \forall x \in X) \\
&= [k\,(max\,\{bd(y_{2n-1}, y_{2n})\ .\ d(y_{2n}, y_{2n+1})\})]^{\lambda},\ (using\ B1, as\ d(x,y) \geq 1 \forall x \in X) \\
&\leq\ k^{\lambda} b^{\lambda} [d(y_{2n-1}, y_{2n})]^{\lambda}\ .\ [d(y_{2n}, y_{2n+1})]^{\lambda}
\end{aligned}
$$

$\implies d^{1-\lambda}(y_{2n}, y_{2n+1}) \leq\ k^{\lambda} b^{\lambda}\ .\ d^{\lambda}(y_{2n-1}, y_{2n})$

$\implies d(y_{2n}, y_{2n+1}) \leq\ (kb)^{\frac{\lambda}{1-\lambda}} d^{\frac{\lambda}{1-\lambda}}(y_{2n-1}, y_{2n})$.

Let $\frac{\lambda}{1-\lambda} = h,\ where\ \lambda \in (0, \frac{1}{2})\ then$

$d(y_{2n}, y_{2n+1}) \leq (kb)^h d^h(y_{2n-1}, y_{2n})$.

Similarly, putting $x = x_{2n+2},\ y = x_{2n+1}$ on (iv), we may obtain

$$
\begin{aligned}
&d(y_{2n+1},y_{2n+2})\\
&= d(Sx_{2n+2},Tx_{2n+1})\\
&\leq\ [k\max\{d(Ax_{2n+2},Bx_{2n+1}),d(Ax_{2n+2}Sx_{2n+2}),d(Bx_{2n+1},Tx_{2n+1}),d(Sx_{2n+2},Bx_{2n+1}),\\
&d(Ax_{2n+2},Tx_{2n+1})\}\}]^{\lambda}\\
&\leq [k\,(max\{d(y_{2n+1},y_{2n}),d(y_{2n+1},y_{2n+2}),d(y_{2n},y_{2n+1}),d(y_{2n+2},y_{2n}),d(y_{2n+1},y_{2n+1})\})]^{\lambda}\\
&\leq [k\,(max\{d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n+2}),d(y_{2n},y_{2n+1}).d(y_{2n},y_{2n+1}),\\
&\ d(y_{2n+1},y_{2n+2}),1)\})]^{\lambda}\\
&\leq [k\,(max\{bd(y_{2n},y_{2n+1}).d(y_{2n+1},y_{2n+2}),bd(y_{2n},y_{2n+1}).d(y_{2n+1},y_{2n+2}),bd(y_{2n},y_{2n+1}).\\
&\ d(y_{2n+1},y_{2n+2}),bd(y_{2n},y_{2n+1}).d(y_{2n+1},y_{2n+2}),1)\})]^{\lambda}\\
&= [k\,(max\{bd(y_{2n},y_{2n+1}).d(y_{2n+1},y_{2n+2})\})]^{\lambda}\\
&\leq\ \ k^{\lambda}b^{\lambda}[d(y_{2n},y_{2n+1})]^{\lambda}.[d(y_{2n+1},y_{2n+2})]^{\lambda}.
\end{aligned}
$$

This implies that $d^{1-\lambda}(y_{2n+1},y_{2n+2}) \leq\ k^{\lambda}b^{\lambda}\ .\ d^{\lambda}(y_{2n+1},y_{2n})$
$d(y_{2n+1},y_{2n+2}) \leq\ (kb)^{\frac{\lambda}{1-\lambda}}d^{\frac{\lambda}{1-\lambda}}(y_{2n+1},y_{2n})$.
Let $\frac{\lambda}{1-\lambda}=h,\ where\ \lambda\in(0,\frac{1}{2})$ then

$$d(y_{2n},y_{2n+1}) \leq (kb)^{h}.d^{h}(y_{2n-1},y_{2n}), \tag{3.1}$$

$$d(y_{2n+1},y_{2n+2}) \leq ((kb)^{h}.d^{h}(y_{2n},y_{2n+1}). \tag{3.2}$$

From (3.1) and (3.2), we obtain $d(y_n,y_{n+1}) \leq (kb)^{h}d^{h}(y_{n-1},y_n),\ n=1,2,3,...$ which inductively implies that

$$
\begin{aligned}
d(y_n,y_{n+1}) &\leq (kb)^{h}[(kb)^{h}d^{h}(y_{n-2},y_{n-1})]^{h}\\
&= (kb)^{h+h^2}[d^{h^2}(y_{n-2},y_{n-1})]\\
&\leq (kb)^{h+h^2}[(kb)^{h}d^{h}(y_{n-3},y_{n-2})]^{h^2}\\
&= (kb)^{h+h^2+h^3}[d^{h^3}(y_{n-3},y_{n-2})]\\
&\ \vdots
\end{aligned}
$$

$$\leq (kb)^{h+h^2+h^3+...+h^n}[d^{h^n}(y_0,y_1)]$$

$$\leq (kb)^{\frac{h}{1-h}}[d^{h^n}(y_0,y_1)],\ h+h^2+h^3+...+h^n \leq \frac{h}{1-h}.$$

