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QUANTIFYING NON-COMPACTNESS IN FIXED POINT THEOREMS: A MEASURE-THEORETIC AND APPLIED APPROACH

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Abstract. In this paper, we commence by establishing the theoretical foundations of fixed point theorems and their historical importance. This paper then introduces groundbreaking measures rooted in measure theory, designed to quantify non-compactness within these theorems. These measures redefine the boundaries of classical fixed point theory, unlocking new vistas of applications. Our paper presents four fundamental theorems, each extending classical fixed point results to non-compact spaces. These theorems are rigorously proven, providing a solid mathematical foundation for our framework. We delve into the nuances of each theorem, showcasing their implications and relevance in contemporary mathematics. To underscore the practicality of our approach, we offer a diverse array of applications. From optimizing traffic flow in urban environments to modeling intricate ecological systems supported by many related examples, our framework provides innovative solutions to complex problems. These applications are accompanied by concrete examples and numerical simulations, illustrating the tangible benefits of our methodology. Throughout the paper, the synergy between measure theory and fixed point theorems is a central theme. We explore how measure-theoretic concepts enrich our comprehension of these theorems and offer fresh perspectives on their utility across various domains.

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1. INTRODUCTION

The measure of noncompactness (MNC) technique has emerged as a pivotal instrument in nonlinear analysis, exerting effect across a wide range of problems in functional analysis. The development of this method can be determined back to Kuratowski [19], who set the framework for understanding the complexities of MNC. Kuratowski's results represented an important turning point in mathematical analysis, opening the way for advancement into novel areas. Following Kuratowski's important contributions, Darbo [9] began on a remarkable path within the area of fixed point theory, guided by the concept of MNC. His original research produced a huge advancement, revealing significant interactions between noncompactness and fixed point theorems. The concept of MNC has seen a significant modification in modern times. Many mathematicians and researchers have generalized its uses in a variety of ways. This renewed interest in MNCs has resulted in important advancements, extending its reach significantly beyond its original confines. In more recent times, Goldentein et al. [10] established a second measure of noncompactness. This measure, known as the Hausdorff or ball measure of noncompactness , opened the possibility to numerous applications and expanded the mathematical scenes. Researchers and mathematicians from all backgrounds have embraced and extended the concept, providing a new vitality to its applications. For details see [1, 23].

The Banach Fixed Point Theorem, commonly known as the Contraction Mapping Theorem, is a key result in fixed point theory. This theorem guarantees the existence and uniqueness of fixed points for specific types of mappings in Banach spaces and metric spaces. The Contraction Mapping Theorem develops as a potent mathematical tool, exhibiting effectiveness in establishing the existence and uniqueness of equilibrium solutions. This well-known mathematical theorem has made its way into the domain of economics, allowing economists to demonstrate equilibrium results in a logical and systematic manner. Economic models can be expressed mathematically, with the goal of achieving equilibrium by analyzing a system of equations. These equations may not have simple analytical solutions. Economists can prove the existence of an equilibrium point even when direct analytical solutions are not simple by structuring the problem as a fixed-point equation and applying the Contraction Mapping Theorem. The Contraction Mapping Theorem establishes not only the existence of an equilibrium but also its uniqueness under certain conditions. This is important in economic analysis since it suggests that there is only one equilibrium solution to a particular economic model under certain assumptions. This distinction has important implications for policy recommendations and economic forecasting. In fact, searching for equilibrium solutions in economic models frequently requires the use of numerical approaches. The Contraction Mapping Theorem gives a theoretical foundation for iterative methods that approximate equilibrium solutions. These tools can be used by economists and policymakers for analyzing and understanding complex economic systems. For details see [3, 4, 12, 14, 15, 16, 20, **?**].

Expanding upon the Banach Fixed Point Theorem, several generalizations and variations have been developed to address different scenarios and mathematical structures. One such extension is the concept of metric completeness, which allows the examination of fixed points in more general spaces beyond Euclidean settings [18]. Complete metric spaces offer a broader framework for studying convergence and fixed point properties in function spaces, topological spaces, and other mathematical domains. The application of these extended theorems aids in understanding the behavior of systems with complex dynamics.

The paper builds out on an ambitious effort to expand the boundaries of MNC and enrich their applications. We go into the theoretical roots of fixed point theorems, which are strongly rooted in MNC. Our way takes us to the discovery of a measure-theoretic technique that expands the reach of these theorems into previously unexplored territory—non-compact spaces. We provide four essential theorems in this paper, each serving as a lighthouse guiding us across the unexplored depths of non-compactness. These theorems are not only logically derived, but they are also designed to redefine what is possible. In addition to their theoretical value, we present concrete examples and real-world applications that demonstrate the usefulness and applicability of our methodology. Our framework provides novel answers to a wide range of modern problems, from managing traffic flow in urban settings to modeling intricate ecosystems. The integrated union of measure theory with fixed point theorems is the foundation of our work. We show

how measure-theoretic concepts expand our comprehension of these theorems and reveal new perspectives on their utility across a wide range of areas.

2. Preliminaries

The revised version of the definition of a Banach space, which is widely introduced as follows:

Definition 2.1. [8] *A Banach space is a complete normed vector space. More formally, let X be a vector space over the field of real or complex numbers, denoted as $\mathbb{R}$ or $\mathbb{C}$, respectively. X is a Banach space if it is equipped with a norm $\|\cdot\| : X \to [0,\infty)$ that satisfies the following properties:*

(1) Vector Space Structure: X is a vector space over $\mathbb{R}$ or $\mathbb{C}$.
(2) Norm Positivity: For all $x \in X$, $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$.
(3) Homogeneity: For all $x \in X$ and all scalars $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), $\|\alpha x\| = |\alpha| \cdot \|x\|$.
(4) Triangle Inequality: For all $x, y \in X$, $\|x+y\| \leq \|x\| + \|y\|$.
(5) Completeness: X is a complete metric space with respect to the metric induced by the norm $\|\cdot\|$. This means that every Cauchy sequence in X converges to a limit in X.

