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SOME COMMON FIXED POINT THEOREMS IN B-METRIC SPACES VIA $\mathscr{F}$-CLASS FUNCTION WITH APPLICATIONS

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Abstract. This article explores the existence of common fixed points for two pairs of self-maps satisfying a contractive condition involving rational expression using $\mathscr{F}$-class function in complete b-metric spaces. In order to support our findings, we draw some corollaries and give examples. Finally, we present applications to nonlinear integral, functional equations and fractional differential equations.

Keywords: $\mathscr{F}$-class functions; integral equations; functional equations; fractional differential equations.
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1. Introduction

Fixed point theory plays a vital role in solving nonlinear equations. Czerwik [7] established the concept of b-metric space or metric type space as a generalization of metric space. Aamari and Moutawakil [1] introduced the concept of property (E.A) in 2002.

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Definition 1.1. [7] Let $\mathfrak{S}$ be a non-empty set and $s \geq 1$ is a given real number. A function $\mathfrak{d} : \mathfrak{S} \times \mathfrak{S} \to [0,\infty)$ is said to be a b-metric if the following conditions are satisfied: for any $\xi, \zeta, \eta \in \mathfrak{S}$

(*i*) $0 \leq \mathfrak{d}(\xi,\zeta)$ and $\mathfrak{d}(\xi,\zeta) = 0$ if and only if $\xi = \zeta$,

(*ii*) $\mathfrak{d}(\xi,\zeta) = \mathfrak{d}(\zeta,\xi)$,

(*iii*) $\mathfrak{d}(\xi,\eta) \leq s[\mathfrak{d}(\xi,\zeta) + \mathfrak{d}(\zeta,\eta)]$.

The pair $(\mathfrak{S},\mathfrak{d})$ is called a b-metric space with coefficient s.

Definition 1.2. [8] Let $\mathfrak{f},\mathfrak{g} : \mathfrak{S} \to \mathfrak{S}$ be two self-maps. If $\mathfrak{f}\xi = \mathfrak{g}\xi$ implies that $\mathfrak{f}\mathfrak{g}\xi = \mathfrak{g}\mathfrak{f}\xi$ for $\xi \in \mathfrak{S}$, then we say that the pair $(\mathfrak{f},\mathfrak{g})$ is weakly compatible.

Definition 1.3. [11] Two self-maps $\mathfrak{f}$ and $\mathfrak{g}$ of a metric space $(\mathfrak{S},\mathfrak{d})$ are called reciprocally continuous if $\lim_{n\to\infty} \mathfrak{f}\mathfrak{g}\xi_n = \mathfrak{f}\eta$ and $\lim_{n\to\infty} \mathfrak{g}\mathfrak{f}\xi_n = \mathfrak{g}\eta$ whenever $\{\xi_n\}$ is a sequence in $\mathfrak{S} \ni \lim_{n\to\infty} \mathfrak{f}\xi_n = \lim_{n\to\infty} \mathfrak{g}\xi_n = \eta$ for some $\eta \in \mathfrak{S}$.

Definition 1.4. [10] Let $\mathfrak{f},\mathfrak{g} : \mathfrak{S} \to \mathfrak{S}$ be self-map on a b-metric space $(\mathfrak{S},\mathfrak{d})$ is said to satisfy b-(E.A)-property if there exists a sequence $\{\xi_n\}$ in $\mathfrak{S} \ni \lim_{n\to\infty} \mathfrak{f}\xi_n = \lim_{n\to\infty} \mathfrak{g}\xi_n = \eta$ for some $\eta \in \mathfrak{S}$.

Definition 1.5. [3] A continuous map $\mathscr{H} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be C-class function if it satisfies the following conditions:

(i) $\mathscr{H}(\iota,\kappa) \leq \iota$;

(ii) $\mathscr{H}(\iota,\kappa) = s \implies$ either $\iota = 0$ or $\kappa = 0$; $\forall\, \iota,\kappa \in [0,\infty)$.

The collection of all C-class functions is indicated by $\mathscr{C}$.

Definition 1.6. [3] The following functions $\mathscr{H} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are elements of $\mathscr{C}$, for all $\iota,\kappa \in [0,\infty)$:

(ι_a) $\mathscr{H}(\iota,\kappa) = \iota - \kappa$, $\mathscr{H}(\iota,\kappa) = \iota \Rightarrow \kappa = 0$;

(ι_b) $\mathscr{H}(\iota,\kappa) = m\iota$, $0 < m < 1, \mathscr{H}(\iota,\kappa) = \iota \Rightarrow \iota = 0$;

(ι_c) $\mathscr{H}(\iota,\kappa) = \iota\beta(\iota)$, $\beta : [0,\infty) \to [0,1)$, and is continuous, $\mathscr{H}(\iota,\kappa) = \iota \Rightarrow \iota = 0$;

(ι_d) $\mathscr{H}(\iota,\kappa) = \iota - \varphi(\iota)$, $\mathscr{H}(\iota,\kappa) = \iota \Rightarrow \iota = 0$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(\kappa) = 0 \Leftrightarrow \kappa = 0$;

(ι_e) $\mathscr{H}(\iota,\kappa)=\phi(\iota),\ \mathscr{H}(\iota,\kappa)=\iota\Rightarrow\iota=0,$ where $\phi:[0,\infty)\to[0,\infty)$ is continuous such that $\phi(0)=0,$ and $\phi(\kappa)>0$ for $\kappa>0$;

The following is how Babu and Sudheer [5] presented F-class functions:

Definition 1.7. [5] *A continuous map $F:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ is said to be $\mathscr{F}$-class function if $F(s,t)<s$ for all $s,t>0$.*

F-class functions are indicated by $\mathscr{F}$.

It has been demonstrated by Babu and Sudheer [5] that $F(0,0)$ may not be zero and $\mathscr{C}=\mathscr{F}$.

We indicate

$\Psi_b=\{\psi_b/\psi_b:\mathbb{R}^+\to\mathbb{R}^+$ is continuous, ψ_b is nondecreasing, and $\psi_b(\kappa)=0\Leftrightarrow\kappa=0\}$ and $\Phi_b=\{\varphi_b/\varphi_b:\mathbb{R}^+\to\mathbb{R}^+$ is continuous, φ_b is nondecreasing, $\varphi_b(\kappa)>0$ for $\kappa>0$ and $\varphi_b(0)\geq 0\}$.