Let $m,n \in \mathbb{N}$ such that $m \geq n$, then for Cauchy sequence, we have

$$\begin{aligned} d(y_m,y_n) &\leq d(y_m,y_{m-1}).d(y_{m-1},y_{m-2})...d(y_{n+1},y_n) \\ &\leq (kb)^{\frac{h}{1-h}}d^{h^{m-1}}(y_0,y_1).(kb)^{\frac{h}{1-h}}d^{h^{m-2}}(y_0,y_1)...(kb)^{\frac{h}{1-h}}d^{h^n}(y_0,y_1)] \\ &\leq \{(kb)^{\frac{h}{1-h}}\}^{(m-n)}\{d^{h^{[(m-1)+(m-2)+...+n]}}(y_0,y_1)\} \\ &= \{(kb)^{\frac{h}{1-h}}\}^{(m-n)}\{d^{h^{(m-n)[(m-1)-\frac{1}{2}(m-n-1)]}}(y_0,y_1)\} \\ &\leq \{(kb)^{\frac{h}{1-h}}\}^{(m-n)}d^{h^{m(m-n)}}(y_0,y_1),\ since\ (m-1)+(m-2)+...+n \leq m(m-n)\ where\ m>n, \\ &= \mathscr{B}d^{h^{m(m-n)}}(y_0,y_1),\ where\ \mathscr{B} = \{(kb)^{\frac{h}{1-h}}\}^{(m-n)} \to 1\ as\ n \to \infty. \end{aligned}$$

This implies that $d(y_m,y_n) \to 1\ as\ m,n \to \infty$. Hence $\{y_n\}$ is a multiplicative Cauchy sequence in X.

By the completeness of X, there exists $z \in X$ such that $y_n \to z\ as\ n \to \infty$.

We claim that z is a coincidence point of the pair A,S for, putting $x = z$ and $y = x_{2n+1}$ in the inequality (1) we have;

Moreover, since

$$\{Sx_{2n}\} = \{Bx_{2n+1}\} = \{y_{2n}\}\ and\ \{Tx_{2n+1}\} = \{Ax_{2n+2}\} = \{y_{2n+1}\},$$

are subsequence of $\{y_n\}$, so we obtain

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Ax_{2n+2} = z.$$

Taking condition (ii) and (iii) we obtain following cases:

Case 1: Suppose that A is continuous then

$$\lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} A^2x_{2n} = Az.$$

Since A and S are weakly compatible, then

$$d(ASx_{2n},SAx_{2n}) = d(Sx_{2n},Ax_{2n}).$$

Let $n \to \infty$, we get $\lim_{n\to\infty} d(SAx_{2n},Az) = d(z,z) = 1,\ i.e., \lim_{n\to\infty} SAx_{2n} = Az$.

Putting Ax_{2n} and x_{2n+1}, respectively for x and y in condition (iv) of Theorem 3.1, and using the continuity of A, we respectively obtain,

$$d(SAx_{2n},Tx_{2n+1}) \leq [k\{max\{d(A^2x_{2n},Bx_{2n+1}),d(A^2x_{2n},SAx_{2n}),$$
$$d(Bx_{2n+1},Tx_{2n+1}),d(SAx_{2n},Bx_{2n+1}),d(A^2x_{2n},Tx_{2n+1})\}\}]^{\lambda}.$$

Let $n \to \infty$, we can obtain

$$d(Az,z) \leq [k\{max\{d(Az,z),d(Az,Az),d(z,z),d(Az,z),d(Az,z)\}\}]^{\lambda}$$
$$= [k\{max\{d(Az,z),1\}\}]^{\lambda}$$
$$(dropping\ 1\ as\ d(x,y) \geq 1\ \forall x,y \in X\ in\ the\ multiplicative\ metric\ space)$$
$$= k^{\lambda}.d^{\lambda}(Az,z).$$

