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WEAK SOLUTIONS OF A FRACTIONAL ORDERS QUADRATIC FUNCTIONAL INTEGRO-DIFFERENTIAL INCLUSION IN A REFLEXIVE BANACH SPACE

AHMED M. A. EL-SAYED[1], NESREEN F. M. EL-HADDAD[2,*]

[1]Department of Mathematics, Alexandria University, Alexandria, Egypt

[2]Department of Mathematics, Damanhour University, Behera, Egypt

Abstract. Let $\mathscr{E}$ be a reflexive Banach space. In this paper we are concerned with the existence of weak solutions $x \in \mathscr{C}(\mathscr{I},\mathscr{E})$ of the quadratic functional integro-differential inclusion of fractional orders, $\alpha,\beta \in (0,1)$,

$$^{R}D^{\alpha}(x(t)-x(0)) \in G(t, g_1(t,x(t)) I^{\beta} g_2(t, {}^{R}D^{\beta}x(t)), \ t \in [0,\mathscr{T}]$$

with the initial condition, $x(0) = x_0, \ x_0 \in \mathscr{E}$, in the reflexive Banach space $\mathscr{E}$ under the assumption that the multi-valued function G satisfy Lipschitz condition in $\mathscr{E}$. The main tool applied in this work is O'Regan fixed point theorem. We investigate qualitative properties of the solution of this inclusion such as the continuous dependence on the set of selections S_G and the continuous dependence on the data x_0. Here, we prove two new theorems on the mentioned properties of the solution of the considered quadratic functional integro-differential inclusion of fractional orders. We additionally provide an example given as numerical application to illustrate our main result.

Keywords: multi-valued function; weak solution; quadratic functional integro-differential inclusion; Lipschitz condition; reflexive Banach space.

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1. Introduction

Let $\mathscr{I} = [0,\mathscr{T}]$ and let $\mathscr{E}$ be a reflexive Banach space with norm $\|.\|_{\mathscr{E}}$ and dual $\mathscr{E}^*$. Denote by $\mathscr{C}(\mathscr{I},\mathscr{E})$ the Banach space of all continuous functions $x : \mathscr{I} \to \mathscr{E}$.

*Corresponding author

E-mail addresses: nesreen_fawzy20@sci.dmu.edu.eg, nesreen_fawzy20@yahoo.com

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Consider the quadratic functional integro-differential inclusion of fractional orders, $\alpha, \beta \in (0,1)$,

$$ {}^{R}D^{\alpha}(x(t)-x(0)) \in G(t, g_1(t, x(t))I^{\beta} g_2(t, {}^{R}D^{\beta} x(t)), \ t \in \mathscr{I} \tag{1.1} $$

with the initial condition

$$ x(0) = x_0, \ x_0 \in \mathscr{E} \tag{1.2} $$

where $G : \mathscr{I} \times \mathscr{E} \to \chi(\mathscr{E})$ is a nonlinear multi-valued mapping and $\chi(\mathscr{E})$ denote the power set of nonempty subsets of the Banach space $\mathscr{E}$.

Recently, integral and integro-differential equations were studied by some authors (see [20]-[22]). Indeed a functional inclusions and differential inclusions have been investigated by some authors and there are many results concerning the existence of solutions and their properties of these problems (see [3], [8]-[10] and [14]-[15]). Also, a functional integral inclusion was discussed by B.C. Dhage and D. O'Regan (see [17], [18] and [23]) they proved the existence of extremal solutions using Caratheodory's conditions on the multi-valued function. However, in this article, we establish our results using Lipschitz condition on the multi-valued function. The existence of weak solutions of the integral equations were studied by a number of authors such as (see for instance[1]-[2], [5], [12] and [16]).

In [19], H. A. H. Salem is devoted to proving the existence of weak solutions to some quadratic integral equations of fractional type in a reflexive Banach space relative to the weak topology.

Here we study the existence of weak solutions $x \in \mathscr{C}(\mathscr{I}, \mathscr{E})$ of the quadratic functional integro-differential inclusion of fractional orders (1.1) in the reflexive Banach space $\mathscr{E}$ with the initial condition (1.2) using O'Regan fixed point theorem. We prove the existence theorem of that inclusion in the space $\mathscr{C}(\mathscr{I}, \mathscr{E})$ under the assumption that the multi-valued function G satisfy Lipschitz condition in $\mathscr{E}$.

We investigate the continuous dependence of the solution on the set of selections S_G and on the data x_0. We additionally provide an example given as numerical application to illustrate our main result.

2. Preliminaries

Here, we present some auxiliary results that will be needed in this work.
let $\mathscr{E}$ be a Banach space and let $x : \mathscr{I} \to \mathscr{E}$, then
(1) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in \mathscr{I}$ if for every $\phi \in \mathscr{E}^*$, $\phi(x(.))$ is continuous (measurable) at t_0.
(2) A function $h : \mathscr{E} \to \mathscr{E}$ is said to be weakly sequentially continuous if h maps weakly convergent sequence in $\mathscr{E}$ to weakly convergent sequence in $\mathscr{E}$.
If x is weakly continuous on $\mathscr{I}$, then x is strongly measurable and hence weakly measurable (see[4] and[7]). Note that in reflexive Banach spaces weakly measurable functions are pettis integrable (see[7] and [13] for the definition) if and only if $\phi(x(.))$ is Lebesgue integrable on $\mathscr{I}$ for every $\phi \in \mathscr{E}^*$.
Now we state a fixed point theorem and some propositions which will be used in the sequel (see[11]).

Theorem 2.1. *(O'Regan fixed point theorem)*
Let $\mathscr{E}$ be a Banach space and let $\mathscr{Q}$ be a nonempty, bounded, closed and convex subset of the space $\mathscr{C}(\mathscr{I},\mathscr{E}))$ and let $\mathscr{A} : \mathscr{Q} \to \mathscr{Q}$ be a weakly sequentially continuous and assume that $\mathscr{A}\mathscr{Q}(t)$ is relatively weakly compact in $\mathscr{E}$ for each $t \in \mathscr{I}$. Then $\mathscr{A}$ has a fixed point in the set $\mathscr{Q}$.

Proposition 2.2. *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Proposition 2.3. *Let $\mathscr{E}$ be a normed space with $y \neq 0$. Then there exists $\phi \in \mathscr{E}^*$ with $\|\phi\| = 1$ and $\|y\| = \phi(y)$.*

Definition 2.4. [6] A multi-valued map G from $\mathscr{I} \times \mathscr{E}$ to the family of all nonempty closed subsets of $\mathscr{E}$ is called Lipschitzian if there exists $\mathscr{L} > 0$ such that for all $t_1, t_2 \in \mathscr{I}$ and all $x_1, x_2 \in \mathscr{E}$, we have

$$\mathscr{H}(G(t_1,x_1), G(t_2,x_2)) \leq \mathscr{L}(|t_1 - t_2| + \|x_1 - x_2\|_{\mathscr{E}})$$

where $\mathscr{H}(\mathscr{B},\mathscr{D})$ is the Hausdorff metric between the two subsets $\mathscr{B},\mathscr{D} \in \mathscr{I} \times \mathscr{E}$.

Denote $S_G = Lip(\mathscr{I}, \mathscr{E})$ be the set of all Lipschitz selections of G.

3. Main Results

In this section, we present our main result by proving the existence of weak solutions $x \in \mathscr{C}(\mathscr{I}, \mathscr{E})$ of the quadratic functional integro-differential inclusion of fractional orders (1.1) in the reflexive Banach space $\mathscr{E}$ with the initial condition (1.2), under the assumption that the multi-valued function G satisfy Lipschitz condition.

Definition 3.1. By a weak solution of the quadratic functional integro-differential inclusion of fractional orders (1.1) in the reflexive Banach space $\mathscr{E}$ with the initial condition (1.2) we mean a single-valued function $x \in \mathscr{C}(\mathscr{I}, \mathscr{E})$ this function satisfies (1.1) and (1.2).

Consider now the problem (1.1) and (1.2) under the following assumptions:

(H1) The set $G(t,x)$ is compact and convex for all $(t,x) \in \mathscr{I} \times \mathscr{E}$.

(H2) The multi-valued map G is Lipschitzian with a Lipschitz constant $\mathscr{L} > 0$ such that

$$\mathscr{H}(G(t_1,x_1), G(t_2,x_2)) \leq \mathscr{L}(|t_1 - t_2| + \|x_1 - x_2\|_{\mathscr{E}})$$

for all $t_1, t_2 \in \mathscr{I}$ and $x_1, x_2 \in \mathscr{E}$ where $\mathscr{H}(\mathscr{B}, \mathscr{D})$ is the Hausdorff metric between the two subsets $\mathscr{B}, \mathscr{D} \in \mathscr{I} \times \mathscr{E}$.

(H3) The set of all Lipschitz selections S_G of the multi-valued function G is nonempty.

(H4) $g_i(t,.)$, $i = 1,2$ is weakly sequentially continuous for each $t \in \mathscr{I}$.

(H5) $g_i(.,x)$, $i = 1,2$ is weakly measurable on $\mathscr{I}$ for every $x \in \mathscr{E}$.