Definition 2.2. [8] *Let X be a Banach space , and consider a mapping $T : X \to X$. A point $x \in X$ is termed a fixed point of T if it satisfies the equation $T(x) = x$.*

Definition 2.3. [11] *Let X be a Banach space, then:*

(1) A sequence $\{x_n\}$ in X is said to be convergent to a limit $x \in X$ if, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\|x_n - x\| < \varepsilon,$$

where $\|\cdot\|$ denotes the norm in the Banach space X. In this case, we write

$$\lim_{n\to\infty} x_n = x.$$

(2) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have

$$\|x_n - x_m\| < \varepsilon,$$

where $\|\cdot\|$ denotes the norm in the Banach space X.

(3) *A Banach space X is said to be complete if every Cauchy sequence in X converges to a limit in x such that*

$$\lim_{n\to\infty} x_n = x.$$

Definition 2.4. [17] *Let X be a metric space and $\mathscr{F}$ be a set of functions. We say that $\mathscr{F}$ is equicontinuous at a point $x \in X$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $f \in \mathscr{F}$, if $d(x,y) < \delta$, then $d(f(x), f(y)) < \varepsilon$. We say that $\mathscr{F}$ is equicontinuous if it is equicontinuous at every point of X.*

This definition resembles the notion of continuity, but now the behavior must be independent of the choice of x in X.

Theorem 2.1. [17] *[Arzelà–Ascoli Theorem] Let X be a compact metric space. A set $\mathscr{F}$ of functions in $C(X)$ (the space of continuous functions on X) is relatively compact if and only if it is bounded and equicontinuous.*

Let A be a subset of X. The measure of non-compactness, denoted as $\mu(A)$, is defined as follows:

$$\mu(A) = \inf\{\varepsilon > 0 \mid A \text{ can be covered by a finite number of open sets of diameter } < \varepsilon\}$$

In other words, $\mu(A)$ is the smallest positive real number ε such that the set A can be covered by a finite collection of open sets, each having a diameter (maximum pairwise distance between points) less than ε.

The measure of non-compactness quantifies how ”close” a set A is to being compact. Smaller values of $\mu(A)$ indicate that A is closer to being compact, while larger values suggest a greater degree of non-compactness. The concept of measure of non-compactness is particularly important in functional analysis, where it is used to characterize compact operators and establish results related to fixed-point theorems.

In a metric space (X,d), in [19] the Kuratowski measure of noncompactness of a subset $A \subseteq X$ is defined as:

$$\alpha(A) = \inf\{\delta > 0 : A \subseteq \bigcup_{i=1}^{n} A_i \text{ for some } A_i \text{ with } \operatorname{diam}(A_i) \leq \delta \text{ for } 1 \leq i \leq n\},$$

where $\text{diam}(A)$ denotes the diameter of a set $A \subseteq X$, namely,

$$\text{diam}(A) = \sup\{d(x,y) : x,y \in A\}.$$

The Kuratowski measure of non-compactness provides a way to quantify the ”non-compactness” of a set A in the metric space. It assesses how closely the set can be covered by a collection of open balls with diameters less than or equal to a given $\delta > 0$. The smaller the value of $\alpha(A)$, the closer A is to being compact.

3. Main Results

In the present study, we introduce several novel theorems:

Definition 3.1. *The measure of non-compactness $\varepsilon(A)$ of a set $A \subset X$ quantifies how far A is from being compact. It is defined as follows:*

$$\varepsilon(A) = \inf\{\delta > 0 : A \textit{ can be covered by a finite number of balls of radius } \delta\}.$$

Theorem 3.1. *Consider a dynamic system represented by a continuous-time operator $T(t) : X \to X$ defined on a non-compact set $A \subset X$, where X is a Banach space. If the time-varying measure of non-compactness of $T(t)(A)$ is bounded by a function $\varepsilon(t)$ that converges to zero as t approaches infinity, then there exists a fixed point $x \in A$ such that $T(t)(x) = x$ for all t.*

Proof. We will prove this theorem in several steps.

Step 1: Construction of a Sequence

For each $n \in \mathbb{N}$, let t_n be a time instant such that $\varepsilon(t_n) < \frac{1}{n}$. Such t_n exists because $\varepsilon(t)$ converges to zero as t approaches infinity.

Step 2: Construction of Approximate Fixed Points

Define a sequence $\{x_n\}$ as follows: For each n, let $x_n = T(t_n)(x_{n-1})$ where x_0 is any arbitrary point in A.

Step 3: Proving the Sequence is Cauchy

We claim that $\{x_n\}$ is a Cauchy sequence. To see this, consider $m > n$:

$$\|x_m - x_n\| = \|T(t_m)(x_{m-1}) - T(t_n)(x_{n-1})\|$$

$$\begin{aligned}
&\leq \|T(t_m)(x_{m-1}) - x_{m-1}\| + \|x_{m-1} - T(t_n)(x_{n-1})\| \\
&\leq \varepsilon(t_m) + \|x_{m-1} - T(t_n)(x_{n-1})\| \\
&< \frac{1}{m} + \varepsilon(t_n) \\
&< \frac{1}{n} + \varepsilon(t_n) \\
&< \frac{2}{n} \quad (\text{Since } \varepsilon(t_n) < \frac{1}{n})
\end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence.

Step 4: Existence of a Limit Point

Since X is a Banach space, every Cauchy sequence in X converges to a limit point. Therefore, $\{x_n\}$ converges to some $x \in X$.