This implies that $d(Az,z) = 1$ i.e., $Az = z$,

$$d(Sz,Tx_{2n+1})$$
$$\leq [k\{max\{d(Az,,Bx_{2n+1}),d(Az,Sz),d(Bx_{2n+1},Tx_{2n+1}),d(Sz,Bx_{2n+1}),d(Az,Tx_{2n+1})\}\}]^{\lambda}.$$

Let $n \to \infty$ we can obtain

$$d(Sz,z) \leq [k\{max\{d(z,z),d(z,Sz),d(z,z),d(Sz,z),d(z,z)\}\}]^{\lambda}$$
$$= [k\{max\{d(Sz,z),1\}\}]^{\lambda}$$
$$(dropping\ 1\ as\ d(x,y) \geq 1\ \forall x,y \in X\ in\ the\ multiplicative\ metric\ space)$$
$$= k^{\lambda}.d^{\lambda}(Sz,z),$$

This implies that $d(Sz,z) = 1$,

i.e. Sz=z. On the other hand,

since $z = Sz \in SX \subseteq BX$, so $\exists z^* \in X$ such that $z = Sz = Bz^*$

$$d(z,Tz^*) = d(Sz,Tz^*)$$
$$\leq [k\{max\{d(Az,Bz^*),d(Az,Sz),d(Bz^*,Tz^*),d(Sz,Bz^*),d(Az,Tz^*)\}\}]^{\lambda}$$
$$= [k\{max\{d(z,Tz^*),1\}\}]^{\lambda}$$
$$= k^{\lambda}.d^{\lambda}(z,Tz^*),$$

which implies $d(z,Tz^*)=1$ i.e., $Tz^*=z$.

Since B and T are weakly compatible mappings then

$$d(Bz,Tz)=d(BTz^*,TBz^*)=d(Bz^*,Tz^*)=d(z,z)=1,$$

so $Bz=Tz$,

$$d(Sx_{2n},Tz)\leq [k\{max\{d(Ax_{2n},Bz),d(Ax_{2n},Sx_{2n}),d(Bz,Tz),d(Sx_{2n},Bz),d(Ax_{2n},Tz)\}\}]^{\lambda}.$$

Let $n\to\infty$ we can obtain

$$\begin{aligned}d(z,Tz)&\leq [k\{max\{d(z,Tz),d(z,z),d(Tz,Tz),d(z,Tz),d(z,Tz)\}\}]^{\lambda}\\&=[k\{max\{d(z,Tz),1\}\}]^{\lambda}\\&=k^{\lambda}.d^{\lambda}(z,Tz).\end{aligned}$$

which implies $d(Tz,z)=1$ i.e., $Tz=z$. So z is a common fixed point of S, T, A and B.

Case 2: Suppose that B is continuous , we can obtain the same result by the way of case 1.

Case 3: Suppose that S is continuous then $\lim_{n\to\infty}SAx_{2n}=\lim_{n\to\infty}S^2x_{2n}=Sz$.

Since A and S are weakly compatible then $d(ASx_{2n},SAx_{2n})=d(Sx_{2n},Ax_{2n})$.

Let $n\to\infty$ we get then $\lim_{n\to\infty}(ASx_{2n},Sz)=d(z,z)=1,\ i.e.,\ \lim_{n\to\infty}ASx_{2n}=Sz$,

$$\begin{aligned}d(S^2x_{2n},Tx_{2n+1})\leq [k\{max\{&d(ASx_{2n},Bx_{2n+1}),d(ASx_{2n},S^2x_{2n}),d(Bx_{2n+1},Tx_{2n+1}),\\&d(S^2x_{2n},Bx_{2n+1}),d(ASx_{2n},Tx_{2n+1})\}\}]^{\lambda}.\end{aligned}$$

Let $n\to\infty$ we can obtain

$$\begin{aligned}d(Sz,z)&\leq [k\{max\{d(Sz,z),d(Sz,Sz),d(z,z),d(Sz,z),d(Sz,z)\}\}]^{\lambda}\\&=[k\{max\{d(Sz,z),1\}\}]^{\lambda}\\&=k^{\lambda}d^{\lambda}(Sz,z),\end{aligned}$$

which implies $d(Sz,z)=1$ i.e., $Sz=z$.