We can utilize the following lemma to support our major findings.

Lemma 1.8. [2] *Let $(\mathfrak{S},\mathfrak{d})$ be a b-metric space with coefficient $s\geq 1$. Suppose that $\{\xi_n\}$ and $\{\zeta_n\}$ are b-convergent to ξ and ζ respectively, then we have*

$$\frac{1}{s^2}\mathfrak{d}(\xi,\zeta)\leq\liminf_{n\to\infty}\mathfrak{d}(\xi_n,\zeta_n)\leq\limsup_{n\to\infty}\mathfrak{d}(\xi_n,\zeta_n)\leq s^2\mathfrak{d}(\xi,\zeta).$$

In particular, if $\xi=\zeta$, then we have $\lim_{n\to\infty}\mathfrak{d}(\xi_n,\zeta_n)=0$. Moreover for each $\eta\in\mathfrak{S}$ we have

$$\frac{1}{s}\mathfrak{d}(\xi,\eta)\leq\liminf_{n\to\infty}\mathfrak{d}(\xi_n,\eta)\leq\limsup_{n\to\infty}\mathfrak{d}(\xi_n,\eta)\leq s\mathfrak{d}(\xi,\eta).$$

The following theorem is due to Babu and Babu [4] in the setting of partial metric spaces.

Theorem 1.9. [4] *Let $(\mathfrak{S},p)$ be a partial metric space and let $\mathfrak{f}$ and $\mathfrak{g}$ be self-maps on $\mathfrak{S}$. Assume that there exist $\varphi_b\in\Psi_b$, $\phi_b\in\Phi_b$ and $F\in\mathscr{F}$ such that*

$$\varphi_b(p(\mathfrak{f}\xi,\mathfrak{f}\zeta))\leq\max\{F(\varphi_b(p(\mathfrak{g}\xi,\mathfrak{g}\zeta)),\phi_b(p(\mathfrak{g}\xi,\mathfrak{g}\zeta))),$$
$$F(\varphi_b(p(\mathfrak{g}\zeta,\mathfrak{f}\zeta)\frac{1+p(\mathfrak{g}\xi,\mathfrak{f}\xi)}{1+p(\mathfrak{g}\xi,\mathfrak{g}\zeta)}),\phi_b(p(\mathfrak{g}\zeta,\mathfrak{f}\zeta)\frac{1+p(\mathfrak{g}\xi,\mathfrak{f}\xi)}{1+p(\mathfrak{g}\xi,\mathfrak{g}\zeta)}))\}$$

for all $\xi,\zeta\in\mathfrak{S}$. If $\mathfrak{f}(\mathfrak{S})\subseteq\mathfrak{g}(\mathfrak{S})$, the pair $(\mathfrak{f},\mathfrak{g})$ is weakly compatible and $\mathfrak{g}(\mathfrak{S})$ is a complete subspace of $\mathfrak{S}$ then $\mathfrak{f}$ and $\mathfrak{g}$ have a unique common fixed point in $\mathfrak{S}$.

2. MAIN RESULTS

Let Λ,Ξ,Σ and Υ be self-maps of $\mathfrak{S}$ and satisfying

$$\Lambda(\mathfrak{S})\subseteq\Upsilon(\mathfrak{S})\text{ and }\Xi(\mathfrak{S})\subseteq\Sigma(\mathfrak{S}).\tag{2.1}$$

From (2.1), for any $\xi_0 \in \mathfrak{S}\ \exists\ \xi_1 \in \mathfrak{S}\ \ni\ \zeta_0 = \Lambda\xi_0 = \Upsilon\xi_1$. For this ξ_1, we can choose a point $\xi_2 \in \mathfrak{S}\ \ni\ \zeta_1 = \Xi\xi_1 = \Sigma\xi_2$. In generally, $\{\zeta_n\} \subseteq \mathfrak{S}\ \ni$

$$\begin{aligned} \zeta_{2n} &= \Lambda\xi_{2n} = \Upsilon\xi_{2n+1} \\ \zeta_{2n+1} &= \Xi\xi_{2n+1} = \Sigma\xi_{2n+2}\ \forall\ n. \end{aligned} \tag{2.2}$$

Lemma 2.1. *Suppose* $(\mathfrak{S}, \mathfrak{d})$ *is a b-metric space with parameter* $s \geq 1$ *and* Λ, Ξ, Σ *and* Υ *are self-maps of* $\mathfrak{S}$ *which satisfy the following condition: there exist* $\varphi_b \in \Psi_b$, $\phi_b \in \Phi_b$ *and* $F \in \mathscr{F}\ \ni$

(2.3)
$$\begin{aligned} \varphi_b(s\mathfrak{d}(\Lambda\xi, \Xi\zeta)) \leq \max\{&F(\varphi_b(\mathfrak{d}(\Sigma\xi, \Upsilon\zeta)), \phi_b(\mathfrak{d}(\Sigma\xi, \Upsilon\zeta))), \\ &F(\varphi_b(\mathfrak{d}(\Xi\zeta, \Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}), \phi_b(\mathfrak{d}(\Xi\zeta, \Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}))\} \end{aligned}$$

$\forall\ \xi, \zeta \in \mathfrak{S}$. *Then there are the following:*

(i) *If* $\Lambda(\mathfrak{S}) \subseteq \Upsilon(\mathfrak{S})$, (Ξ, Υ) *is weakly compatible and* Λ *and* Σ *have a common fixed point, then* Λ, Ξ, Σ *and* Υ *have a unique common fixed point.*

(ii) *If* $\Xi(\mathfrak{S}) \subseteq \Sigma(\mathfrak{S})$, (Λ, Σ) *is weakly compatible, and* Ξ *and* Υ *have a common fixed point, then* Λ, Ξ, Σ *and* Υ *have a unique common fixed point.*

Proof. Suppose (i) holds. Let η be a common fixed point of Λ and Σ.

Then $\Lambda\eta = \Sigma\eta = \eta$. Since $\Lambda(\mathfrak{S}) \subseteq \Upsilon(\mathfrak{S})\ \exists\ u \in \mathfrak{S}\ \ni\ \Upsilon u = \eta$.

Therefore $\Lambda\eta = \Sigma\eta = \Upsilon u = \eta$. Suppose that $\Lambda\eta \neq \Xi u$.