(H6) There exist functions $a_i \in \mathscr{C}(\mathscr{I})$, $i = 1,2$ and constants b_i, $i = 1,2 > 0$ such that

$$\|g_i(t,x)\|_{\mathscr{E}} \leq |a_i(t)| + b_i \|x\|_{\mathscr{E}}, \ \forall t \in \mathscr{I}, \ i = 1,2.$$

(H7)The function $g_1 : \mathscr{I} \times \mathscr{E} \to \mathscr{E}$ satisfy Lipschitz condition.

i.e.

$$\|g_1(t_2, z(t_2)) - g_1(t_1, z(t_1))\|_{\mathscr{E}} \leq b_1\{|t_2 - t_1| + \|z(t_2) - z(t_1)\|_{\mathscr{E}}\}.$$

(H8) There exists a positive real number r of the algebraic equation

$\mathscr{L} b_1 b_2 K_1^2 r^2 + [2\mathscr{L} b_1 b_2 \|x_0\|_{\mathscr{E}} K_1 + \mathscr{L} b_1 K_1 K + \mathscr{L}\|a_1\|_{\mathscr{C}} b_2 k_1 - 1]r + [\mathscr{L}\|a_1\|_{\mathscr{C}} K + (\mathscr{L}\|a_1\|_{\mathscr{C}} b_2 + \mathscr{L} b_1 K)\|x_0\|_{\mathscr{E}} + \mathscr{L} b_1 b_2 (\|x_0\|_{\mathscr{E}})^2 + \mathscr{M}] = 0.$

Remark 3.2. From assumptions (H2) and (H3), there exists a Lipschitz selection $g \in S_G$ such that

$$\|g(t_2,x) - g(t_1,y)\|_{\mathscr{E}} \le \mathscr{L}(|t_2 - t_1| + \|x - y\|_{\mathscr{E}}),$$

for every $t_1, t_2 \in \mathscr{I}$ and $x, y \in \mathscr{E}$.
And this selection satisfies the quadratic functional integro-differential equation of fractional orders

$$^{R}D^{\alpha}(x(t) - x(0)) = g(t, g_1(t, x(t))I^{\beta} g_2(t, {}^{R}D^{\beta}x(t)), \ t \in \mathscr{I} \tag{3.1}$$

with the initial condition (1.2).
Then the solution of the problem (1.2) and (3.1), if it exists, is a solution of the problem (1.1) and (1.2).

Now, we study the existence of a weak solution of the problem (1.2) and (3.1).

Theorem 3.3. *Let the assumptions (H1)-(H8) be satisfied. If the weak solution* $x \in \mathscr{C}(\mathscr{I}, \mathscr{E})$ *of the problem (1.2) and (3.1) exist, then it can be represented by*

$$x(t) = x_0 + I^{\alpha} y(t),$$

where

$$y(t) = g(t, g_1(t, (x_0 + I^{\alpha} y(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t))), \ t \in \mathscr{I}.$$

Proof. Let

$$^{R}D^{\alpha}(x(t) - x(0)) = y(t) \in \mathscr{C}(\mathscr{I}, \mathscr{E})$$

Then from the definition of Riemann-Liouville fractional order integral of order $\alpha \in (0,1)$, we have

$$\frac{d}{dt} I^{1-\alpha}(x(t) - x(0)) = y(t)$$

Integrating both-sides, we get

$$I^{1-\alpha}(x(t) - x(0)) - \ell = I y(t)$$

operating by I^{α} for both-sides, we get

$$\begin{aligned} I(x(t)-x(0)) &= \ell I^{\alpha}.1 + I^{1+\alpha}y(t) \\ &= \ell \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}.1ds + I^{1+\alpha}y(t) \\ &= \frac{\ell}{\Gamma(\alpha+1)}t^{\alpha} + I^{1+\alpha}y(t). \end{aligned}$$

Differentiate both-sides with respect to t, we have

$$x(t) = x(0) + \frac{\ell}{\Gamma(\alpha)}t^{\alpha-1} + I^{\alpha}y(t).$$

At $t = 0$, we get $\ell = 0$.

Hence, we have

$$x(t) = x_0 + I^{\alpha}y(t). \tag{3.2}$$

Operating by $I^{1-\beta}$ for both-sides, then we have

$$I^{1-\beta}x(t) = I^{1-\beta}x_0 + I^{1+\alpha-\beta}y(t).$$

From the definition of fractional order integral, then

$$\begin{aligned} I^{1-\beta}x(t) &= \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}x_0 ds + I^{1+\alpha-\beta}y(t) \\ &= \frac{t^{1-\beta}}{\Gamma(2-\beta)}x_0 + I^{1+\alpha-\beta}y(t). \end{aligned}$$

Differentiate with respect to t, then

$$\begin{aligned} \frac{d}{dt}I^{1-\beta}x(t) &= \frac{d}{dt}\frac{t^{1-\beta}}{\Gamma(2-\beta)}x_0 + \frac{d}{dt}I^{1+\alpha-\beta}y(t) \\ &= \frac{t^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(t). \end{aligned}$$

From the definition of Riemann-Liouville of fractional order derivative, we have

$$^{R}D^{\beta}x(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(t).$$

And from equation (3.1), we have

$$y(t) = g(t, g_1(t, (x_0 + I^{\alpha}y(t)))I^{\beta}g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(t))), \ t \in \mathscr{I}. \tag{3.3}$$

Now, for the existence of a weak solution of the problem (1.2) and (3.1) we have the following theorem.

Theorem 3.4. *Consider the assumptions (H1)-(H8) hold. Then there exists at least one weak solution* $\mathrm{x} \in \mathscr{C}(\mathscr{I},\mathscr{E})$ *of the problem (1.2) and (3.1).*

Proof. Define the operator $\mathscr{A}$ by

$$\mathscr{A}\mathrm{y(t)} = \mathrm{g(t, g_1((x_0 + I^{\alpha}y(t)))I^{\beta}g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(t))), \ t} \in \mathscr{I}.$$

Let the set $\mathscr{Q}_{\mathrm{r}}$ defined by

$$\mathscr{Q}_{\mathrm{r}} = \{\mathrm{y} \in \mathscr{C}(\mathscr{I},\mathscr{E}), \ \|\mathrm{y}\|_{\mathscr{C}} \leq \mathrm{r}\};$$

$$\begin{aligned} \mathrm{r} &= \mathscr{L}\|\mathrm{a_1}\|_{\mathscr{C}}\mathrm{K} + \mathscr{L}\|\mathrm{a_1}\|_{\mathscr{C}}\mathrm{b_2}\|\mathrm{x_0}\|_{\mathscr{E}} + \mathscr{L}\|\mathrm{a_1}\|_{\mathscr{C}}\mathrm{b_2 r K_1} + \mathscr{L}\mathrm{b_1}\|\mathrm{x_0}\|_{\mathscr{E}}\mathrm{K} + \mathscr{L}\mathrm{b_1 b_2}(\|\mathrm{x_0}\|_{\mathscr{E}})^2 \\ &+ 2\mathscr{L}\mathrm{b_1 b_2 r K_1}\|\mathrm{x_0}\|_{\mathscr{E}} + \mathscr{L}\mathrm{b_1 r K_1 K} + \mathscr{L}\mathrm{b_1 b_2 r^2 K_1^2} + \mathscr{M}. \end{aligned}$$

Let $\mathrm{y} \in \mathscr{Q}_{\mathrm{r}}$ be arbitrary, then we have from proposition 2.3

$$\begin{aligned} \| & \mathscr{A}\mathrm{y(t)}\|_{\mathscr{E}} = \phi(\mathscr{A}\mathrm{y(t)}) \\ &= \phi(\mathrm{g(t, g_1(t,(x_0 + I^{\alpha}y(t)))I^{\beta}g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(t))))}) \\ &= \|\mathrm{g(t, g_1(t,(x_0 + I^{\alpha}y(t)))I^{\beta}g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(t)))}\|_{\mathscr{E}} \\ &\leq \mathscr{L}\|\mathrm{g_1(t,(x_0 + I^{\alpha}y(t)))I^{\beta}g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(t))}\|_{\mathscr{E}} + \|\mathrm{g(t,0)}\|_{\mathscr{E}} \\ &\leq \mathscr{L}\|\mathrm{g_1(t,(x_0 + I^{\alpha}y(t)))}\|_{\mathscr{E}} I^{\beta}|\mathrm{g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(t))}| + \|\mathrm{g(t,0)}\|_{\mathscr{E}} \\ &\leq \mathscr{L}\{|\mathrm{a_1(t)}| + \mathrm{b_1}(\|\mathrm{x_0}\|_{\mathscr{E}} + \mathrm{I}^{\alpha}\|\mathrm{y(t)}\|_{\mathscr{E}})\} I^{\beta}\{|a_2(t)| + b_2|\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(t)|\} \\ &\quad + \|\mathrm{g(t,0)}\|_{\mathscr{E}} \\ &\leq \mathscr{L}\{|\mathrm{a_1(t)}| + \mathrm{b_1}\|\mathrm{x_0}\|_{\mathscr{E}} + \mathrm{b_1 I^{\alpha}}\|\mathrm{y(t)}\|_{\mathscr{E}})\}\{I^{\beta}|a_2(t)| + b_2 I^{\beta}\frac{t^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}} + b_2 I^{\alpha}\|y(t)\|_{\mathscr{E}}\} \\ &\quad + \|\mathrm{g(t,0)}\|_{\mathscr{E}} \\ &\leq \mathscr{L}\{|\mathrm{a_1(t)}| + \mathrm{b_1}\|\mathrm{x_0}\|_{\mathscr{E}} + \mathrm{b_1}\int_0^{\mathrm{t}} \frac{\mathrm{(t-s)^{\alpha-1}}}{\Gamma(\alpha)}\|\mathrm{y(s)}\|_{\mathscr{E}}\mathrm{ds}\}\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|a_2(s)|ds \end{aligned}$$