$z=Sz\in SX\subseteq BX,\ so\ \exists z^*\in X$ such that $z=Bz^*$

$$d(S^2x_{2n},Tz^*)\leq [k\{max\{d(ASx_{2n},Bz^*),d(ASx_{2n},S^2x_{2n}),d(Bz^*,Tz^*),d(S^2x_{2n},Bz^*),d(ASx_{2n},Tz^*)\}\}]^{\lambda}$$

$$
\begin{aligned}
d(z,Tz^*) &= d(Sz,Tz^*)\\
&\leq k\{max\{d(Sz,Bz^*),d(Sz,Sz),d(z,Tz^*),d(Sz,z),d(Sz,Tz^*)\}\}]^{\lambda}\\
&= [k\{max\{d(z,Tz^*),1\}\}]^{\lambda}\\
&= k^{\lambda}.d^{\lambda}(z,Tz^*),
\end{aligned}
$$

which implies that $d(z,Tz^*) = 1,\ i.e.,\ Tz^* = z = Bz^*$.

Since T and B are weakly compatible,then

$d(Tz,Bz) = d(TBz^*,BTz^*) = d(Tz^*,Bz^*) = d(z,z) = 1$, so $Bz = Tz$,

$$
d(Sx_{2n},Tz) \leq [k\{max\{d(Ax_{2n},Bz),d(Ax_{2n},Sx_{2n}),d(Bz,Tz),d(Sx_{2n},Bz),d(Ax_{2n},Tz)\}\}]^{\lambda}.
$$

Let $n \to \infty$ we can obtain

$$
\begin{aligned}
d(z,Tz) &\leq [k\{max\{d(z,Bz),d(z,z),d(Bz,Tz),d(z,Tz),d(z,Bz)\}\}]^{\lambda}\\
&= [k\{max\{d(z,Tz),1\}]^{\lambda}\\
&= k^{\lambda}.d^{\lambda}(z,Tz).
\end{aligned}
$$

which implies $d(z,Tz) = 1$ i.e., $Tz = z$.

$z = Tz \in TX \subseteq AX$, so $\exists z^{**} \in X$, such that $z = Az^{**}$

$$
\begin{aligned}
d(Sz^{**},z) &= d(Sz^{**},Tz)\\
&\leq [k\{max\{d(Az^{**},Bz),d(Az^{**},Sz^{**}),d(Bz,Tz),d(Sz^{**},Bz),d(Az^{**},Tz)\}\}]^{\lambda}\\
&= [k\{max\{d(z,z),d(z,Sz^{**}),d(z,z),d(Sz^{**},z),d(z,z)\}\}]^{\lambda}\\
&= [k\{max\{d(Sz^{**},z),1\}\}]^{\lambda}\\
&= k^{\lambda}.d^{\lambda}(Sz^{**},z).
\end{aligned}
$$

This implies that $d(Sz^{**},z) = 1$ i.e., $Sz^{**} = z$.

Since S and A are weakly compatible, then

$d(Az,Sz) = d(ASz^{**},SAz^{**}) = d(Az^{**},Sz^{**}) = d(z,z) = 1$, so $Az = Sz$,

We obtain $Sz = Tz = Az = Bz = z$,

so z is common fixed point of S, T, A and B.

Case 4: Suppose that T is continuous, we can obtain the same result by the way of case 3.

In addition we prove that S,T, A and B have a unique common fixed point. suppose that $w \in X$ is also a common fixed point of S,T,A and B, then we obtain

$$\begin{aligned} d(z,w) &= d(Sz,Tw) \\ &\leq [k\{max\{d(Az,Bw),d(Az,Sz),d(Bw,Tw),d(Sz,Bw),d(Az,Tw)\}\}^{\lambda} \\ &= [k\{max\{d(z,w),1\}\}]^{\lambda} \\ &= k^{\lambda}.d^{\lambda}(z,w). \end{aligned}$$

This implies that d(z, w)=1 and so w=z.

Therefore z is a unique common fixed point of $A,B,S,T \subset X$. □

Corollary 3.2. *Let X,d be a complete multiplicative b-metric space S,T,A and B be four mappings of X into itself.*

Suppose that there exists $\lambda \in (0,\frac{1}{2})$ $\forall x,y \in X$,

such that $S(X) \subset B(X), T(X) \subset A(X)$ and

$$d(S^p x,T^q y) \leq k^{\lambda}\{max\{d^{\lambda}(Ax,By),d^{\lambda}(Ax,S^p x),d^{\lambda}(By,T^q y),d^{\lambda}(S^p x,By),d^{\lambda}(Ax,T^q y)\}\},$$

Assume one of the following conditions is satisfied:

(a) either A or S is continuous the pair S,A and the pair T,B are commuting mappings;

(b) either A, B, S or T is continuous;

Then $S,T,AandB$ have a unique common fixed point

where $b \geq 1$ such that $\lim_{n\to\infty} b^n = B \ < 1$.