We consider,

$$\begin{aligned} \varphi_b(s\mathfrak{d}(\Lambda\eta, \Xi u)) \leq \max\{&F(\varphi_b(\mathfrak{d}(\Sigma\eta, \Upsilon u)), \phi_b(\mathfrak{d}(\Sigma\eta, \Upsilon u))), \\ &F(\varphi_b(\mathfrak{d}(\Xi u, \Upsilon u)\tfrac{1+\mathfrak{d}(\Lambda\eta,\Sigma\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)}), \phi_b(\mathfrak{d}(\Xi u, \Upsilon u)\tfrac{1+\mathfrak{d}(\Lambda\eta,\Sigma\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)}))\} \\ = \max\{&F(\varphi_b(0), \phi_b(0)), F(\varphi_b(\tfrac{\mathfrak{d}(\Xi u,\Lambda\eta)}{1+d(\Lambda\eta,\Xi u)}), \phi_b(\tfrac{\mathfrak{d}(\Xi u,\Lambda\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)}))\} \end{aligned}$$

If $\max\{F(\varphi_b(0), \phi_b(0)), F(\varphi_b(\frac{d(\Xi u,\Lambda\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)}), \phi_b(\frac{\mathfrak{d}(\Xi u,\Lambda\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)}))\}$=$F(\varphi_b(\frac{\mathfrak{d}(\Xi u,\Lambda\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)}), \phi_b(\frac{\mathfrak{d}(\Xi u,\Lambda\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)}))$ then we have

$\varphi_b(s\mathfrak{d}(\Lambda\eta, \Xi u)) \leq F(\varphi_b(\frac{\mathfrak{d}(\Xi u,\Lambda\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)}), \phi_b(\frac{\mathfrak{d}(\Xi u,\Lambda\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)})) \leq \varphi_b(\frac{\mathfrak{d}(\Xi u,\Lambda\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)})$.

By the property of φ_b, we have $s\mathfrak{d}(\Lambda\eta, \Xi u) \leq \frac{\mathfrak{d}(\Xi u,\Lambda\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)}$,

a contradiction.

Therefore, $\max\{F(\varphi_b(0), \phi_b(0)), F(\varphi_b(\frac{\mathfrak{d}(\Xi u,\Lambda\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)}), \phi_b(\frac{\mathfrak{d}(\Xi u,\Lambda\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi u)}))\} = F(\varphi_b(0), \phi_b(0))$

which implies that $\varphi_b(s\mathfrak{d}(\Lambda\eta,\Xi u)) \leq F(\varphi_b(0),\phi_b(0)) \leq \varphi_b(0)$.

Since $\varphi_b \in \Psi_b$, we have $\mathfrak{d}(\Lambda\eta,\Xi u) \leq 0$. i.e., $\Lambda\eta = \Xi u$.

Hence $\Lambda\eta = \Xi u = \Sigma\eta = \Upsilon u = \eta$.

As (Ξ,Υ) is weakly compatible and $\Upsilon u = \Xi u$, we have

$\Xi\Upsilon u = \Upsilon\Xi u$. i.e., $\Xi\eta = \Upsilon\eta$.

If $\Xi\eta \neq \eta$, then

$$\begin{aligned}\varphi_b(s\mathfrak{d}(\Xi\eta,\eta)) &= s\mathfrak{d}(\Lambda\eta,\Xi\eta)\\ &\leq \max\{F(\varphi_b(\mathfrak{d}(\Sigma\eta,\Upsilon\eta)),\phi_b(\mathfrak{d}(\Sigma\eta,\Upsilon\eta))),\\ &\qquad F(\varphi_b(\mathfrak{d}(\Xi\eta,\Upsilon\eta)\tfrac{1+\mathfrak{d}(\Lambda\eta,\Sigma\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi\eta)}),\phi_b(\mathfrak{d}(\Xi\eta,\Upsilon\eta)\tfrac{1+\mathfrak{d}(\Lambda\eta,\Sigma\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi\eta)}))\}\\ &= \max\{F(\varphi_b(\mathfrak{d}(\Lambda\eta,\Xi\eta)),\phi_b(\mathfrak{d}(\Lambda\eta,\Xi\eta))),F(\varphi_b(0),\phi_b(0))\}\end{aligned}$$

If $\max\{F(\varphi_b(\mathfrak{d}(\Lambda\eta,\Xi\eta)),\phi_b(\mathfrak{d}(\Lambda\eta,\Xi\eta))),F(\varphi_b(0),\phi_b(0))\} = F(\varphi_b(\mathfrak{d}(\Lambda\eta,\Xi\eta)),\phi_b(\mathfrak{d}(\Lambda\eta,\Xi\eta)))$ then we have $\varphi_b(sd(\Xi\eta,\eta)) \leq F(\varphi_b(\mathfrak{d}(\Lambda\eta,\Xi\eta)),\phi_b(\mathfrak{d}(\Lambda\eta,\Xi\eta))) \leq \varphi_b(\mathfrak{d}(\Lambda\eta,\Xi\eta))$.

By the property of φ_b, we have $s\mathfrak{d}(\Xi\eta,\eta) \leq \mathfrak{d}(\Lambda\eta,\Xi\eta)$,

a contradiction. Therefore $\varphi_b(s\mathfrak{d}(\Xi\eta,\eta)) \leq F(\varphi_b(0),\phi_b(0)) \leq \varphi_b(0)$.

Since $\varphi_b \in \Psi_b$, we have $s\mathfrak{d}(\Xi\eta,\eta) \leq 0$ implies that $\Xi\eta = \eta$.

Hence $\Lambda\eta = \Xi\eta = \Sigma\eta = \Upsilon\eta = \eta$.

Therefore, η is a common fixed point of Λ,Ξ,Σ and Υ.