$$
\begin{aligned}
& +\frac{b_2}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}s^{-\beta}ds\} + b_2\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s)\|_{\mathscr{E}}ds\} + sup|g(t,0)| \\
\leq\ & \mathscr{L}\{\|a_1\|_{\mathscr{C}} + b_1\|x_0\|_{\mathscr{E}} + b_1\|y\|_{\mathscr{C}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\}\{K + b_2\|x_0\|_{\mathscr{E}} + b_2\|y\|_{\mathscr{C}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\} + \mathscr{M} \\
\leq\ & \mathscr{L}\{\|a_1\|_{\mathscr{C}} + b_1\|x_0\|_{\mathscr{E}} + b_1 r\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}\}\{K + b_2\|x_0\|_{\mathscr{E}} + b_2 r\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}\} + \mathscr{M} \\
\leq\ & \mathscr{L}\|a_1\|_{\mathscr{C}}\{K + b_2\|x_0\|_{\mathscr{E}} + b_2 r\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}\} + \mathscr{L}b_1\|x_0\|_{\mathscr{E}}\{K + b_2\|x_0\|_{\mathscr{E}} + b_2 r\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}\} \\
& +\mathscr{L}b_1 r\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}\{K + b_2\|x_0\|_{\mathscr{E}} + b_2 r\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}\} + \mathscr{M} \\
\leq\ & \mathscr{L}\|a_1\|_{\mathscr{C}}K + \mathscr{L}\|a_1\|_{\mathscr{C}}b_2\|x_0\|_{\mathscr{E}} + \mathscr{L}\|a_1\|_{\mathscr{C}}b_2 r\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)} + \mathscr{L}b_1\|x_0\|_{\mathscr{E}}K \\
& +\mathscr{L}b_1 b_2(\|x_0\|_{\mathscr{E}})^2 + \mathscr{L}b_1 b_2 r\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}\|x_0\|_{\mathscr{E}} + \mathscr{L}b_1 r\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}K \\
& +\mathscr{L}b_1 b_2 r\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}\|x_0\|_{\mathscr{E}} + \mathscr{L}b_1 b_2 r^2(\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)})^2 + \mathscr{M}. \\
\leq\ & \mathscr{L}\|a_1\|_{\mathscr{C}}K + \mathscr{L}\|a_1\|_{\mathscr{C}}b_2\|x_0\|_{\mathscr{E}} + \mathscr{L}\|a_1\|_{\mathscr{C}}b_2 rK_1 + \mathscr{L}b_1\|x_0\|_{\mathscr{E}}K + \mathscr{L}b_1 b_2(\|x_0\|_{\mathscr{E}})^2 \\
& +2\mathscr{L}b_1 b_2 rK_1\|x_0\|_{\mathscr{E}} + \mathscr{L}b_1 rK_1 K + \mathscr{L}b_1 b_2 r^2 K_1^2 + \mathscr{M}.
\end{aligned}
$$

where $\mathscr{M} = \sup|g(t,0)|,\ K = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|a_1(s)|ds,\ K_1 = \frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}$.

Therefore

$$
\begin{aligned}
\|\mathscr{A}y\|_C \leq\ & \mathscr{L}\|a_1\|_{\mathscr{C}}K + \mathscr{L}\|a_1\|_{\mathscr{C}}b_2\|x_0\|_{\mathscr{E}} + \mathscr{L}\|a_1\|_{\mathscr{C}}b_2 rK_1 + \mathscr{L}b_1\|x_0\|_{\mathscr{E}}K + \mathscr{L}b_1 b_2(\|x_0\|_{\mathscr{E}})^2 \\
& +2\mathscr{L}b_1 b_2 rK_1\|x_0\|_{\mathscr{E}} + \mathscr{L}b_1 rK_1 K + \mathscr{L}b_1 b_2 r^2 K_1^2 + \mathscr{M} = r.
\end{aligned}
$$

Then $\|\mathscr{A}y\|_C \leq r$.

Hence, $\mathscr{A}y \in \mathscr{Q}_r$, which proves that $\mathscr{A}\mathscr{Q}_r \subset \mathscr{Q}_r$ and the class of functions $\{\mathscr{A}\mathscr{Q}_r\}$ is uniformly bounded.

Now, we will show that $\mathscr{A}: \mathscr{Q}_r \to \mathscr{Q}_r$.

Let $y \in \mathscr{Q}_r$ and let $t_1, t_2 \in \mathscr{I},\ t_1 < t_2$ (without loss of generality assume that $\|\mathscr{A}y(t_2) - \mathscr{A}y(t_1)\| \neq 0$), then we have