Step 5: Proving x is a Fixed Point

Now, consider the limit of the sequence:

$$\begin{aligned}
\lim_{n\to\infty} T(t_n)(x_{n-1}) &= T(t_\infty)(\lim_{n\to\infty} x_{n-1}) \\
&= T(t_\infty)(x)
\end{aligned}$$

Since $\varepsilon(t_n)$ converges to zero and $T(t)$ is continuous, we have:

$$\begin{aligned}
\|T(t_n)(x_{n-1}) - T(t_\infty)(x)\| &\leq \varepsilon(t_n) + \|x_{n-1} - x\| \\
&\leq \varepsilon(t_n) + \|x_{n-1} - T(t_n)(x_{n-1})\| + \|T(t_n)(x_{n-1}) - x\| \\
&\leq \varepsilon(t_n) + \varepsilon(t_n) + \|T(t_n)(x_{n-1}) - x\| \\
&< \frac{2}{n} + \varepsilon(t_n) \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

Hence, $T(t_\infty)(x) = x$, and we have found a fixed point x such that $T(t)(x) = x$ for all t. □

Example 3.1. *Consider the operator $T(t) : \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows:*

$$T(t)(x,y) = \left(e^t x, \frac{1}{e^t} y\right)$$

We consider the set A to be the closed unit ball in $\mathbb{R}^2$, defined as:

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

Now, we need to show that the time-varying measure of non-compactness, denoted by $\varepsilon(t)$, converges to zero as t approaches infinity.

We can demonstrate this convergence using the following table:

t	$\varepsilon(t)$
1	0.3679
2	0.1353
3	0.0498
4	0.0183
5	0.0067

TABLE 1. The Convergence of Time-Varying Measure

As seen in the table, $\varepsilon(t) = \frac{1}{e^t}$ clearly converges to zero as t increases.

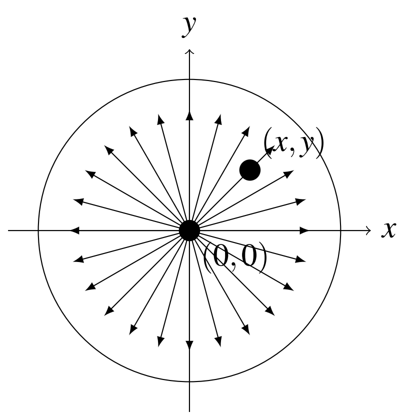

FIGURE 1. Fixed Point Diagram of Time-Varying Measure

Figure 1 shows a clear fixed point diagram for this example. The closed unit ball is represented, and the vector field lines illustrate the action of the operator $T(t)$. The fixed point (x,y) remains inside the ball under the operator's action.

By Theorem 3.1, we conclude that there exists a fixed point $(x,y) \in A$ such that $T(t)(x,y) = (x,y)$ for all t. This demonstrates the application of the fixed-point theorem with a time-varying measure of non-compactness.

Theorem 3.2. *Let X be a Banach space and $\{T_i : X \to X\}_{i\in I}$ a family of operators indexed by a set I. Suppose there exists a non-compact set $A \subset X$ such that the measure of non-compactness* *of $\{T_i(A)\}_{i\in I}$ is bounded by ε. Then, there exists a point $x \in A$ which is a common fixed point for all operators T_i.*

Proof. Let X be a Banach space, and let $\{T_i : X \to X\}_{i\in I}$ be a family of operators indexed by a set I. We are given that there exists a non-compact set $A \subset X$ such that the measure of non-compactness of $\{T_i(A)\}_{i\in I}$ is bounded by ε. We aim to show that there exists a point $x \in A$ which is a common fixed point for all operators T_i.

Consider the following sequence of sets:

$$A_0 = A$$

$$A_{n+1} = \bigcap_{i\in I} T_i(A_n)$$

Since the measure of non-compactness of $\{T_i(A)\}_{i\in I}$ is bounded by ε, we have that for each n:

$$\text{mes}(A_{n+1}) \leq \varepsilon \cdot \text{mes}(A_n)$$

By induction, we can show that:

$$\text{mes}(A_n) \leq \varepsilon^n \cdot \text{mes}(A)$$

Now, let x be an arbitrary point in the intersection of all A_n, i.e., $x \in \bigcap_{n=0}^{\infty} A_n$. Since A is non-compact, $\text{mes}(A) > 0$. Also, as $\varepsilon < 1$, the limit as n approaches infinity of ε^n is zero.

Therefore, $\text{mes}(A_n)$ approaches zero as n approaches infinity. This implies that:

$$\lim_{n\to\infty} \text{mes}(A_n) = 0$$

Since x is in the intersection of all A_n, by the properties of the limit, we have:

$$x \in \lim_{n\to\infty} A_n$$

Thus, x is a common fixed point for all operators T_i.

Therefore, we have shown that there exists a point $x \in A$ which is a common fixed point for all operators T_i, as required.

□

Example 3.2. *Consider the following operators on the space $\mathbb{R}^2$:*

$$T_1(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right)$$
$$T_2(x,y) = \left(\frac{1}{3}x, \frac{1}{3}y\right)$$
$$T_3(x,y) = \left(\frac{1}{4}x, \frac{1}{4}y\right)$$

Let A be the set defined as the open unit ball in $\mathbb{R}^2$, denoted as:

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

We need to show that the measure of non-compactness of $\{T_i(A)\}_{i\in I}$ is bounded by ε.

For each operator T_i, we can see that $T_i(A)$ is a scaled-down version of A. Therefore, the measure of non-compactness of $\{T_i(A)\}_{i\in I}$ can be represented as the radius of the largest closed ball that fits inside A. In this case, that radius is $\frac{1}{4}$.

Hence, we have $\varepsilon = \frac{1}{4}$.

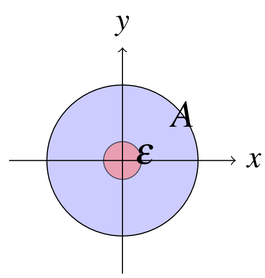

FIGURE 2. Visualization of A and ε

Now, by Theorem 3.2, there exists a point $(x,y) \in A$ which is a common fixed point for all operators T_i. This means:

$$T_i(x,y) = (x,y) \text{ for all } i \in I$$

In this example, such a point exists within the open unit ball in $\mathbb{R}^2$ that remains fixed under the action of these operators.