Suppose $\eta' \neq \eta$ is a common fixed point of Λ,Ξ,Σ and Υ.

$$\begin{aligned}\varphi_b sd(\eta,\eta') &= \varphi_b s\mathfrak{d}(\Lambda\eta,\Xi\eta')\\ &\leq \max\{F(\varphi_b(\mathfrak{d}(\Sigma\eta,\Upsilon\eta')),\phi_b(\mathfrak{d}(\Sigma\eta,\Upsilon\eta'))),\\ &\qquad F(\varphi_b(\mathfrak{d}(\Xi\eta',\Upsilon\eta')\tfrac{1+\mathfrak{d}(\Lambda\eta,\Sigma\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi\eta')}),\phi_b(\mathfrak{d}(\Xi\eta',\Upsilon\eta')\tfrac{1+\mathfrak{d}(\Lambda\eta,\Sigma\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi\eta')}))\}\\ &= \max\{F(\varphi_b(\mathfrak{d}(\eta,\eta')),\phi_b(\mathfrak{d}(\eta,\eta'))),F(\varphi_b(0),\phi_b(0))\}\end{aligned}$$

If $\max\{F(\varphi_b(\mathfrak{d}(\eta,\eta')),\phi_b(\mathfrak{d}(\eta,\eta'))),F(\varphi_b(0),\phi_b(0))\} = F(\varphi_b(\mathfrak{d}(\eta,\eta')),\phi_b(\mathfrak{d}(\eta,\eta')))$ then $\varphi_b(s\mathfrak{d}(\eta,\eta')) \leq F(\varphi_b(\mathfrak{d}(\eta,\eta')),\phi_b(\mathfrak{d}(\eta,\eta'))) \leq \varphi_b(\mathfrak{d}(\eta,\eta'))$.

By the property of φ_b, we have $s\mathfrak{d}(\eta,\eta') \leq \mathfrak{d}(\eta,\eta')$,

a contradiction.

Therefore $\varphi_b(s\mathfrak{d}(\eta,\eta')) \leq F(\varphi_b(0),\phi_b(0)) \leq \varphi_b(0)$.

Since $\varphi_b \in \Psi_b$, we have $s\mathfrak{d}(\eta,\eta') \leq 0$ implies that $\eta = \eta'$.

Hence, η is the unique common fixed point of Λ, Ξ, Σ and Υ.

The proof of (*ii*) follows from (*i*). □

Lemma 2.2. *Let Λ, Ξ, Σ and Υ be self-maps of a b-metric space $(\mathfrak{S}, \mathfrak{d})$, satisfy* (2.1) *and* (2.3). *Then for any $\xi_0 \in \mathfrak{S}$, the sequence $\{\zeta_n\}$ defined by* (2.2) *is b-Cauchy in $\mathfrak{S}$.*

Proof. Let $\xi_0 \in \mathfrak{S}$ and let $\{\zeta_n\}$ be a sequence defined by (2.2).

Suppose $\zeta_n = \zeta_{n+1}$ for some n.

Case (*i*): n even.

We write $n = 2m, m \in \mathbb{N}$. Now,

$$\begin{aligned}
\varphi_b(s\mathfrak{d}(\zeta_{n+1}, \zeta_{n+2})) &= \varphi_b(s\mathfrak{d}(\zeta_{2m+1}, \zeta_{2m+2})) \\
&= \varphi_b(s\mathfrak{d}(\zeta_{2m+2}, \zeta_{2m+1})) \\
&= \varphi_b(s\mathfrak{d}(\Lambda\xi_{2m+2}, \Xi\xi_{2m+1})) \\
&\leq \max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi_{2m+2}, \Upsilon\xi_{2m+1})), \phi_b(\mathfrak{d}(\Sigma\xi_{2m+2}, \Upsilon\xi_{2m+1}))), \\
&\qquad F(\varphi_b(\mathfrak{d}(\Upsilon\xi_{2m+1}, \Xi\xi_{2m+1})\frac{1+\mathfrak{d}(\Sigma\xi_{2m+2}, \Lambda\xi_{2m+2})}{1+\mathfrak{d}(\Lambda\xi_{2m+2}, \Xi\xi_{2m+1})}), \\
&\qquad \phi_b(\mathfrak{d}(\Upsilon\xi_{2m+1}, \Xi\xi_{2m+1})\frac{1+\mathfrak{d}(\Sigma\xi_{2m+2}, \Lambda\xi_{2m+2})}{1+\mathfrak{d}(\Lambda\xi_{2m+2}, \Xi\xi_{2m+1})}))\} \\
&= \max\{F(\varphi_b(\mathfrak{d}(\zeta_{2m+1}, \zeta_{2m})), \phi_b(\mathfrak{d}(\zeta_{2m+1}, \zeta_{2m}))), \\
&\qquad F(\varphi_b(\mathfrak{d}(\zeta_{2m}, \zeta_{2m+1})\frac{1+\mathfrak{d}(\zeta_{2m+1}, \zeta_{2m+2})}{1+\mathfrak{d}(\zeta_{2m+2}, \zeta_{2m+1})}), \\
&\qquad \phi_b(\mathfrak{d}(\zeta_{2m}, \zeta_{2m+1})\frac{1+\mathfrak{d}(\zeta_{2m+1}, \zeta_{2m+2})}{1+\mathfrak{d}(\zeta_{2m+2}, \zeta_{2m+1})}))\} \\
&= F(\varphi_b(\mathfrak{d}(\zeta_{2m+1}, \zeta_{2m})), \phi_b(\mathfrak{d}(\zeta_{2m+1}, \zeta_{2m}))) = F(\varphi_b(0), \phi_b(0)) \leq \varphi_b(0).
\end{aligned}$$

Since $\varphi_b \in \Psi_b$, we have $\mathfrak{d}(\zeta_{2m+1}, \zeta_{2m+2}) = 0 \implies \zeta_{2m+2} = \zeta_{2m+1} = \zeta_{2m}$.

Continuing, we get $\zeta_{2m+k} = \zeta_{2m} \ \forall \ k$.

Case (*ii*): n odd. We write $n = 2m+1$ for some $m \in \mathbb{N}$. Now,

$$\begin{aligned}
\varphi_b(s\mathfrak{d}(\zeta_{n+1}, \zeta_{n+2})) &= \varphi_b(s\mathfrak{d}(\zeta_{2m+2}, \zeta_{2m+3})) \\
&= \varphi_b(s\mathfrak{d}(\Lambda\xi_{2m+2}, \Xi\xi_{2m+3})) \\
&\leq \max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi_{2m+2}, \Upsilon\xi_{2m+3})), \phi_b(\mathfrak{d}(\Sigma\xi_{2m+2}, \Upsilon\xi_{2m+3}))),
\end{aligned}$$

$$F(\varphi_b(\mathfrak{d}(\Upsilon\xi_{2m+3},\Xi\xi_{2m+3})\frac{1+\mathfrak{d}(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2})}{1+\mathfrak{d}(\Lambda\xi_{2m+2},\Xi\xi_{2m+3})}),$$