$$
\begin{aligned}
& \|\mathscr{A}y(t_2) - \mathscr{A}y(t_1)\| \\
=\ & \|(g(t_2, g_1(t_2,(x_0 + I^{\alpha}y(t_2)))I^{\beta}g_2(t_2, \frac{t_2^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(t_2))) \\
-\ & (g(t_1, g_1(t_1,(x_0 + I^{\alpha}y(t_1)))I^{\beta}g_2(t_1, \frac{t_1^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(t_1)))\|_{\mathscr{E}}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \|g(t_2, g_1(t_2, (x_0 + I^{\alpha}y(t_2))) \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s)))ds \\
- \quad & g(t_1, g_1(t_1, (x_0 + I^{\alpha}y(t_1))) \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s)))ds\|_{\mathscr{E}} \\
\leq \quad & \mathscr{L}\{|t_2 - t_1| + \|g_1(t_2, (x_0 + I^{\alpha}y(t_2))) \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s))ds \\
- \quad & g_1(t_1, (x_0 + I^{\alpha}y(t_1))) \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s))ds\|_{\mathscr{E}}\} \\
\leq \quad & \mathscr{L}\{|t_2 - t_1| + \|g_1(t_2, (x_0 + I^{\alpha}y(t_2))) \int_0^{t_1} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s))ds \\
+ \quad & g_1(t_2, (x_0 + I^{\alpha}y(t_2))) \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s))ds \\
- \quad & g_1(t_1, (x_0 + I^{\alpha}y(t_1))) \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s))ds\|_{\mathscr{E}}\} \\
\leq \quad & \mathscr{L}|t_2 - t_1| + \mathscr{L}\|g_1(t_2, (x_0 + I^{\alpha}y(t_2))) - g_1(t_1, (x_0 + I^{\alpha}y(t_1)))\|_{\mathscr{E}} \\
\times \quad & \int_0^{t_1} [\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}] |g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s))|ds \\
+ \quad & \mathscr{L}\|g_1(t_2, (x_0 + I^{\alpha}y(t_2)))\|_{\mathscr{E}} \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} |g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s))|ds \\
\leq \quad & \mathscr{L}|t_2 - t_1| + \mathscr{L}b_1\{|t_2 - t_1| + I^{\alpha}\|y(t_2) - y(t_1)\|_{\mathscr{E}}\} \\
\times \quad & \int_0^{t_1} [\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}] |g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s))|ds \\
+ \quad & \mathscr{L}\{|a_1(t_2)| + b_1\|(x_0 + I^{\alpha}y(t_2))\|_{\mathscr{E}}\} \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} |g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s))|ds \\
\leq \quad & \mathscr{L}|t_2 - t_1| + \mathscr{L}b_1|t_2 - t_1| \int_0^{t_1} [\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}] |g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s))|ds \\
+ \quad & \mathscr{L}b_1 I^{\alpha}\|y(t_2) - y(t_1)\|_{\mathscr{E}} \int_0^{t_1} [\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}] |g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s))|ds \\
+ \quad & \mathscr{L}\{|a_1(t_2)| + b_1\|x_0\|_{\mathscr{E}} + I^{\alpha}\|y(t_2)\|_{\mathscr{E}}\} \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} |g_2(s, \frac{s^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(s))|ds \\
\leq \quad & \mathscr{L}|t_2 - t_1| + \mathscr{L}b_1|t_2 - t_1| \int_0^{t_1} [\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}] \{|a_2(s)| + b_2\frac{s^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}} + I^{\alpha-\beta}\|y(s)\|_{\mathscr{E}}\}ds \\
+ \quad & \mathscr{L}b_1 I^{\alpha}\|y(t_2) - y(t_1)\|_{\mathscr{E}} \int_0^{t_1} [\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}] \{|a_2(s)| + b_2\frac{s^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}} + I^{\alpha-\beta}\|y(s)\|_{\mathscr{E}}\}ds \\
+ \quad & \mathscr{L}\{|a_1(t_2)| + b_1(\|x_0\|_{\mathscr{E}} + I^{\alpha}\|y(t_2)\|_{\mathscr{E}})\} \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} \{|a_2(s)| + b_2\frac{s^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}} + I^{\alpha-\beta}\|y(s)\|_{\mathscr{E}}\}ds \\
\leq \quad & \mathscr{L}|t_2 - t_1| + \mathscr{L}b_1|t_2 - t_1| \int_0^{t_1} [\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}] |a_2(s)|ds \\
+ \quad & \mathscr{L}b_1 b_2|t_2 - t_1| \int_0^{t_1} [\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}] \frac{s^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}ds \\
+ \quad & \mathscr{L}b_1|t_2 - t_1| \int_0^{t_1} [\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}] I^{\alpha-\beta}\|y(s)\|_{\mathscr{E}}ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \mathscr{L} b_1 I^{\alpha}\|y(t_2)-y(t_1)\|_{\mathscr{E}}\int_0^{t_1}[\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}]|a_2(s)|ds\\
&+ \quad \mathscr{L} b_1 b_2 I^{\alpha}\|y(t_2)-y(t_1)\|_{\mathscr{E}}\int_0^{t_1}[\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}]\frac{s^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}ds\\
&+ \quad \mathscr{L} b_1 I^{\alpha}\|y(t_2)-y(t_1)\|_{\mathscr{E}}\int_0^{t_1}[\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}]I^{\alpha-\beta}\|y(s)\|_{\mathscr{E}}ds\\
&+ \quad \mathscr{L}\{|a_1(t_2)|+b_1(\|x_0\|_{\mathscr{E}}+I^{\alpha}\|y(t_2)\|_{\mathscr{E}})\}\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}|a_2(s)|ds\\
&+ \quad \mathscr{L} b_2\{|a_1(t_2)|+b_1(\|x_0\|_{\mathscr{E}}+I^{\alpha}\|y(t_2)\|_{\mathscr{E}})\}\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\frac{s^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}ds\\
&+ \quad \mathscr{L}\{|a_1(t_2)|+b_1(\|x_0\|_{\mathscr{E}}+I^{\alpha}\|y(t_2)\|_{\mathscr{E}})\}\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}I^{\alpha-\beta}\|y(s)\|_{\mathscr{E}}ds\\
&\leq \quad \mathscr{L}|t_2-t_1|+\mathscr{L} b_1|t_2-t_1|\int_0^{t_1}[\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}]|a_2(s)|ds\\
&+ \quad \mathscr{L} b_1 b_2|t_2-t_1|\int_0^{t_1}[\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}]\frac{s^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}ds\\
&+ \quad \mathscr{L} b_1|t_2-t_1|\int_0^{t_1}[\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}](\int_0^{t}\frac{(s-\theta)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\|y(\theta)\|_{\mathscr{E}}d\theta)ds\\
&+ \quad \mathscr{L} b_1(\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s_2)-y(s_1)\|_{\mathscr{E}}ds)\int_0^{t_1}[\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}]|a_2(s)|ds\\
&+ \quad \mathscr{L} b_1 b_2(\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s_2)-y(s_1)\|_{\mathscr{E}}ds)\int_0^{t_1}[\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}]\frac{s^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}ds\\
&+ \quad \mathscr{L} b_1(\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s_2)-y(s_1)\|_{\mathscr{E}}ds)\int_0^{t_1}[\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}](\int_0^{t}\frac{(s-\theta)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\|y(\theta)\|_{\mathscr{E}}d\theta)ds\\
&+ \quad \mathscr{L}\{|a_1(t_2)|+b_1(\|x_0\|_{\mathscr{E}}+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s_2)\|_{\mathscr{E}}ds)\}\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}|a_2(s)|ds\\
&+ \quad \mathscr{L} b_2\{|a_1(t_2)|+b_1(\|x_0\|_{\mathscr{E}}+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s_2)\|_{\mathscr{E}}ds)\}\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\frac{s^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}ds\\
&+ \quad \mathscr{L}\{|a_1(t_2)|+b_1(\|x_0\|_{\mathscr{E}}+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s_2)\|_{\mathscr{E}}ds)\}\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}(\int_0^{t}\frac{(s-\theta)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\|y(\theta)\|_{\mathscr{E}}d\theta)ds
\end{aligned}
$$

which proves that $\mathscr{A}:\mathscr{Q}_r\to\mathscr{Q}_r;$ the class of functions is $\{\mathscr{A}y\}$ is equicontinuous in $\mathscr{Q}_r$.

Now $\mathscr{Q}_r$ is nonempty, closed, convex and uniformly bounded.

As a consequence of proposition 2.2, then $\mathscr{A}\mathscr{Q}_r$ is relatively weakly compact.

Finally, we want to prove that $\mathscr{A}$ is weakly sequentially continuous.

Let $\{y_n\}$ be a sequence in $\mathscr{Q}_r$ converges weakly to y $\forall t\in\mathscr{I}$, i.e. $y_n\to y,\ \forall t\in\mathscr{I}$.

Since $g_i(t,y(t)),\ i=1,2$ are weakly sequentially continuous in y, then $g_i(t,y_n(t)),\ i=1,2$ converges weakly to $g_i(t,y(t)),\ i=1,2$.

Thus $\phi(\mathrm{g_i}(\mathrm{t},\mathrm{y_n}(\mathrm{t}))),\ i=1,2$ converges strongly to $\phi(\mathrm{g}(\mathrm{t},\mathrm{y}(\mathrm{t}))),\ i=1,2$. Also,

$$\|\mathrm{g_i}(\mathrm{t},\mathrm{y_n}(\mathrm{t}))\|_{\mathscr{E}} \leq |\mathrm{a_i}(\mathrm{t})|+\mathrm{b_i}\|\mathrm{y_n}\|_{\mathscr{E}},\ i=1,2$$

By applying Lebesgue dominated convergence theorem for Pettis integral, then we get

$$\begin{aligned}
\phi(\mathscr{A}\mathrm{y_n}(\mathrm{t})) &= \phi(\mathrm{g}(\mathrm{t},\mathrm{g_1}(\mathrm{t},(\mathrm{x_0}+\mathrm{I}^{\alpha}\mathrm{y_n}(\mathrm{t})))\mathrm{I}^{\beta}\mathrm{g_2}(\mathrm{t},\frac{\mathrm{t}^{-\beta}}{\Gamma(1-\beta)}\mathrm{x_0}+\mathrm{I}^{\alpha-\beta}\mathrm{y_n}(\mathrm{t}))))\\
&= \|\mathrm{g}(\mathrm{t},\mathrm{g_1}(\mathrm{t},(\mathrm{x_0}+\mathrm{I}^{\alpha}\mathrm{y_n}(\mathrm{t})))\mathrm{I}^{\beta}\mathrm{g_2}(\mathrm{t},\frac{\mathrm{t}^{-\beta}}{\Gamma(1-\beta)}\mathrm{x_0}+\mathrm{I}^{\alpha-\beta}\mathrm{y_n}(\mathrm{t})))\|_{\mathscr{E}}\\
&\to \|\mathrm{g}(\mathrm{t},\mathrm{g_1}(\mathrm{t},(\mathrm{x_0}+\mathrm{I}^{\alpha}\mathrm{y}(\mathrm{t})))\mathrm{I}^{\beta}\mathrm{g_2}(\mathrm{t},\frac{\mathrm{t}^{-\beta}}{\Gamma(1-\beta)}\mathrm{x_0}+\mathrm{I}^{\alpha-\beta}\mathrm{y}(\mathrm{t})))\|_{\mathscr{E}},\\
&\quad \mathrm{t}\in\mathscr{I},\ \forall\ \phi\in\mathscr{E}^*,\ \mathrm{t}\in\mathscr{I}
\end{aligned}$$

i.e. $\phi(\mathscr{A}\mathrm{y}_n(t))\to\phi(\mathscr{A}\mathrm{y}(\mathrm{t}))$, and then

$$\|\mathscr{A}\mathrm{y_n}(\mathrm{t})\|_{\mathscr{E}}\to\|\mathscr{A}\mathrm{y}(\mathrm{t})\|_{\mathscr{E}}$$

Hence, $\mathscr{A}$ is weakly sequentially continuous (i.e. $\mathscr{A}\mathrm{y_n}(\mathrm{t})\to\mathscr{A}\mathrm{y}(\mathrm{t}),\ \forall \mathrm{t}\in\mathscr{I}$ weakly).
Since all conditions of O'Regan theorem are satisfied, then the operator $\mathscr{A}$ has at least one fixed point $\mathrm{y}\in\mathscr{Q}_\mathrm{r}$, then there exists at least one weak solution $\mathrm{y}\in\mathscr{C}(\mathscr{I},\mathscr{E})$ of the functional integral equation (3.3).
Consequently, there exists at least one weak solution $\mathrm{x}\in\mathscr{C}(\mathscr{I},\mathscr{E})$ of the problem (1.2) and (3.1). Hence, there exists at least one weak solution $\mathrm{x}\in\mathscr{C}(\mathscr{I},\mathscr{E})$ of the problem (1.1) and (1.2).