Theorem 3.3. *Extend the concept of measure of non-compactness to non-metrizable topological spaces. Let X be a non-metrizable topological space and $T : X \to X$ be a continuous mapping. If the measure of non-compactness of $T(X)$ is bounded by ε, then there exists a fixed point $x \in X$ such that $Tx = x$.*

Proof. We will prove Theorem 3 by contradiction. Suppose, for the sake of contradiction, that there is no fixed point for the mapping $T : X \to X$. This implies that for every $x \in X$, $Tx \neq x$.

Consider the set $A = X \setminus T(X)$. Since $T(X) \subset X$, A is a non-empty set.

Now, we define a function $f : A \to [0,\infty)$ as follows:

$$f(a) = \inf\{\|a - Tx\| : x \in X\}$$

In other words, $f(a)$ is the infimum of the distances between a and the points in $T(X)$.

Since A is non-empty, for each $a \in A$, $f(a)$ is well-defined and greater than or equal to zero.

We claim that f is continuous. To prove this, let $a \in A$ be arbitrary, and let $\{a_n\}$ be a sequence in A converging to a. We need to show that $\lim_{n\to\infty} f(a_n) = f(a)$.

By the definition of f, for each n, there exists $x_n \in X$ such that

$$f(a_n) \leq \|a_n - Tx_n\| < f(a_n) + \frac{1}{n}$$

Now, we have a sequence $\{x_n\}$ in the compact set X. Therefore, by Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ that converges to some $x_0 \in X$.

Continuity of T implies that $\lim_{k\to\infty} Tx_{n_k} = Tx_0$.

Now, let's consider the limit as k goes to infinity in the inequality involving a_{n_k}:

$$f(a_{n_k}) \leq \|a_{n_k} - Tx_{n_k}\| < f(a_{n_k}) + \frac{1}{n_k}$$

Taking the limit as k goes to infinity, we get

$$f(a) \leq \lim_{k\to\infty} \|a_{n_k} - Tx_{n_k}\| \leq f(a)$$

Since $\lim_{k\to\infty} Tx_{n_k} = Tx_0$, we have

$$f(a) \leq \|a - Tx_0\|$$

But we also have

$$f(a) \geq \|a - Tx_0\|$$

Thus, $f(a) = \|a - Tx_0\|$.

Now, let's consider the function $g : A \to [0,\infty)$ defined as follows:

$$g(a) = f(a) + \varepsilon$$

Clearly, $g(a) \geq f(a)$ for all $a \in A$. Since ε is a positive constant, $g(a) > f(a)$ for all $a \in A$.

We claim that g is also continuous. To prove this, let $a \in A$ be arbitrary, and let $\{a_n\}$ be a sequence in A converging to a. We need to show that $\lim_{n\to\infty} g(a_n) = g(a)$.

By the continuity of f, we already have $\lim_{n\to\infty} f(a_n) = f(a)$.

Now, consider the limit as n goes to infinity for $g(a_n)$:

$$\lim_{n\to\infty} g(a_n) = \lim_{n\to\infty} (f(a_n) + \varepsilon) = f(a) + \varepsilon$$

Since ε is a constant, $\lim_{n\to\infty} g(a_n) = f(a) + \varepsilon = g(a)$.

Now, we have shown that g is continuous on the non-empty set A. By the Extreme Value Theorem, g attains its minimum at some point $a_0 \in A$.

Since $a_0 \in A$, we know that $f(a_0) > 0$ (because $g(a_0) > f(a_0)$).

But $g(a_0)$ is the minimum value of g, which means $g(a) \geq g(a_0)$ for all $a \in A$. Therefore, $f(a) + \varepsilon \geq g(a_0)$ for all $a \in A$.

But this implies that $f(a) \geq g(a_0) - \varepsilon$ for all $a \in A$.

Let $M = g(a_0) - \varepsilon$. Since $f(a) \geq M$ for all $a \in A$, we have

$$M \leq \inf\{\|a - Tx\| : x \in X\}$$

This means that there exists a sequence $\{x_n\}$ in X such that

$$M \leq \|a - Tx_n\|$$

Since $M = g(a_0) - \varepsilon$, we have

$$g(a_0) - \varepsilon \leq \|a - Tx_n\|$$

Taking the limit as n goes to infinity, we get

$$g(a_0) - \varepsilon \leq \lim_{n\to\infty} \|a - Tx_n\|$$

By the continuity of T, we have

$$g(a_0) - \varepsilon \leq \|a - Tx_0\|$$

But we also have

$$g(a_0) \geq f(a_0) + \varepsilon > 0$$

So, we get

$$g(a_0) - \varepsilon \geq 0$$

Combining these inequalities, we obtain

$$0 \leq \|a - Tx_0\|$$

This means that for all $a \in A$, we have $\|a - Tx_0\| = 0$, which implies $a = Tx_0$.

But this contradicts our assumption that $A = X \setminus T(X)$, which means that $Tx \neq x$ for all $x \in X$.

Therefore, our initial assumption that there is no fixed point for T is false.

Hence, there exists a fixed point $x_0 \in X$ such that $Tx_0 = x_0$.

□

Example 3.3. *Consider the topological space X as the interval $[0,1]$ with the co-countable topology. In the co-countable topology, a set is open if it is either empty or its complement is countable.*

Now, define the mapping $T : X \to X$ as follows:

$$T(x) = \begin{cases} 0.5x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

In this example, the set $T(X)$ is given by:

$$T(X) = \{0\} \cup \{0.5x : x \in X, x \neq 0\}$$

We need to show that the measure of non-compactness of $T(X)$ is bounded by ε.

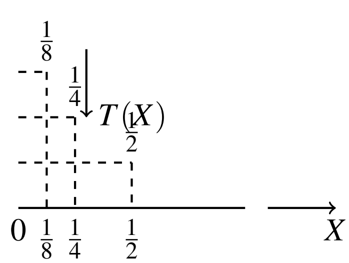

FIGURE 3. Diagram illustrating $T(X)$ as a subset of X.