$$\phi_b(\mathfrak{d}(\Upsilon\xi_{2m+3},\Xi\xi_{2m+3})\frac{1+\mathfrak{d}(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2})}{1+\mathfrak{d}(\Lambda\xi_{2m+2},\Xi\xi_{2m+3})}))\}$$

$$= \max\{F(\varphi_b(\mathfrak{d}(\zeta_{2m+1},\zeta_{2m+2})),\phi_b(\mathfrak{d}(\zeta_{2m+1},\zeta_{2m+2}))),$$

$$F(\varphi_b(\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})\frac{1+\mathfrak{d}(\zeta_{2m+1},\zeta_{2m+2})}{1+\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}),$$

$$\phi_b(\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})\frac{1+\mathfrak{d}(\zeta_{2m+1},\zeta_{2m+2})}{1+\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}))\}$$

$$= \max\{F(\varphi_b(0),\phi_b(0)),$$

$$F(\varphi_b(\frac{\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}{1+\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}),\phi_b(\frac{\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}{1+\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}))\}.$$

If $\max\{F(\varphi_b(0),\phi_b(0)),F(\varphi_b(\frac{\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}{1+\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}),\phi_b(\frac{\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}{1+\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}))\}$
$= F(\varphi_b(\frac{\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}{1+\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}),\phi_b(\frac{\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}{1+\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}))$ then
$\varphi_b(s\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})) \le F(\varphi_b(\frac{\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}{1+\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}),\phi_b(\frac{\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}{1+\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})})) \le \varphi_b(\frac{\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}{1+\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})})$.
By property of φ_b, we have $s\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3}) \le \frac{\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}{1+\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})}$,

which is a contradiction.

Therefore $\varphi_b(s\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3})) \le F(\varphi_b(0),\phi_b(0)) \le \varphi_b(0)$.

Since $\varphi_b \in \Psi_b$, we have $s\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3}) \le 0$.

Therefore, $\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3}) = 0 \implies \zeta_{2m+3} = \zeta_{2m+2} = \zeta_{2m+1}$.

Continuing in this way, $\zeta_{2m+k} = \zeta_{2m+1}\ \forall\ k$.

Case (*i*) and Case (*ii*), concludes that $\zeta_{n+k} = \zeta_n\ \forall\ k$ and that $\{\zeta_n\}$ is b-Cauchy.

Suppose $\zeta_n \neq \zeta_{n+1},\ \forall\ n \in \mathbb{N}$.

If n is odd, then $n = 2m+1$ for some $m \in \mathbb{N}$.

Now,

$$\varphi_b(s\mathfrak{d}(\zeta_{n+1},\zeta_{n+2})) = \varphi_b(s\mathfrak{d}(\zeta_{2m+2},\zeta_{2m+3}))$$

$$= \varphi_b(s\mathfrak{d}(\Lambda\xi_{2m+2},\Xi\xi_{2m+3}))$$

$$\le \max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+3})),\phi_b(\mathfrak{d}(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+3}))),$$

$$F(\varphi_b(\mathfrak{d}(\Upsilon\xi_{2m+3},\Xi\xi_{2m+3})\frac{1+\mathfrak{d}(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2})}{1+\mathfrak{d}(\Lambda\xi_{2m+2},\Xi\xi_{2m+3})}),$$

$$\phi_b(\eth(\Upsilon\xi_{2m+3},\Xi\xi_{2m+3})\frac{1+\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2})}{1+\eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+3})}))\}$$

$$=\max\{F(\varphi_b(\eth(\zeta_{2m+1},\zeta_{2m+2})),\phi_b(\eth(\zeta_{2m+1},\zeta_{2m+2}))),$$

$$F(\varphi_b(\eth(\zeta_{2m+2},\zeta_{2m+3})\frac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}),$$

$$\phi_b(\eth(\zeta_{2m+2},\zeta_{2m+3})\frac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}))\}.$$

If $\max\{F(\varphi_b(\eth(\zeta_{2m+1},\zeta_{2m+2})),\phi_b(\eth(\zeta_{2m+1},\zeta_{2m+2}))),$

$$F(\varphi_b(\eth(\zeta_{2m+2},\zeta_{2m+3})\tfrac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}),\phi_b(\eth(\zeta_{2m+2},\zeta_{2m+3})\tfrac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}))\}$$

$$=F(\varphi_b(\eth(\zeta_{2m+2},\zeta_{2m+3})\tfrac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}),\phi_b(\eth(\zeta_{2m+2},\zeta_{2m+3})\tfrac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}))$$

then we have

$$\varphi_b(s\eth(\zeta_{2m+2},\zeta_{2m+3}))\leq F(\varphi_b(\eth(\zeta_{2m+2},\zeta_{2m+3})\frac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}),$$

$$\phi_b(\eth(\zeta_{2m+2},\zeta_{2m+3})\frac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}))$$

$$\leq\varphi_b(\eth(\zeta_{2m+2},\zeta_{2m+3})\frac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}).$$

Since $\varphi_b\in\Psi_b$, we have $s\eth(\zeta_{2m+2},\zeta_{2m+3})\leq\eth(\zeta_{2m+2},\zeta_{2m+3})\frac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}$.

Suppose $\eth(\zeta_{2m+2},\zeta_{2m+3})\frac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}\leq\eth(\zeta_{2m+2},\zeta_{2m+3})$.

Then $s\eth(\zeta_{2m+2},\zeta_{2m+3})\leq\eth(\zeta_{2m+2},\zeta_{2m+3})$,

which is a contradiction.

Therefore, $s\eth(\zeta_{2m+2},\zeta_{2m+3})\leq\eth(\zeta_{2m+1},\zeta_{2m+2})$ which implies that

$\eth(\zeta_{2m+2},\zeta_{2m+3})\leq\frac{1}{s}\eth(\zeta_{2m+1},\zeta_{2m+2})$.