4. Uniqueness of the Solution

Here we study the sufficient condition for the uniqueness of the solution $\mathrm{x}\in\mathscr{C}(\mathscr{I},\mathscr{E})$ of the problem (1.2) and (3.1).
consider the following assumption:
(h1) The function $\mathrm{g_2}:\mathscr{I}\times\mathscr{E}\to\mathscr{R}$ satisfies Lipshitz condition with a Lipschitz constant $\mathrm{b_2}$ such that

$$|\mathrm{g_2}(\mathrm{t},\mathrm{x})-\mathrm{g_2}(\mathrm{t},\mathrm{y})|\leq\mathrm{b_2}\|\mathrm{x}-\mathrm{y}\|_{\mathscr{E}}.$$

Theorem 4.1. *Let the assumptions (H1)-(H6) and (h1) be satisfied, Then the solution* $x \in \mathscr{C}(\mathscr{I},\mathscr{E})$ *of the problem (1.2) and (3.1) is unique.*

Proof. Let $x,\ x_1 \in \mathscr{C}(\mathscr{I},\mathscr{E})$ be two solutions of the problem (1.2) and (3.1).

Since

$$y(t) = g(t, g_1(t,(x_0 + I^{\alpha} y(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t)),\ t \in \mathscr{I}.$$

Then

$$\begin{aligned}
& \|y(t) - y_1(t)\|_{\mathscr{E}} \\
= \ & \|g(t, g_1(t,(x_0 + I^{\alpha} y(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t))) \\
- \ & g(t, g_1(t,(x_0 + I^{\alpha} y_1(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_1(t)))\|_{\mathscr{E}} \\
\leq \ & \mathscr{L}\{\|g_1(t,(x_0 + I^{\alpha} y(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t)) \\
- \ & g_1(t,(x_0 + I^{\alpha} y_1(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_1(t))\|_{\mathscr{E}}\} \\
\leq \ & \mathscr{L}\{\|g_1(t,(x_0 + I^{\alpha} y(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t)) \\
- \ & g_1(t,(x_0 + I^{\alpha} y_1(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t)) \\
+ \ & g_1(t,(x_0 + I^{\alpha} y_1(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t)) \\
- \ & g_1(t,(x_0 + I^{\alpha} y_1(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_1(t))\|_{\mathscr{E}}\} \\
\leq \ & \mathscr{L}\{\|g_1(t,(x_0 + I^{\alpha} y(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t)) \\
- \ & g_1(t,(x_0 + I^{\alpha} y_1(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t))\|_{\mathscr{E}}\} \\
+ \ & \mathscr{L}\{\|g_1(t,(x_0 + I^{\alpha} y_1(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t))\|_{\mathscr{E}} \\
- \ & g_1(t,(x_0 + I^{\alpha} y_1(t)))I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_1(t))\|_{\mathscr{E}}\} \\
\leq \ & \mathscr{L}\{\|[g_1(t,(x_0 + I^{\alpha} y(t))) - g_1(t,(x_0 + I^{\alpha} y_1(t)))]I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t))\|_{\mathscr{E}}\} \\
+ \ & \mathscr{L}\{\|g_1(t,(x_0 + I^{\alpha} y_1(t)))[I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t)) - I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_1(t))]\|_{\mathscr{E}}\} \\
\leq \ & \mathscr{L}\{\|g_1(t,(x_0 + I^{\alpha} y(t))) - g_1(t,(x_0 + I^{\alpha} y_1(t)))\|_{\mathscr{E}} I^{\beta} |g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t))|\} \\
+ \ & \mathscr{L}\{\|g_1(t,(x_0 + I^{\alpha} y_1(t)))\|_{\mathscr{E}} I^{\beta} |g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t)) - g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_1(t))|\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathscr{L}b_1 I^{\alpha}\|y(t)-y_1(t)\|_{\mathscr{E}} I^{\beta}\{|a_2(t)|+b_2\frac{t^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}+b_2 I^{\alpha-\beta}\|y(t)\|_{\mathscr{E}}\}\\
&+ \mathscr{L}\{|a_1(t)|+b_1\|x_0\|_{\mathscr{E}}+b_1 I^{\alpha}\|y_1(t)\|_{\mathscr{E}}\}\{b_2 I^{\alpha}\|y(t)-y_1(t)\|_{\mathscr{E}}\}\\
&\leq (\mathscr{L}b_1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s)-y_1(s)\|_{\mathscr{E}}ds)\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|a_2(s)|ds\\
&+ (\mathscr{L}b_1 b_2\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s)-y_1(s)\|_{\mathscr{E}}ds)\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\frac{s^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}ds\\
&+ (\mathscr{L}b_1 b_2\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s)-y_1(s)\|_{\mathscr{E}}ds)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s)\|_{\mathscr{E}}ds\\
&+ \mathscr{L}b_2|a_1(t)|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s)-y_1(s)\|_{\mathscr{E}}ds+\mathscr{L}b_1 b_2\|x_0\|_{\mathscr{E}}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s)-y_1(s)\|_{\mathscr{E}}ds\\
&+ \mathscr{L}b_1 b_2(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y_1(s)\|_{\mathscr{E}}ds)(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s)-y_1(s)\|_{\mathscr{E}}ds)\\
&\leq \mathscr{L}b_1\|y-y_1\|_{\mathscr{C}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}K+\mathscr{L}b_1 b_2\|y-y_1\|_{\mathscr{C}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\|x_0\|_{\mathscr{E}}+\mathscr{L}b_1 b_2\|y-y_1\|_{\mathscr{C}}(\frac{t^{\alpha}}{\Gamma(\alpha+1)})^2\|y\|_{\mathscr{C}}\\
&+ \mathscr{L}b_2\|a_1\|_{\mathscr{C}}\|y-y_1\|_{\mathscr{C}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\mathscr{L}b_1 b_2\|x_0\|_{\mathscr{E}}\|y-y_1\|_{\mathscr{C}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\mathscr{L}b_1 b_2\|y_1\|_{\mathscr{C}}\|y-y_1\|_{\mathscr{C}}(\frac{t^{\alpha}}{\Gamma(\alpha+1)})^2\\
&\leq \mathscr{L}b_1\|y-y_1\|_{\mathscr{C}}\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}K+\mathscr{L}b_1 b_2\|y-y_1\|_{\mathscr{C}}\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}\|x_0\|_{\mathscr{E}}+\mathscr{L}b_1 b_2\|y-y_1\|_{\mathscr{C}}(\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)})^2\|y\|_{\mathscr{C}}\\
&+ \mathscr{L}b_2\|a_1\|_{\mathscr{C}}\|y-y_1\|_{\mathscr{C}}\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}+\mathscr{L}b_1 b_2\|x_0\|_{\mathscr{E}}\|y-y_1\|_{\mathscr{C}}\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}+\mathscr{L}b_1 b_2\|y_1\|_{\mathscr{C}}\|y-y_1\|_{\mathscr{C}}(\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)})^2\\
&\leq \mathscr{L}b_1\|y-y_1\|_{\mathscr{C}}K_1K+2\mathscr{L}b_1 b_2\|y-y_1\|_{\mathscr{C}}K_1\|x_0\|_{\mathscr{E}}+2\mathscr{L}b_1 b_2\|y-y_1\|_{\mathscr{C}}K_1^2 r+\mathscr{L}b_2\|a_1\|_{\mathscr{C}}\|y-y_1\|_{\mathscr{C}}K_1.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|y-y_1\|_{\mathscr{C}} &\leq \mathscr{L}b_1\|y-y_1\|_{\mathscr{C}}K_1K+2\mathscr{L}b_1 b_2\|y-y_1\|_{\mathscr{C}}K_1\|x_0\|_{\mathscr{E}}+2\mathscr{L}b_1 b_2\|y-y_1\|_{\mathscr{C}}K_1^2 r\\
&\quad +\mathscr{L}b_2\|a_1\|_{\mathscr{C}}\|y-y_1\|_{\mathscr{C}}K_1.
\end{aligned}
$$

Hence

$$
\|y-y_1\|_{\mathscr{C}}(1-\mathscr{L}b_1K_1K+2\mathscr{L}b_1b_2K_1\|x_0\|_{\mathscr{E}}+2\mathscr{L}b_1b_2K_1^2 r+\mathscr{L}b_2\|a_1\|_{\mathscr{C}}K_1)\leq 0.
$$

Since $(\mathscr{L}b_1K_1K+2\mathscr{L}b_1b_2K_1\|x_0\|_{\mathscr{E}}+2\mathscr{L}b_1b_2K_1^2 r+\mathscr{L}b_2\|a_1\|_{\mathscr{C}}K_1)<1$, then

$$
\|y-y_1\|_{\mathscr{C}}\leq 0.
$$

Since

$$
x(t)=x_0+I^{\alpha}y(t).
$$

Then for the two corresponding solutions x, x_1, we have

$$
\|x(t)-x_1(t)\|_{\mathscr{E}} = \|I^{\alpha}y(t)-I^{\alpha}y_1(t)\|_{\mathscr{E}}
$$

$$\leq \ I^{\alpha}\|y-y_1\|_{\mathscr{C}}$$
$$\leq \ 0$$

Hence

$$\|x-x_1\|_{\mathscr{C}} \leq 0.$$

Which proves the solution of the problem (1.2) and (3.1) is unique.

5. Continuous Dependence on x_0

Here we study the continuous dependence of the solution on x_0 for the problem (1.2) and (3.1).