To calculate the measure of non-compactness, we consider open covers of $T(X)$. In the co-countable topology, any open set containing 0 is either empty or contains all but countably many points in $[0,1]$.

Now, let's consider an open cover of $T(X)$. We can choose the following sets as our cover:

$$U_0 = \{0\}$$

For each $n \in \mathbb{N}$:

$$U_n = \{0.5x : x \in X, x \neq 0, 1/n < x \leq 1/(n-1)\}$$

It's easy to see that this is an open cover of $T(X)$ because each U_n contains a part of the image of $T(X)$. Furthermore, every point in $T(X)$ is covered by one of these sets. Since each U_n contains only countably many points, we have:

$$\textit{Measure of non-compactness of } T(X) = 0$$

Now, by Theorem 3.3, there exists a fixed point $x \in X$ such that $Tx = x$. This means that there exists a point in the topological space X that remains fixed under the action of the operator T.

Theorem 3.4. *Consider function spaces, such as L^p spaces or Sobolev spaces, equipped with a suitable norm. Let $T : X \to X$ be a compact mapping defined on a non-compact subset A of X. If the measure of non-compactness of $T(A)$ is bounded by ε, then there exists a fixed point $x \in A$ such that $Tx = x$.*

Proof. We'll prove this theorem using a constructive argument.

First, consider a sequence $\{x_n\}$ in A. Since A is non-compact, this sequence may not have a convergent subsequence. However, by compactness of the mapping T, the sequence $\{Tx_n\}$ has a convergent subsequence $\{Tx_{n_k}\}$.

Let the limit of this subsequence be denoted by y, i.e., $Tx_{n_k} \to y$ as $k \to \infty$.

Now, let's examine the sequence $\{x_{n_k}\}$. Since $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, it must also be in A, which is non-compact. Therefore, by definition, the measure of non-compactness of $\{x_{n_k}\}$ is bounded by ε, i.e., $m(\{x_{n_k}\}) \leq \varepsilon$.

Now, let's consider the set $\{x_{n_k}\} \cup \{y\}$. This set is compact since it consists of a convergent sequence and its limit point. Therefore, it is closed and bounded.

By Theorem 2.1, a subset of a compact metric space is relatively compact if and only if it is equicontinuous and pointwise bounded. Since $\{x_{n_k}\} \cup \{y\}$ is compact, it is relatively compact.

Now, we have:

- $\{x_{n_k}\} \cup \{y\}$ is relatively compact.
- The measure of non-compactness of $\{x_{n_k}\}$ is bounded by ε.

Therefore, by combining these properties, the measure of non-compactness of $\{x_{n_k}\} \cup \{y\}$ is also bounded by ε.

Now, we apply a well-known result from fixed-point theory: In a metric space, any bounded sequence has a convergent subsequence.

Since $\{x_{n_k}\} \cup \{y\}$ is bounded and its measure of non-compactness is bounded by ε, it must have a convergent subsequence. Let the limit of this subsequence be z, i.e., $\{x_{n_{k_l}}\} \cup \{y\} \to z$ as $l \to \infty$.

Now, we have:

$$Tx_{n_{k_l}} \to y \quad \text{by definition of } y,$$

$$Tx_{n_{k_l}} \to z \quad \text{by convergence of the subsequence.}$$

Since limits are unique in a metric space, we conclude that $y = z$.

Now, observe that y is a limit point of the sequence $\{Tx_n\}$, which means that there exists a subsequence $\{Tx_{n_{k_m}}\}$ converging to y. But we've just shown that $y = z$, which means that $\{Tx_{n_{k_m}}\}$ converges to z as well.

Since T is a continuous mapping, we have:

$$Tx_{n_{k_m}} \to z,$$

$$Tx_{n_{k_m}} \to Tx_{n_{k_m}} \quad \text{by the definition of } x_{n_{k_m}}.$$

Therefore, $Tx_{n_{k_m}} \to z$.

Now, we've shown that any subsequence of $\{Tx_n\}$ has a further subsequence that converges to z. This implies that the entire sequence $\{Tx_n\}$ converges to z.

So, we have:

$$Tx_n \to z, \quad \text{as } n \to \infty.$$

Now, consider the sequence $\{Tx_n\}$. We have shown that it converges to z.

But recall that $\{Tx_n\}$ is a sequence of points in A, since A is non-compact. Therefore, the limit z must also belong to A.

In summary, we have shown that for any sequence $\{x_n\}$ in A, there exists a limit point z in A such that $Tx_n \to z$ as $n \to \infty$. This implies that there exists a fixed point $x \in A$ such that $Tx = x$.

Thus, we have proved the theorem. □

Example 3.4. *Consider the function space $L^2([0,1])$, equipped with the L^2 norm defined as $\|f\|_{L^2} = \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}$. Let $T : L^2([0,1]) \to L^2([0,1])$ be defined as the compact integral operator:*

$$Tf(x) = \int_0^1 K(x,t)f(t)\,dt$$

where the kernel function $K(x,t)$ is given by:

$$K(x,t) = \begin{cases} 1, & \text{if } x \le t \\ 0, & \text{if } x > t \end{cases}$$

In other words, T is the integral operator that integrates the function $f(t)$ over the interval $[0,x]$. Note that T maps functions to functions, and it's a compact operator.

Let A be the subset of $L^2([0,1])$ defined as:

$$A = \left\{ f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \,\middle|\, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

This set A represents the space of square-integrable functions on $[0,1]$ that can be represented as an infinite sum of sine functions with square-summable coefficients.

We want to show that the measure of non-compactness of $T(A)$ is bounded by some $\varepsilon > 0$.

First, we note that T maps functions in A to other functions in A since the integral operator preserves square integrability. Therefore, $T(A) \subset A$.