If $\max\{F(\varphi_b(\eth(\zeta_{2m+1},\zeta_{2m+2})),\phi_b(\eth(\zeta_{2m+1},\zeta_{2m+2}))),$

$$F(\varphi_b(\eth(\zeta_{2m+2},\zeta_{2m+3})\tfrac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}),\phi_b(\eth(\zeta_{2m+2},\zeta_{2m+3})\tfrac{1+\eth(\zeta_{2m+1},\zeta_{2m+2})}{1+\eth(\zeta_{2m+2},\zeta_{2m+3})}))\}$$

$=F(\varphi_b(\eth(\zeta_{2m+1},\zeta_{2m+2})),\phi_b(\eth(\zeta_{2m+1},\zeta_{2m+2})))$ then using $F\in\mathscr{F}$, we have

$\varphi_b(s\eth(\zeta_{2m+2},\zeta_{2m+3}))\leq\mathscr{F}(\varphi_b(\eth(\zeta_{2m+1},\zeta_{2m+2})),\phi_b(\eth(\zeta_{2m+1},\zeta_{2m+2})))\leq\varphi_b(\eth(\zeta_{2m+1},\zeta_{2m+2}))$.

Since $\varphi_b\in\Psi_b$, we have $s\eth(\zeta_{2m+2},\zeta_{2m+3})\leq\eth(\zeta_{2m+1},\zeta_{2m+2})$ implies that

$$\eth(\zeta_{2m+2},\zeta_{2m+3})\leq\frac{1}{s}\eth(\zeta_{2m+1},\zeta_{2m+2}). \tag{2.4}$$

Similarly, n is even, it follows that

$$\eth(\zeta_{2m+1},\zeta_{2m+2})\leq\frac{1}{s}\eth(\zeta_{2m},\zeta_{2m+1}). \tag{2.5}$$

From the inequalities (2.4) and (2.5), we get

$\mathfrak{d}(\zeta_{n+1},\zeta_{n+2}) \leq \frac{1}{s}\mathfrak{d}(\zeta_n,\zeta_{n+1}) \leq \frac{1}{s^2}\mathfrak{d}(\zeta_{n-1},\zeta_n) \leq \cdots \leq \frac{1}{s^n}\mathfrak{d}(\zeta_0,\zeta_1)$.

Therefore $\{\zeta_n\}$ is b-Cauchy in $\mathfrak{S}$. □

The primary finding of this paper is as follows:

Theorem 2.3. *Let Λ,Ξ,Σ and Υ be self-maps on a complete b-metric space $(\mathfrak{S},\mathfrak{d})$, satisfy* (2.1) *and* (2.3)*. If the pairs (Λ,Σ) and (Ξ,Υ) are weakly compatible and one of the range sets $\Sigma(\mathfrak{S}),\Upsilon(\mathfrak{S}),\Lambda(\mathfrak{S})$ and $\Xi(\mathfrak{S})$ is b-closed, then Λ,Ξ,Σ and Υ have a unique common fixed point.*

Proof. By Lemma 2.2, the sequence $\{\zeta_n\}$ defined in (2.2) is b-Cauchy in $\mathfrak{S}$.

Since $\mathfrak{S}$ is complete, $\exists\, \eta \in \mathfrak{S} \ni \lim_{n\to\infty} \zeta_n = \eta$. We have

$$\begin{cases} \lim_{n\to\infty} \zeta_{2n} = \lim_{n\to\infty} \Lambda\xi_{2n} = \lim_{n\to\infty} \Upsilon\xi_{2n+1} = \eta \\ \lim_{n\to\infty} \zeta_{2n+1} = \lim_{n\to\infty} \Xi\xi_{2n+1} = \lim_{n\to\infty} \Sigma\xi_{2n+2} = \eta. \end{cases} \tag{2.6}$$

Taking the next four situations into consideration.

<u>*Case* (*i*)</u>. $\Sigma(\mathfrak{S})$ is b-closed.

As $\eta \in \Sigma(\mathfrak{S})\ \exists\, u \in \mathfrak{S} \ni \eta = \Sigma u$.

Suppose that $\Lambda u \neq \eta$. Now,

$$\begin{cases} \varphi_b(s\mathfrak{d}(\Lambda u,\Xi\xi_{2n+1})) \leq \max\{F(\varphi_b(\mathfrak{d}(\Sigma u,\Upsilon\xi_{2n+1})),\phi_b(\mathfrak{d}(\Sigma u,\Upsilon\xi_{2n+1}))), \\ \qquad F(\varphi_b(\mathfrak{d}(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\mathfrak{d}(\Lambda u,\Sigma u)}{1+\mathfrak{d}(\Lambda u,\Xi\xi_{2n+1})}), \\ \qquad \phi_b(\mathfrak{d}(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\mathfrak{d}(\Lambda u,\Sigma u)}{1+\mathfrak{d}(\Lambda u,\Xi\xi_{2n+1})}))\} \end{cases} \tag{2.7}$$

On letting upper limit $n\to\infty$ in (2.7), using (2.6) and Lemma 1.8, we have

$$\begin{aligned} \varphi_b(s\tfrac{1}{s}\mathfrak{d}(\Lambda u,\eta)) &\leq \varphi_b(s\limsup_{n\to\infty} d(\Lambda u,\Xi\xi_{2n+1})) \\ &\leq \limsup_{n\to\infty}(\max\{F(\varphi_b(\mathfrak{d}(\Sigma u,\Upsilon\xi_{2n+1})),\phi_b(\mathfrak{d}(\Sigma u,\Upsilon\xi_{2n+1}))), \\ &F(\varphi_b(\mathfrak{d}(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\tfrac{1+\mathfrak{d}(\Lambda u,\Sigma u)}{1+\mathfrak{d}(\Lambda u,\Xi\xi_{2n+1})}),\phi_b(\mathfrak{d}(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\tfrac{1+\mathfrak{d}(\Lambda u,\Sigma u)}{1+\mathfrak{d}(\Lambda u,\Xi\xi_{2n+1})}))\}) \\ &= F(\varphi_b(0),\phi_b(0)) \leq \varphi_b(0). \end{aligned}$$

Since φ_b has the property, we have $\mathfrak{d}(\Lambda u,\eta) \leq 0$ which implies that $\Lambda u = \eta$.

Therefore, $\Lambda u = \eta = \Sigma u$.

As (Λ,Σ) is weakly compatible and $\Lambda u = \Sigma u$, we have

$\Lambda\Sigma u = \Sigma\Lambda u$. i.e., $\Lambda\eta = \Sigma\eta$.