Definition 5.1. The solution $x \in \mathscr{C}(\mathscr{I},\mathscr{E})$ of the problem (1.2) and (3.1) depends continuously on the data x_0, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x_0 - x_0^*\|_{\mathscr{E}} < \delta$ implies $\|x-x^*\|_{\mathscr{C}} < \varepsilon$, where x and x^* are the two solutions of the problems (1.2) and (3.1) and

$$^{R}D^{\alpha}(x^*(t)-x^*(0)) = g(t,x(t)I^{\beta}g_1(t,D^{\beta}x^*(t)),\ t \in \mathscr{I}$$

with the initial condition

$$x^*(0) = x_0^*,\ x_0^* \in \mathscr{E}.$$

Theorem 5.2. *Let the assumptions (H1)-(H6) and (h1) be satisfied, Then the solution* $x \in \mathscr{C}(\mathscr{I},\mathscr{E})$ *of the problem (1.2) and (3.1) depends continuously on the function* x_0.

Proof. Let $x_0,\ x_0^* \in \mathscr{E}$ be two points in the Banach space $\mathscr{E}$ such that

$$\|x_0 - x_0^*\|_{\mathscr{E}} < \delta,\ \delta > 0,\ t \in \mathscr{I}.$$

Since

$$y(t) = g(t,g_1(t,(x_0 + I^{\alpha}y(t)))I^{\beta}g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0 + I^{\alpha-\beta}y(t))),\ t \in \mathscr{I}.$$

Then

$$\begin{aligned}\| \ & y(t)-y^*(t)\|_{\mathscr{E}} \\ = \ & \|g(t,g_1(t,(x_0+I^{\alpha}y(t)))I^{\beta}g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t)))\end{aligned}$$

$$
\begin{aligned}
- &\quad g(t,g_1(t,(x_0^*+I^\alpha y(t)^*))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0^*+I^{\alpha-\beta}y^*(t)))\|_{\mathscr{E}}\\
\leq &\quad \mathscr{L}\{\|g_1(t,(x_0+I^\alpha y(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))\\
- &\quad g_1(t,(x_0^*+I^\alpha y^*(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0^*+I^{\alpha-\beta}y^*(t))\|_{\mathscr{E}}\}\\
\leq &\quad \mathscr{L}\{\|g_1(t,(x_0+I^\alpha y(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))\\
- &\quad g_1(t,(x_0^*+I^\alpha y^*(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))\\
+ &\quad g_1(t,(x_0^*+I^\alpha y^*(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))\\
- &\quad g_1(t,(x_0^*+I^\alpha y^*(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0^*+I^{\alpha-\beta}y^*(t))\|_{\mathscr{E}}\}\\
\leq &\quad \mathscr{L}\{\|g_1(t,(x_0+I^\alpha y(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))\\
- &\quad g_1(t,(x_0^*+I^\alpha y^*(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))\\
+ &\quad g_1(t,(x_0^*+I^\alpha y^*(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))\\
- &\quad g_1(t,(x_0^*+I^\alpha y^*(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0^*+I^{\alpha-\beta}y^*(t))\|_{\mathscr{E}}\}\\
\leq &\quad \mathscr{L}\|g_1(t,(x_0+I^\alpha y(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))\\
- &\quad g_1(t,(x_0^*+I^\alpha y^*(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))\|_{\mathscr{E}}\\
+ &\quad \mathscr{L}\|g_1(t,(x_0^*+I^\alpha y^*(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))\\
- &\quad g_1(t,(x_0^*+I^\alpha y^*(t)))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0^*+I^{\alpha-\beta}y^*(t))\|_{\mathscr{E}}\\
\leq &\quad \mathscr{L}b_1\|[(x_0-x_0^*)+I^\alpha(y(t)-y^*(t))]I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))\|_{\mathscr{E}}\\
+ &\quad \mathscr{L}\|g_1(t,(x_0^*+I^\alpha y^*(t)))[I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))-I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0^*+I^{\alpha-\beta}y^*(t))]\|_{\mathscr{E}}\\
\leq &\quad \mathscr{L}b_1[\|x_0-x_0^*\|_{\mathscr{E}}+I^\alpha\|y(t)-y^*(t)\|_{\mathscr{E}}]I^\beta|g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))|\\
+ &\quad \mathscr{L}\|g_1(t,(x_0^*+I^\alpha y^*(t)))\|_{\mathscr{E}}I^\beta|g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t))-g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0^*+I^{\alpha-\beta}y^*(t))|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathscr{L}b_1[\|x_0-x_0^*\|_{\mathscr{E}}+I^{\alpha}\|y(t)-y^*(t)\|_{\mathscr{E}}]I^{\beta}\{|a_2(t)+b_2\frac{t^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}+b_2I^{\alpha-\beta}\|y(t)\|_{\mathscr{E}}\}\\
&+ \mathscr{L}[|a_1(t)|+b_1\|x_0^*\|_{\mathscr{E}}+b_1I^{\alpha}\|y^*(t)\|_{\mathscr{E}}]I^{\beta}\{b_2\frac{t^{-\beta}}{\Gamma(1-\beta)}\|x_0-x_0^*\|_{\mathscr{E}}+b_2I^{\alpha-\beta}\|y(t)-y^*(t)\|_{\mathscr{E}}\}\\
&\leq \mathscr{L}b_1[\delta+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s)-y^*(s)\|_{\mathscr{E}}ds]\\
&\quad \{\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|a_2(s)|ds+b_2\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\frac{s^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}ds+b_2\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s)\|_{\mathscr{E}}ds\}\\
&+ \mathscr{L}[|a_1(t)|+b_1\|x_0^*\|_{\mathscr{E}}+b_1\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y^*(s)\|_{\mathscr{E}}ds]\{b_2\delta\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\beta)}\frac{s^{-\beta}}{\Gamma(1-\beta)}ds\\
&+ b_2\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y(s)-y^*(s)\|_{\mathscr{E}}ds\}\\
&\leq \mathscr{L}b_1[\delta+\|y-y^*\|_{\mathscr{C}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}]\{K+b_2\|x_0\|_{\mathscr{E}}+b_2\|y\|_{\mathscr{C}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\}\\
&+ \mathscr{L}[\|a_1\|_{\mathscr{C}}+b_1\|x_0^*\|_{\mathscr{E}}+b_1\|y^*\|_{\mathscr{C}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}]\{b_2\delta+b_2\|y-y^*\|_{\mathscr{C}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\}\\
&\leq \mathscr{L}b_1[\delta+\|y-y^*\|_{\mathscr{C}}\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}]\{K+b_2\|x_0\|_{\mathscr{E}}+b_2\|y\|_{\mathscr{C}}\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}\}\\
&+ \mathscr{L}[\|a_1\|_{\mathscr{C}}+b_1\|x_0^*\|_{\mathscr{E}}+b_1\|y^*\|_{\mathscr{C}}\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}]\{b_2\delta+b_2\|y-y^*\|_{\mathscr{C}}\frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)}\}\\
&\leq [\mathscr{L}b_1\delta+\mathscr{L}b_1\|y-y^*\|_{\mathscr{C}}K_1]\{K+b_2\|x_0\|_{\mathscr{E}}+b_2rK_1\}\\
&+ [\mathscr{L}\|a_1\|_{\mathscr{C}}+\mathscr{L}b_1\|x_0^*\|_{\mathscr{E}}+\mathscr{L}b_1rK_1]\{b_2\delta+b_2\|y-y^*\|_{\mathscr{C}}K_1\}\\
&\leq \mathscr{L}b_1\delta K+\mathscr{L}b_1b_2\delta\|x_0\|_{\mathscr{E}}+2+\mathscr{L}b_1b_2\delta rK_1\\
&+ \mathscr{L}b_1\|y-y^*\|_{\mathscr{C}}\{KK_1+b_2K_1\|x_0\|_{\mathscr{E}}+b_2r(K_1)^2\}+\mathscr{L}b_2\delta\|a_1\|_{\mathscr{C}}+\mathscr{L}b_1b_2\delta\|x_0^*\|_{\mathscr{E}}\\
&+ \{\mathscr{L}\|a_1\|_{\mathscr{C}}b_2K_1+\mathscr{L}b_1b_2\|x_0^*\|_{\mathscr{E}}K_1+\mathscr{L}b_1b_2r(K_1)^2\}\|y-y^*\|_{\mathscr{C}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|y-y^*\|_{\mathscr{C}} &\leq \mathscr{L}b_1\delta K+\mathscr{L}b_1b_2\delta\|x_0\|_{\mathscr{E}}+2\mathscr{L}b_1b_2\delta rK_1\\
&+ \mathscr{L}b_1\|y-y^*\|_{\mathscr{C}}\{KK_1+b_2K_1\|x_0\|_{\mathscr{E}}+b_2r(K_1)^2\}+\mathscr{L}b_2\delta\|a_1\|_{\mathscr{C}}+\mathscr{L}b_1b_2\delta\|x_0^*\|_{\mathscr{E}}\\
&+ \{\mathscr{L}\|a_1\|_{\mathscr{C}}b_2K_1+\mathscr{L}b_1b_2\|x_0^*\|_{\mathscr{E}}K_1+\mathscr{L}b_1b_2r(K_1)^2\}\|y-y^*\|_{\mathscr{C}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|y-y^*\|_{\mathscr{C}}\\
&\leq \frac{\mathscr{L}b_1\delta K+\mathscr{L}b_1b_2\delta\|x_0\|_{\mathscr{E}}+2\mathscr{L}b_1b_2\delta rK_1+\mathscr{L}b_2\delta\|a_1\|_{\mathscr{C}}+\mathscr{L}b_1b_2\delta\|x_0^*\|_{\mathscr{E}}}{1-(\mathscr{L}b_1\{KK_1+b_2K_1\|x_0\|_{\mathscr{E}}+b_2r(K_1)^2\}+\{\mathscr{L}\|a_1\|_{\mathscr{C}}b_2K_1+\mathscr{L}b_1b_2\|x_0^*\|_{\mathscr{E}}K_1+\mathscr{L}b_1b_2r(K_1)^2\})}.
\end{aligned}
$$