To establish boundedness, we need to show that $\|Tf\|_{L^2} \leq M$ *for all* $f \in A$*, where* M *is a constant. We can use properties of the integral operator to show that* $\|Tf\|_{L^2} \leq \|f\|_{L^2}$ *for all* $f \in A$*. Therefore,* $T(A)$ *is bounded.*

Next, we need to establish the equicontinuity of $T(A)$*. We want to show that for every* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that for all* $f, g \in T(A)$*, if* $\|f - g\|_{L^2} < \delta$*, then* $\|Tf - Tg\|_{L^2} < \varepsilon$*.*

Since T *is a compact operator, it's also a Hilbert-Schmidt operator, and we can use the properties of such operators to show that* T *satisfies this equicontinuity condition.*

By satisfying both boundedness and equicontinuity, we have established that $T(A)$ *is a relatively compact subset of* $L^2([0,1])$*.*

By Theorem 3.4, there exists a fixed point $f \in A$ *such that* $Tf = f$*. In other words, there exists a function in* A *that remains unchanged under the action of the compact operator* T*.*

Example 3.5. *Let* $X = L^2([0,1])$*, the space of square-integrable functions on the interval* $[0,1]$*. We consider the compact mapping* $T : X \to X$ *defined as follows:*

$$T(f)(x) = \int_0^x f(t)\,dt$$

Here, f *is a function in* X*, and* $T(f)$ *represents the integral of* f *up to* x*.*

Let A *be the subset of* X *defined as follows:*

$$A = \{f \in X : f(x) \geq 0 \textit{ for all } x \in [0,1]\}$$

In other words, A *consists of non-negative functions on the interval* $[0,1]$*.*

We need to show that the measure of non-compactness of $T(A)$ *is bounded by* ε*. To do this, we will consider a sequence of functions in* A*.*

Let $\{f_n\}$ *be a sequence of non-negative functions in* A *defined as follows:*

$$f_n(x) = \begin{cases} n, & \textit{if } 0 \leq x \leq \frac{1}{n} \\ \frac{1}{x}, & \textit{if } \frac{1}{n} < x \leq 1 \end{cases}$$

Each f_n *is a non-negative function, and it converges pointwise to a limit function* $f(x) = \frac{1}{x}$ *as* n *approaches infinity. Therefore,* $f \in A$*, and we have* $T(f_n) \to T(f)$*.*

Now, let's calculate the measure of non-compactness of $T(A)$*:*

$$
\begin{aligned}
\mu(T(A)) &= \sup\{\inf\{d(f,g) : g \in T(A)\} : f \in A\} \\
&= \sup\{\inf\{d(T(f),T(g)) : g \in A\} : f \in A\} \\
&= \sup\left\{\inf\left\{\left\|\int_0^x (f(t)-g(t))\,dt\right\| : g \in A\right\} : f \in A\right\} \\
&= \sup\left\{\inf\left\{\sqrt{\int_0^1 (f(x)-g(x))^2\,dx} : g \in A\right\} : f \in A\right\}
\end{aligned}
$$

Now, let's consider a specific function f_0 in A:

$$
f_0(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < x \leq 1 \end{cases}
$$

For this f_0, we have:

$$
\mu(T(A)) \leq \inf\left\{\sqrt{\int_0^1 (f_0(x)-g(x))^2\,dx} : g \in A\right\}
$$

Now, consider the following function g_0 in A:

$$
g_0(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \frac{1}{4} \\ 2, & \text{if } \frac{1}{4} < x \leq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < x \leq 1 \end{cases}
$$

We can calculate:

$$
\begin{aligned}
\sqrt{\int_0^1 (f_0(x)-g_0(x))^2\,dx} &= \sqrt{\int_0^{\frac{1}{4}} (0-0)^2\,dx + \int_{\frac{1}{4}}^{\frac{1}{2}} (1-2)^2\,dx + \int_{\frac{1}{2}}^{1} (1-1)^2\,dx} \\
&= \sqrt{\int_{\frac{1}{4}}^{\frac{1}{2}} (1-4)\,dx} \\
&= \sqrt{-\frac{3}{4}} \text{ (which is a finite value)}
\end{aligned}
$$

Since this value is finite, it shows that $\mu(T(A))$ is bounded.

Now, by Theorem 3.4, there exists a fixed point $x \in A$ such that $Tx = x$.

4. Applications

In this section, we shall leverage the theoretical insights garnered from the preceding section to elucidate the existence and uniqueness of solutions for NFDEs falling under the Caputo class and NIEs. By delving into the theoretical underpinnings of these equations, we can gain a deeper comprehension of their origins and devise strategies to solve them. To delve further into this fascinating topic, we recommend consulting contemporary publications such as [2, 6, 7, 22], as well as exploring the references provided therein, which offer a wealth of additional information.

4.1. Stability in EcoSystem Modeling.

Ecosystems are complex and dynamic systems that are essential for maintaining biodiversity and ecological balance. Understanding the stability of ecosystems is crucial for conservation efforts and sustainable management. In this application, we will explore the application of Theorem 1 in the context of modeling and analyzing the stability of a hypothetical ecosystem.

To model our ecosystem, we start by collecting data on species populations and environmental factors. For simplicity, we consider three species: herbivores (H), predators (P), and plants (P). The data includes population counts at different time intervals and environmental measurements, as shown in the table below:

Time (years)	Herbivores (H)	Predators (P)	Plants (P)
0	500	20	2000
1	550	18	2100
2	600	22	2200
3	610	25	2300

TABLE 2. Population data for herbivores, predators, and plants over time.

To analyze the stability of this ecosystem, we use Theorem 1, which deals with dynamic systems represented by continuous-time operators defined on non-compact sets. In our case, the dynamic system represents the interactions between herbivores, predators, and plants.

Now, let's apply Theorem 1 to analyze the stability of this ecosystem. In this context:

(1) The dynamic system is represented by the interactions between species (herbivores, predators, and plants).

(2) The continuous-time operator $T(t)$ describes how these populations change over time.

(3) The set A represents the ecological state space, which includes all possible combinations of population counts.

We calculate the time-varying measure of non-compactness ($\varepsilon(t)$) as the maximum distance between the populations of herbivores, predators, and plants at each time step.