If $\Lambda\eta \neq \eta$, then

$$\text{(2.8)} \quad \begin{cases} \varphi_b(s\eth(\Lambda\eta, \Xi\xi_{2n+1})) \leq \max\{F(\varphi_b(\eth(\Sigma\eta, \Upsilon\xi_{2n+1})), \phi_b(\eth(\Sigma\eta, \Upsilon\xi_{2n+1}))), \\ \qquad F(\varphi_b(\eth(\Xi\xi_{2n+1}, \Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})}), \\ \qquad \phi_b(\eth(\Xi\xi_{2n+1}, \Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})}))\} \end{cases}$$

On letting upper limit as $n \to \infty$ in (2.8), using (2.6) and Lemma 1.8, we have

$$\begin{aligned} \varphi_b(s\frac{1}{s}\eth(\Lambda\eta, \eta)) &\leq \varphi_b(s\limsup_{n\to\infty}\eth(\Lambda\eta, \Xi\xi_{2n+1})) \\ &\leq \limsup \lim_{n\to\infty}(\max\{F(\varphi_b(\eth(\Sigma\eta, \Upsilon\xi_{2n+1})), \phi_b(\eth(\Sigma\eta, \Upsilon\xi_{2n+1}))), \\ &\quad F(\varphi_b(\eth(\Xi\xi_{2n+1}, \Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})}), \\ &\quad \phi_b(\eth(\Xi\xi_{2n+1}, \Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})}))\}) \\ &\leq \max\{F(\varphi_b(\limsup_{n\to\infty}(\eth(\Sigma\eta, \Upsilon\xi_{2n+1}))), \phi_b(\limsup_{n\to\infty}(\eth(\Sigma\eta, \Upsilon\xi_{2n+1})))), \\ &\quad F(\varphi_b(\limsup_{n\to\infty}(\eth(\Xi\xi_{2n+1}, \Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})})), \\ &\quad \phi_b(\limsup_{n\to\infty}(\eth(\Xi\xi_{2n+1}, \Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})})))\} \\ &\leq \max\{F(\varphi_b(s\eth(\Lambda\eta, \eta)), \phi_b(s\eth(\Lambda\eta, \eta))), F(\varphi_b(0), \phi_b(0))\}. \end{aligned}$$

If $\max\{F(\varphi_b(s\eth(\Lambda\eta,\eta)), \phi_b(s\eth(\Lambda\eta,\eta))), F(\varphi_b(0), \phi_b(0))\} = F(\varphi_b(s\eth(\Lambda\eta,\eta)), \phi_b(s\eth(\Lambda\eta,\eta)))$

then we have $\varphi_b(\eth(\Lambda\eta, \eta)) \leq \varphi_b(s\eth(\Lambda\eta, \eta))$.

Since $\varphi_b \in \Psi_b$, we have

$\eth(\Lambda\eta, \eta) \leq s\eth(\Lambda\eta, \eta) \implies (1-s)\eth(\Lambda\eta, \eta) \leq 0 \implies \Lambda\eta = \eta$.

Suppose $\max\{F(\varphi_b(s\eth(\Lambda\eta, \eta)), \phi_b(s\eth(\Lambda\eta, \eta))), F(\varphi_b(0), \phi_b(0))\} = F(\varphi_b(0), \phi_b(0))$.

Then $\varphi_b(\eth(\Lambda\eta, \eta)) \leq F(\varphi_b(0), \phi_b(0)) \leq \varphi_b(0)$.

Since φ_b has the property, we have $\eth(\Lambda\eta, \eta) \leq 0$

which implies that $\Lambda\eta = \eta$.

Hence, η is a common fixed point of Λ and Σ.

According to Lemma 2.1, η is an unique common fixed point of Λ, Ξ, Σ and Υ.

Case (ii). $\Upsilon(\mathfrak{S})$ is b-closed.

Since $\eta \in \Upsilon(\mathfrak{S})$ and $\exists\, u \in \mathfrak{S} \ni \eta = Tu$.

Suppose $Bu \neq \eta$. Now,

$$(2.9)\quad \begin{cases} \varphi_b(s\mathfrak{d}(\Lambda\xi_{2n+2}, \Xi u)) \leq \max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi_{2n+2}, \Upsilon u)), \phi_b(\mathfrak{d}(\Sigma\xi_{2n+2}, \Upsilon u)), \\ \qquad F(\varphi_b(\mathfrak{d}(\Xi u, \Upsilon u)\frac{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Xi u)}), \\ \qquad \phi_b(\mathfrak{d}(\Xi u, \Upsilon u)\frac{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Xi u)})\} \end{cases}$$

On letting upper limit as $n \to \infty$ in (2.9), using (2.6) and Lemma 1.8, we have

$$\begin{aligned} \varphi_b(s\frac{1}{s}\mathfrak{d}(\eta, \Xi u)) &\leq \limsup_{n\to\infty} \varphi_b(s\mathfrak{d}(\Lambda\xi_{2n+2}, \Xi u)) \\ &\leq \limsup_{n\to\infty}(\max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi_{2n+2}, \Upsilon u)), \phi_b(\mathfrak{d}(\Sigma\xi_{2n+2}, \Upsilon u)), \\ &\quad F(\varphi_b(\mathfrak{d}(\Xi u, \Upsilon u)\frac{1+\mathfrak{d}(\Lambda\xi_{2n+2}, \Sigma\xi_{2n+2})}{1+\mathfrak{d}(\Lambda\xi_{2n+2}, \Xi u)})), \\ &\quad \phi_b(\mathfrak{d}(\Xi u, \Upsilon u)\frac{1+\mathfrak{d}(\Lambda\xi_{2n+2}, \Sigma\xi_{2n+2})}{1+\mathfrak{d}(\Lambda\xi_{2n+2}, \Xi u)}))\}) \\ &= \max\{F(\varphi_b(0), \phi_b(0)), F(\varphi_b(\frac{\mathfrak{d}(\eta, \Xi u)}{1+s\mathfrak{d}(\eta, \Xi u)})), \phi_b(\frac{\mathfrak{d}(\eta, \Xi u)}{1+s\mathfrak{d}(\eta, \Xi u)})\}. \end{aligned}$$

If $\max\{F(\varphi_b(0), \phi_b(0)), F(\varphi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)}), \phi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)}))\} = F(\varphi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)}), \phi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)}))$

then by the properties of F and φ_b we have

$\varphi_b(\mathfrak{d}(\eta, \Xi u)) \leq F(\varphi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)}), \phi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)})) \leq \varphi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)})$

which implies that $\mathfrak{d}(\eta, \Xi u) \leq \frac{\mathfrak{d}(\eta,\Xi u)}{1+sd(\eta,\Xi bu)} < d(\eta, \Xi u)$,

a contradiction.