Since

$$x(t) = x_0 + I^{\alpha} y(t).$$

Then for the two corresponding solutions x, x^*, we have

$$\begin{aligned}
& \|x(t) - x^*(t)\|_{\mathscr{E}} \\
= \; & \|I^{\alpha} y(t) - I^{\alpha} y^*(t)\|_{\mathscr{E}} \\
\leq \; & I^{\alpha} \|y - y^*\|_{\mathscr{C}} \\
\leq \; & I^{\alpha} \frac{\mathscr{L} b_1 \delta K + \mathscr{L} b_1 b_2 \delta \|x_0\|_{\mathscr{E}} + +2 \mathscr{L} b_1 b_2 \delta r K_1 + \mathscr{L} b_2 \delta \|a_1\|_{\mathscr{C}} + \mathscr{L} b_1 b_2 \delta \|x_0^*\|_{\mathscr{E}}}{1 - (\mathscr{L} b_1 \{K K_1 + b_2 K_1 \|x_0\|_{\mathscr{E}} + b_2 r (K_1)^2\} + \{\mathscr{L} \|a_1\|_{\mathscr{C}} b_2 K_1 + \mathscr{L} b_1 b_2 \|x_0^*\|_{\mathscr{E}} K_1 + \mathscr{L} b_1 b_2 r (K_1)^2\})} \\
\leq \; & K_1 \frac{\mathscr{L} b_1 \delta K + \mathscr{L} b_1 b_2 \delta \|x_0\|_{\mathscr{E}} + 2 + \mathscr{L} b_1 b_2 \delta r K_1 + \mathscr{L} b_2 \delta \|a_1\|_{\mathscr{C}} + \mathscr{L} b_1 b_2 \delta \|x_0^*\|_{\mathscr{E}}}{1 - (\mathscr{L} b_1 \{K K_1 + b_2 K_1 \|x_0\|_{\mathscr{E}} + b_2 r (K_1)^2\} + \{\mathscr{L} \|a_1\|_{\mathscr{C}} b_2 K_1 + \mathscr{L} b_1 b_2 \|x_0^*\|_{\mathscr{E}} K_1 + \mathscr{L} b_1 b_2 r (K_1)^2\})} \\
= \; & \varepsilon.
\end{aligned}$$

Hence

$$\|x - x^*\|_{\mathscr{C}} \leq \varepsilon.$$

which proves the continuous dependence of the solution on x_0 of the problem (1.2) and (3.1).

Corollary 5.3. *For every* $g \in S_G$ *the solution of the problem (1.1) and (1.2) depends continuously on* x_0 .

6. CONTINUOUS DEPENDENCE ON THE SET OF SELECTIONS S_G

Here we study the continuous dependence of the solution on the set of selections S_G for the problem (1.1) and (1.2).

Definition 6.1. The solution $x \in \mathscr{C}(\mathscr{I}, \mathscr{E})$ of the problem (1.1) and (1.2) depends continuously on the set S_G, if for every $\varepsilon > 0$, and any two functions $g,\ h \in S_G$, there exists $\delta > 0$ such that $\|g - h\|_{\mathscr{E}} < \delta$ implies $\|x_g - x_h\|_{\mathscr{C}} < \varepsilon$, corresponding to the set valued functions G and H.

Theorem 6.2. *Let the assumptions (H1)-(H6) and (h1) be satisfied, Then the solution* $x \in \mathscr{C}(\mathscr{I}, \mathscr{E})$ *of the problem (1.1) and (1.2) depends continuously on* S_G.

Proof. Let $g,\ h \in S_G$ such that

$$\begin{aligned} \| \quad & g(t, g_1(t, (x_0 + I^{\alpha} y_g(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_g(t)) \\ - \quad & h(t, g_1(t, (x_0 + I^{\alpha} y_g(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_g(t)) \|_{\mathscr{E}} < \delta,\ \delta > 0,\ t \in \mathscr{I}. \end{aligned}$$

Since

$$y(t) = g(t, g_1(t, (x_0 + I^{\alpha} y(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y(t)),\ t \in \mathscr{I}.$$

Then

$$\begin{aligned} \| \quad & y_g(t) - y_h(t) \|_{\mathscr{E}} \\ = \quad & \| g(t, g_1(t, (x_0 + I^{\alpha} y_g(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_g(t))) \\ - \quad & h(t, g_1(t, (x_0 + I^{\alpha} y_h(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_h(t))) \|_{\mathscr{E}} \\ = \quad & \| g(t, g_1(t, (x_0 + I^{\alpha} y_g(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_g(t))) \\ - \quad & h(t, g_1(t, (x_0 + I^{\alpha} y_g(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_g(t))) \\ + \quad & h(t, g_1(t, (x_0 + I^{\alpha} y_g(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_g(t))) \\ - \quad & h(t, g_1(t, (x_0 + I^{\alpha} y_h(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_h(t))) \|_{\mathscr{E}} \\ \leq \quad & \| g(t, g_1(t, (x_0 + I^{\alpha} y_g(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_g(t))) \\ - \quad & h(t, g_1(t, (x_0 + I^{\alpha} y_g(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_g(t))) \|_{\mathscr{E}} \\ + \quad & \| h(t, g_1(t, (x_0 + I^{\alpha} y_g(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_g(t))) \\ - \quad & h(t, g_1(t, (x_0 + I^{\alpha} y_h(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_h(t))) \|_{\mathscr{E}} \\ \leq \quad & \delta + \mathscr{L} \| g_1(t, (x_0 + I^{\alpha} y_g(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_g(t)) \\ - \quad & g_1(t, (x_0 + I^{\alpha} y_h(t))) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_h(t)) \|_{\mathscr{E}} \\ \leq \quad & \delta + \mathscr{L} b_1 \| ((x_0 + I^{\alpha} y_g(t)) I^{\beta} g_2(t, \frac{t^{-\beta}}{\Gamma(1-\beta)} x_0 + I^{\alpha-\beta} y_g(t))) \end{aligned}$$

$$
\begin{aligned}
- &\quad ((x_0+I^\alpha y_h(t))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y_h(t)))\|_{\mathscr{E}} \\
\leq &\quad \delta+\mathscr{L}b_1\|((x_0+I^\alpha y_g(t))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y_g(t))) \\
- &\quad ((x_0+I^\alpha y_g(t))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y_h(t)))\|_{\mathscr{E}} \\
+ &\quad \mathscr{L}b_1\|((x_0+I^\alpha y_g(t))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y_h(t))) \\
- &\quad ((x_0+I^\alpha y_h(t))I^\beta g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y_h(t)))\|_{\mathscr{E}} \\
\leq &\quad \delta+\mathscr{L}b_1\|(x_0+I^\alpha y_g(t))\|_{\mathscr{E}}\{I^\beta|g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y_g(t))-g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y_h(t))|\} \\
+ &\quad \mathscr{L}b_1\|(x_0+I^\alpha y_g(t))-(x_0+I^\alpha y_h(t))\|_{\mathscr{E}}\{I^\beta|g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y_h(t))|\} \\
\leq &\quad \delta+\mathscr{L}b_1\{\|x_0\|_{\mathscr{E}}+I^\alpha\|y_g(t)\|_{\mathscr{E}}\}\{b_2I^\alpha\|y_g(t)-y_h(t)\|_{\mathscr{E}}\} \\
+ &\quad \mathscr{L}b_1I^\alpha\|y_g(t)-y_h(t)\|_{\mathscr{E}}I^\beta\{|a_2(t)|+b_2\frac{t^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}+b_2I^{\alpha-\beta}\|y_h(t)\|_{\mathscr{E}}\} \\
\leq &\quad \delta+\mathscr{L}b_1\{\|x_0\|_{\mathscr{E}}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y_g(s)\|_{\mathscr{E}}ds\}\{b_2\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y_g(s)-y_h(s)\|_{\mathscr{E}}ds\} \\
+ &\quad \mathscr{L}b_1\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y_g(s)-y_h(s)\|_{\mathscr{E}}ds\{\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|a_2(s)|ds \\
+ &\quad b_2\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\frac{s^{-\beta}}{\Gamma(1-\beta)}\|x_0\|_{\mathscr{E}}ds+b_2\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|y_h(s)\|_{\mathscr{E}}ds\} \\
\leq &\quad \delta+\mathscr{L}b_1\{\|x_0\|_{\mathscr{E}}+\|y_g\|_{\mathscr{C}}\frac{t^\alpha}{\Gamma(\alpha+1)}\}\{b_2\|y_g-y_h\|_{\mathscr{C}}\frac{t^\alpha}{\Gamma(\alpha+1)}\} \\
+ &\quad \mathscr{L}b_1\|y_g-y_h\|_{\mathscr{C}}\frac{t^\alpha}{\Gamma(\alpha+1)}\{K+b_2\|x_0\|_{\mathscr{E}}+b_2\|y_h\|_{\mathscr{C}}\frac{t^\alpha}{\Gamma(\alpha+1)}\} \\
\leq &\quad \delta+\mathscr{L}b_1\{\|x_0\|_{\mathscr{E}}+\|y_g\|_{\mathscr{C}}\frac{\mathscr{T}^\alpha}{\Gamma(\alpha+1)}\}\{b_2\|y_g-y_h\|_{\mathscr{C}}\frac{\mathscr{T}^\alpha}{\Gamma(\alpha+1)}\} \\
+ &\quad \mathscr{L}b_1\|y_g-y_h\|_{\mathscr{C}}\frac{\mathscr{T}^\alpha}{\Gamma(\alpha+1)}\{K+b_2\|x_0\|_{\mathscr{E}}+b_2\|y_h\|_{\mathscr{C}}\frac{\mathscr{T}^\alpha}{\Gamma(\alpha+1)}\} \\
\leq &\quad \delta+\mathscr{L}b_1\{\|x_0\|_{\mathscr{E}}+rK_1\}\{b_2\|y_g-y_h\|_{\mathscr{C}}K_1\}+\mathscr{L}b_1\|y_g-y_h\|_{\mathscr{C}}K_1\{K+b_2\|x_0\|_{\mathscr{E}}+b_2rK_1\}
\end{aligned}
$$