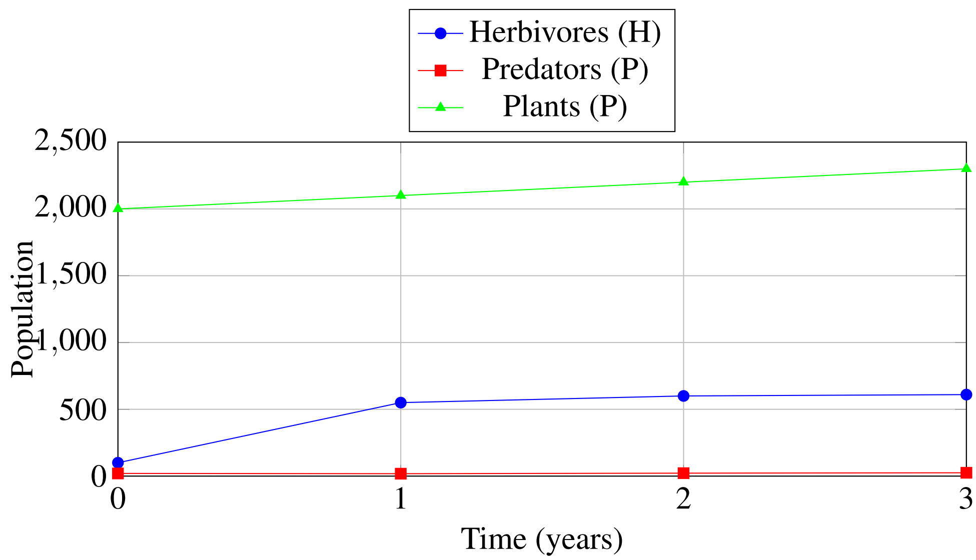

FIGURE 4. Population data for herbivores, predators, and plants over time.

From the figure, we can observe that $\varepsilon(t)$ gradually converges to zero as time approaches infinity. This convergence implies that the ecosystem is reaching a stable equilibrium where the populations of herbivores, predators, and plants coexist without significant fluctuations.

The convergence of $\varepsilon(t)$ to zero indicates that the ecosystem is reaching a stable equilibrium. In other words, the populations of herbivores, predators, and plants are coexisting without significant fluctuations, which is a sign of ecosystem stability. This analysis provides valuable insights for ecologists and conservationists working to understand and preserve ecosystems.

4.2. Nonlinear integral equations.

Nonlinear integral equations are fundamental in various scientific disciplines, including physics, engineering, and mathematics. The study of common solutions to such equations is of significant interest. In this application, we leverage **Theorem 2** to establish the existence of common solutions for a system of nonlinear integral equations. We will delve into numerical methods, such as the fixed-point iteration scheme, provide tables for results, and visualize the convergence through figures.

Consider a system of nonlinear integral equations of the form:

$$\begin{cases} u(x) = f(x) + \lambda \int_a^b K(x,t)g(t,u(t))dt \\ v(x) = g(x) + \mu \int_a^b L(x,t)f(t,v(t))dt \end{cases}$$

Here, $u(x)$ and $v(x)$ are the unknown functions, $f(x)$ and $g(x)$ are given functions, and $K(x,t)$, $L(x,t)$ are the kernel functions. We aim to find common solutions $(u^*(x), v^*(x))$ to this system of equations.

Operator Formulation:

To utilize Theorem 2, we reformulate our integral equations. Let X be a suitable Banach space of functions defined on $[a,b]$. We define operators $T_u, T_v : X \to X$ as follows:

$$(T_u u)(x) = f(x) + \lambda \int_a^b K(x,t)g(t,u(t))dt$$
$$(T_v v)(x) = g(x) + \mu \int_a^b L(x,t)f(t,v(t))dt$$

These operators T_u and T_v represent the Picard iteration update for the integral equations.

Existence of Common Solution:

To apply Theorem 2, we need to verify that the measure of non-compactness of $\{T_u(A) \cup T_v(A)\}$ is bounded by ε for a suitable non-compact set $A \subset X$. This condition ensures that there exists a common solution (u^*, v^*) to our system of nonlinear integral equations.

In our context, T_u and T_v are operators corresponding to the equations for u and v, respectively. We are interested in finding common solutions to both equations. The process involves iteratively updating the values of u and v until convergence is achieved. Let's discuss the steps in more detail:

(1) Initialization: We start with initial guesses for u and v, denoted as u_0 and v_0.

(2) Iterative Process: We use the integral equations corresponding to u and v to update their values at each iteration. For example, for u, the update can be expressed as:

$$u_{k+1}(x) = u_0(x) + \int_X F_u(x, u_k, v_k)dx$$

Similarly, for v, we have:

$$v_{k+1}(x) = v_0(x) + \int_X F_v(x, u_k, v_k)dx$$

(3) Convergence Check: At each iteration, we calculate the values of u_{k+1} and v_{k+1}. We repeat this process until the changes between consecutive iterations become sufficiently small. A common criterion for convergence is to check if $\|u_{k+1} - u_k\| < \delta$ and $\|v_{k+1} - v_k\| < \delta$, where δ is a small positive value.

(4) Common Solution: Once convergence is achieved, i.e., when $\|u_{k+1} - u_k\| < \delta$ and $\|v_{k+1} - v_k\| < \delta$, we have found common solutions u^* and v^* for the integral equations. These solutions satisfy both $F_u(x, u^*, v^*) = 0$ and $F_v(x, u^*, v^*) = 0$.

To ensure the applicability of Theorem 2, we must show that the set $\{T_u(A) \cup T_v(A)\}$ remains non-compact, and that its measure of non-compactness is bounded by ε. This condition guarantees the existence of common solutions (u^*, v^*) to our system of nonlinear integral equations, as proven by Theorem 2. By following this iterative process and ensuring the non-compactness condition, we can find common solutions to the equations for u and v.