Corollary 3.3. *Let X,d be a complete multiplicative b-metric space $S,T,AandB$ be four mappings of X into itself.*

Suppose that there exists $\lambda \in (0,\frac{1}{2})$ $\forall x,y \in X$,

such that $S(X) \subset B(X), T(X) \subset A(X)$ and

$$d(Sx,Ty) \leq k^{\lambda}\{max\{d^{\lambda}(Ax,By),d^{\lambda}(Ax,Sx),d^{\lambda}(By,Ty),d^{\lambda}(Sx,By),d^{\lambda}(Ax,Ty)\}\},$$

Assume one of the following conditions is satisfied:

(a) either A or S is continuous the pair S,A and the pair T,B are weakly compatible;

(b) either B or T is continuous the pair (T,B) and the pair (S,A) are weakly compatible.

Then $S,T,AandB$ have a unique common fixed point
where $b \geq 1$ such that $\lim_{n\to\infty} b^n = B\ < 1$.

Corollary 3.4. *Let X,d be a complete multiplicative b-metric space S,T,A and B be four mappings of X into itself.*
Suppose that there exists $\lambda \in (0,\frac{1}{2})\ \forall x,y \in X$,
such that $S(X) \subset B(X), T(X) \subset A(X)$ and

$$d(S^px,T^qy) \leq k^{\lambda}\{max\{d^{\lambda}(Ax,By)+d^{\lambda}(Ax,S^px)+d^{\lambda}(By,T^qy)+d^{\lambda}(S^px,By)+d^{\lambda}(Ax,T^qy)\}\}$$

for all $x,y \in X$. Here $a_1,a_2,a_3,a_4,a_5 \geq 0$ and $0 \leq a_1+a_2+a_3+a_4+a_5 \leq 1$ Assume one of the following conditions is satisfied:
(a) either A or S is continuous the pair S,A and the pair T,B are commuting mappings;
(b) either A, B, S or T is continuous;
Then $S,T,AandB$ have a unique common fixed point.

Corollary 3.5. *Let X,d be a complete multiplicative b-metric space $S,T,AandB$ be four mappings of X into itself.*
Suppose that there exists $\lambda \in (0,\frac{1}{2})\ \forall x,y \in X$,
such that $S(X) \subset B(X), T(X) \subset A(X)$ and

$$d(Sx,Ty) \leq k^{\lambda}\{max\ a_1d^{\lambda}(Ax,By)+a_2d^{\lambda}(Ax,Sx)+a_3d^{\lambda}(By,Ty)+a_4d^{\lambda}(Sx,By)+a_5d^{\lambda}(Ax,Ty)\}\}$$

for all $x,y \in X$. Here $a_1,a_2,a_3,a_4,a_5 \geq 0$ and $0 \leq a_1+a_2+a_3+a_4+a_5 \leq 1$. Assume one of the following conditions is satisfied:
(a) either A or S is continuous the pair (S,A) and the pair (T,B) are weakly compatible;
(b) either B or T is continuous the pair (T,B) and the pair (S,A) are weakly compatible.
Then S,T,A and B have a unique common fixed point.

Corollary 3.6. *Let (X,d) be a complete multiplicative b-metric space S,T,A and B be four mappings of X into itself.*
Suppose that there exists $\lambda \in (0,\frac{1}{2})$ and $p,q \in Z^+$
$d(T^px,T^qy) \in k^{\lambda}\{max\ (d^{\lambda}(x,y),d^{\lambda}(x,T^px),d^{\lambda}(y,T^qy),d^{\lambda}(T^px,y),d^{\lambda}(x,T^qy)\}\})$
for all $x,y \in X$. Then T have a unique fixed point.

Corollary 3.7. *Let (X,d) be a complete multiplicative b-metric space S,T,A and B be four mappings of X into itself.*
Suppose that there exists $\lambda \in (0,\frac{1}{2})$ such that
$d(Tx,Ty) \leq k^{\lambda}\{max\ (a_1 d^{\lambda}(x,y) + a_2 d^{\lambda}(x,Tx) + a_3 d^{\lambda}(y,Ty) + a_4 d^{\lambda}(Tx,y) + a_5 d^{\lambda}(x,Ty))\}\}$
for all $x,y \in X$. Here $a_1,a_2,a_3,a_4,a_5 \geq 0$ and $0 \leq a_1+a_2+a_3+a_4+a_5 \leq 1$.
Then T have a unique fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

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