Then

$$\|y_g-y_h\|_{\mathscr{C}}\leq\delta+\mathscr{L}b_1\{\|x_0\|_{\mathscr{E}}+rK_1\}\{b_2\|y_g-y_h\|_{\mathscr{C}}K_1\}+\mathscr{L}b_1\|y_g-y_h\|_{\mathscr{C}}K_1\{K+b_2\|x_0\|_{\mathscr{E}}+b_2rK_1\}$$

Hence

$$\|y_g-y_h\|_{\mathscr{C}}\leq\frac{\delta}{1-(\mathscr{L}b_1\{\|x_0\|_{\mathscr{E}}+rK_1\}b_2K_1+Lb_1K_1\{K+b_2\|x_0\|_{\mathscr{E}}+b_2rK_1)}$$

Since

$$x(t) = \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1} + I^{\alpha} y(t).$$

Then for the two corresponding solutions x_g, x_h, we have

$$\begin{aligned}
&\|x_g(t) - x_h(t)\|_{\mathscr{E}} \\
= \;& \|I^{\alpha} y_g(t) - I^{\alpha} y_h(t)\|_{\mathscr{E}} \\
\leq \;& I^{\alpha} \|y_g - y_h\|_{\mathscr{C}} \\
\leq \;& I^{\alpha} \frac{\delta}{1 - (\mathscr{L} b_1 \{\|x_0\|_{\mathscr{E}} + rK_1\} b_2 K_1 + L b_1 K_1 \{K + b_2 \|x_0\|_{\mathscr{E}} + b_2 r K_1)} \\
\leq \;& \frac{t^{\alpha}}{\Gamma(\alpha+1)} \frac{\delta}{1 - (\mathscr{L} b_1 \{\|x_0\|_{\mathscr{E}} + rK_1\} b_2 K_1 + L b_1 K_1 \{K + b_2 \|x_0\|_{\mathscr{E}} + b_2 r K_1)} \\
\leq \;& \frac{\mathscr{T}^{\alpha}}{\Gamma(\alpha+1)} \frac{\delta}{1 - (\mathscr{L} b_1 \{\|x_0\|_{\mathscr{E}} + rK_1\} b_2 K_1 + L b_1 K_1 \{K + b_2 \|x_0\|_{\mathscr{E}} + b_2 r K_1)} = \varepsilon.
\end{aligned}$$

Hence

$$\|x_g - x_h\|_{\mathscr{C}} \leq \varepsilon.$$

Which proves the continuous dependence of the solution on the set of selections S_G.

7. Example

Now we give an example as numerical application to illustrate main result contained in Theorem 3.4.

Let $\overline{\mathscr{Q}} = [-1, 1]$ and $\mathscr{J} = [0, 1]$. Consider the set-valued function
$G : \mathscr{J} \times \overline{\mathscr{Q}} \to \chi(\mathscr{E})$ defined by

$$G(t, u(t)) = \mathscr{L}(t + u(t))\overline{\mathscr{Q}}$$

is Lipschitzian this is since

$$\begin{aligned}
\mathscr{H}(G(t_1, u_1(t_1)), G(t_2, u_2(t_2))) &= \mathscr{H}((t_1 + u_1(t_1))\overline{\mathscr{Q}}, (t_2 + u_2(t_2))\overline{\mathscr{Q}}) \\
&= \|(t_1 + u_1(t_1)) - (t_2 + u_2(t_2))\| \\
&\leq \mathscr{L}\{|t_1 - t_2| + \|u_1(t_1) - u_2(t_2)\|\}.
\end{aligned}$$

Now let $g(t,u) = \mathscr{L}(t+u(t)) \in G(t,u)$.

Hence, we can apply our results to the problem

$$^{R}D^{\frac{1}{2}}(x(t)-0.1) = \mathscr{L}[t+\frac{t^2}{4}x(t)\ I^{\frac{1}{2}}(t\ ^{R}D^{\frac{1}{2}}x(t))],\ t\in\mathscr{J} \tag{7.1}$$

with the initial condition

$$x(0) = x_0 = 0.1. \tag{7.2}$$

Let

$$^{R}D^{\frac{1}{2}}(x(t)-0.1) = y(t) \in \mathscr{C}(\mathscr{J},\mathscr{E}).$$

Then

$$^{R}D^{\frac{1}{2}}x(t) = 0.1\frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} + I^{0}y(t).$$

Hence

$$y(t) = t+\frac{t^2}{4}(0.1+I^{\frac{1}{2}}y(t))I^{\frac{1}{2}}[t(0.1\frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}+I^{0}y(t))].$$

Here $g_1(t,x(t)) = \frac{t^2}{4} = \frac{t^2}{4}(0.1+I^{\frac{1}{2}}y(t))$,
$g_2(t,^{R}D^{\beta}x(t)) = g_2(t,\frac{t^{-\beta}}{\Gamma(1-\beta)}x_0+I^{\alpha-\beta}y(t)) = t(0.1\frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}+I^{0}y(t)) = (0.1\frac{t^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}+ty(t))$
and $\alpha=\beta=\frac{1}{2}$.

Now

$$\begin{aligned}
\|y(t)\| &= \|t+\frac{t^2}{4}[0.1+I^{\frac{1}{2}}y(t)]I^{\frac{1}{2}}[t(0.1\frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}+I^{0}y(t))]\| \\
&\leq t+[0.1\frac{t^2}{4}+I^{\frac{1}{2}}\|y(t)\|_{\mathscr{E}}\frac{t^2}{4}]I^{\frac{1}{2}}[0.1\frac{t^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}+t\|y(t)\|_{\mathscr{E}}]\| \\
&\leq t+[0.1\frac{t^2}{4}+\|y\|_{\mathscr{C}}I^{\frac{1}{2}}1\frac{t^2}{4}][\frac{0.1}{\sqrt{\pi}}I^{\frac{1}{2}}t^{\frac{1}{2}}+\|y\|_{\mathscr{C}}I^{\frac{1}{2}}t]\| \\
&\leq t+[0.1\frac{t^2}{4}+\|y\|_{\mathscr{C}}\int_0^t\frac{(t-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\frac{s^2}{4}ds][\frac{0.1}{\sqrt{\pi}}\int_0^t\frac{(t-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}s^{\frac{1}{2}}ds+\|y\|_{\mathscr{C}}\int_0^t\frac{(t-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}sds] \\
&\leq t+[0.1\frac{t^2}{4}+\frac{16\|y\|_{\mathscr{C}}}{15\sqrt{\pi}}t^{\frac{5}{2}}][\frac{0.1}{\sqrt{\pi}}\frac{1}{2}\sqrt{\pi}t+\frac{4\|y\|_{\mathscr{C}}}{3\sqrt{\pi}}t^{\frac{3}{2}}] \\
&\leq 1+[\frac{0.1}{4}+\frac{16}{15\sqrt{\pi}}r][\frac{0.1}{2}+\frac{4}{3\sqrt{\pi}}r] \\
&\leq 1+\frac{2.6}{30\sqrt{\pi}}r+\frac{64}{45\pi}r^2
\end{aligned}$$

The assumptions (H1)-(H7) of Theorem 3.4 are satisfies with $a_1(t) = a_2(t) = t,\ b_1 = b_2 = 0.1$ and $\mathscr{L} = 0.01$.

Therefore, by applying to Theorem 3.4, then the problems (7.1) and (7.2) has a solution $x \in \mathscr{J}$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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