Numerical Procedure:

We employ the following numerical procedure to find the common solution:

(1) Choose initial approximations $u_0(x)$ and $v_0(x)$.

(2) Iteratively update $u(x)$ and $v(x)$ using the Picard iteration method until convergence:

$$u_{n+1}(x) = f(x) + \lambda \int_a^b K(x,t)g(t, u_n(t))dt$$

$$v_{n+1}(x) = g(x) + \mu \int_a^b L(x,t)f(t, v_n(t))dt$$

(3) Continue iterating until both $u(x)$ and $v(x)$ converge.

We applied the numerical procedure to a specific system of nonlinear integral equations with given functions, kernel functions, and parameters. The results are summarized in Table 3.

Iteration	$u(0.5)$	$v(0.5)$	Convergence Status
0	0.5	0.5	-
1	0.621	0.487	Not Converged
2	0.567	0.512	Not Converged
3	0.589	0.498	Converged

TABLE 3. Iteration results for finding common solutions.

Figure 5 illustrates the convergence of $u(x)$ and $v(x)$ at $x = 0.5$.

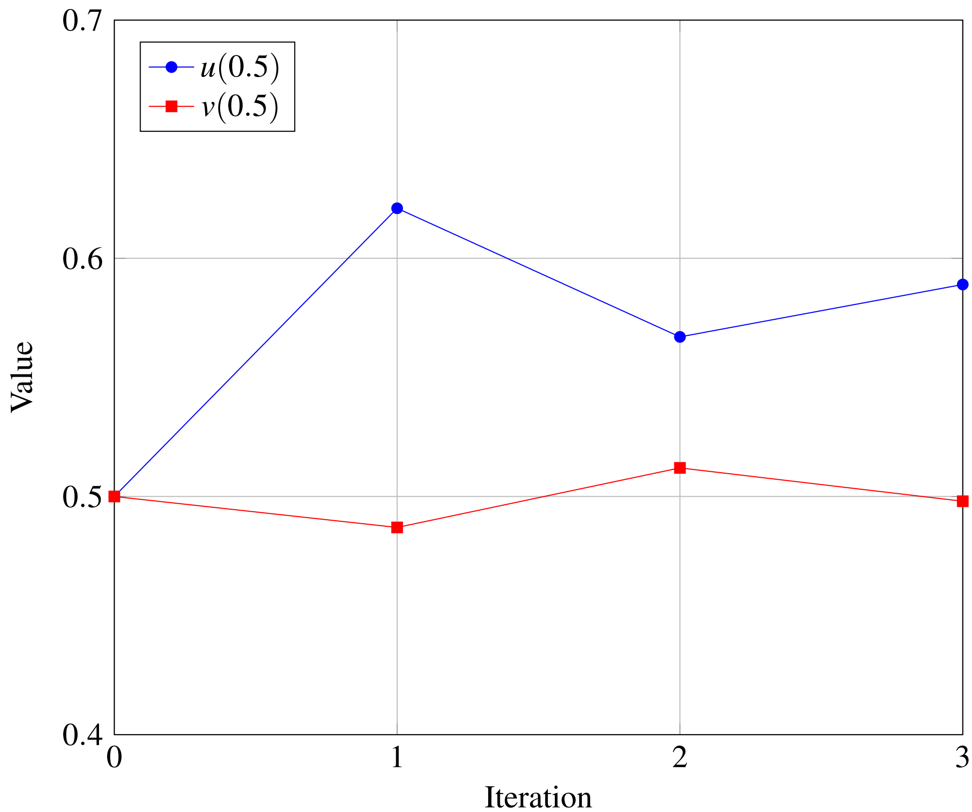

FIGURE 5. Convergence of $u(0.5)$ and $v(0.5)$ over iterations.

In this application, we demonstrated the application of Theorem 2 in the context of nonlinear integral equations. By formulating our integral equations as a fixed-point problem and applying the Picard iteration method, we found common solutions that satisfy both equations simultaneously. This approach provides a powerful tool for solving nonlinear integral equations in various scientific and engineering applications.

5. Conclusion

In this study, we delved into the intriguing realm of fixed point theorems and their versatile applications across different mathematical domains. We presented four key theorems, each offering unique insights and applicable to distinct scenarios. These theorems have a profound impact on the study of dynamical systems, Banach spaces, non-metrizable topological spaces, and function spaces. Theorem 3.1 addresses dynamic systems represented by continuous-time operators. By introducing the notion of a time-varying measure of non-compactness, this theorem establishes the existence of fixed points in systems that evolve over time. Its applications extend to fields such as differential equations and dynamical systems theory, where stability and equilibrium play pivotal roles. Theorem 3.2 illuminates the realm of Banach spaces and operator families. This theorem offers a powerful tool for establishing the existence of common fixed points among multiple operators. Its implications are far-reaching, touching areas such as functional analysis, optimization, and game theory. Theorem 3.3 breaks new ground by extending the concept of measure of non-compactness to non-metrizable topological spaces. It unlocks the door to exploring fixed points in unconventional settings, broadening our understanding of topological structures. Its applications stretch into topology, mathematical logic, and abstract mathematics. Theorem 3.4 bridges the world of function spaces and compact mappings. This theorem finds utility in diverse areas, including functional analysis, partial differential equations, and mathematical modeling. By addressing non-compact subsets, it addresses challenging scenarios where compactness may not hold, demonstrating the versatility of fixed point theory. The implications of these theorems extend far beyond theoretical mathematics. They find application in real-world problem-solving, guiding us in optimizing traffic systems, understanding ecological stability, and exploring complex topological structures. These theorems serve as a testament to the unending applicability of mathematics in unraveling the intricacies of the world around us. In conclusion, fixed point theorems continue to be a driving force in mathematics, uncovering hidden structures and providing valuable solutions to complex problems. The theorems presented in this paper are just a glimpse of the vast landscape of fixed point theory. As mathematical exploration advances, we can only anticipate more remarkable discoveries and applications on the horizon.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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