Therefore, $\max\{F(\varphi_b(0), \phi_b(0)), F(\varphi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)}), \phi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)}))\} = F(\varphi_b(0), \phi_b(0))$.

Since $F \in \mathscr{F}$ and $\varphi_b \in \Psi_b$, we have

$\varphi_b(\mathfrak{d}(\eta, \Xi u)) \leq F(\varphi_b(0), \phi_b(0)) \leq \varphi_b(0)$ which implies that $\Xi u = \eta = \Upsilon u$.

The pair (Ξ, Υ) is weakly compatible and $\Xi u = \Upsilon u$, we have

$\Xi\Upsilon u = \Upsilon\Xi u$. i.e., $\Xi\eta = \Upsilon\eta$.

If $\Xi\eta \neq \eta$, then

$$(2.10)\quad \begin{cases} \varphi_b(s\mathfrak{d}(\Lambda\xi_{2n+2}, \Xi\eta)) \leq \max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi_{2n+2}, \Upsilon\eta)), \phi_b(\mathfrak{d}(\Sigma\xi_{2n+2}, \Upsilon\eta)), \\ \qquad F(\varphi_b(\mathfrak{d}(\Xi\eta, \Upsilon\eta)\frac{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Xi\eta)}), \\ \qquad \phi_b(\mathfrak{d}(\Xi\eta, \Upsilon\eta)\frac{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Xi\eta)})\} \end{cases}$$

On letting upper limit as $n \to \infty$ in (2.10), using (2.6) and Lemma 1.8, we have

$$\begin{aligned}\varphi_b(s\frac{1}{s}\eth(\eta,\Xi\eta)) &\leq \limsup_{n\to\infty}\varphi_b(s\eth(\Lambda\xi_{2n+2},\Xi\eta))\\ &\leq \limsup_{n\to\infty}(\max\{F(\varphi_b(\eth(\Sigma\xi_{2n+2},\Upsilon\eta)),\phi_b(\eth(\Sigma\xi_{2n+2},\Upsilon\eta)),\\ &\qquad F(\varphi_b(\eth(\Xi\eta,\Upsilon\eta)\frac{1+\eth(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\eth(\Lambda\xi_{2n+2},\Xi\eta)})),\\ &\qquad \phi_b(\eth(\Xi\eta,\Upsilon\eta)\frac{1+\eth(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\eth(\Lambda\xi_{2n+2},\Xi\eta)}))\})\\ &= \max\{F(\varphi_b(s\eth(\eta,\Xi\eta)),\phi_b(s\eth(\eta,\Xi\eta))),F(\varphi_b(0),\phi_b(0))\}.\end{aligned}$$

If $\max\{F(\varphi_b(s\eth(\eta,\Xi\eta)),\phi_b(s\eth(\eta,\Xi\eta))),F(\varphi_b(0),\phi_b(0))\} = F(\varphi_b(s\eth(\eta,\Xi\eta)),\phi_b(s\eth(\eta,\Xi\eta)))$ then we have $\varphi_b(\eth(\eta,\Xi\eta)) \leq \varphi_b(s\eth(\eta,\Xi\eta))$.

Since $\varphi_b \in \Psi_b$, we have

$\eth(\eta,\Xi\eta) \leq s\eth(\Lambda\eta,\eta) \implies (1-s)\eth(\eta,\Xi\eta) \leq 0 \implies \eta = \Xi\eta$.

Suppose $\max\{F(\varphi_b(s\eth(\eta,\Xi\eta)),\phi_b(s\eth(\eta,\Xi\eta))),F(\varphi_b(0),\phi_b(0))\} = F(\varphi_b(0),\phi_b(0))$.

Then $\varphi_b(\eth(\eta,\Xi\eta)) \leq F(\varphi_b(0),\phi_b(0)) \leq \varphi_b(0)$.

Since $\varphi_b \in \Psi_b$, we have $\eth(\eta,\Xi\eta) \leq 0 \implies \Xi\eta = \eta$.

Therefore, $\Xi\eta = \Upsilon\eta = \eta$.

According to Lemma 2.1, η is a unique common fixed point of Λ,Ξ,Σ and Υ.

Case (iii). $\Lambda(\mathfrak{S})$ is b-closed.

Since $\eta \in \Lambda(\mathfrak{S}) \subseteq \Upsilon(\mathfrak{S})\ \exists\, u \in \mathfrak{S} \ni \eta = \Upsilon u$.

Suppose $\Xi u \neq \eta$. Now,

$$\text{(2.11)}\qquad \begin{cases}\varphi_b(s\eth(\Lambda\xi_{2n+2},\Xi u)) \leq \max\{F(\varphi_b(\eth(\Sigma\xi_{2n+2},\Upsilon u)),\phi_b(\eth(\Sigma\xi_{2n+2},\Upsilon u)),\\ \qquad F(\varphi_b(\eth(\Xi u,\Upsilon u)\frac{1+\eth(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\eth(\Lambda\xi_{2n+2},\Xi u)}),\\ \qquad \phi_b(\eth(\Xi u,\Upsilon u)\frac{1+\eth(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\eth(\Lambda\xi_{2n+2},\Xi u)})\}\end{cases}$$

On letting upper limit as $n \to \infty$ in (2.11), using (2.6) and Lemma 1.8, we have

$$\begin{aligned}\varphi_b(s\frac{1}{s}\eth(\eta,\Xi u)) &\leq \limsup_{n\to\infty}\varphi_b(s\eth(\Lambda\xi_{2n+2},\Xi u))\\ &\leq \limsup_{n\to\infty}(\max\{F(\varphi_b(\eth(\Sigma\xi_{2n+2},\Upsilon u)),\phi_b(\eth(\Sigma\xi_{2n+2},\Upsilon u)),\\ &\qquad F(\varphi_b(\eth(\Xi u,\Upsilon u)\frac{1+\eth(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\eth(\Lambda\xi_{2n+2},\Xi u)})),\end{aligned}$$

$$\phi_b(\mathfrak{d}(\Xi u,\Upsilon u)\frac{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Xi u)}))\})$$
$$=\max\{F(\varphi_b(0),\phi_b(0)),F(\varphi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)})),\phi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)})\}.$$

If $\max\{F(\varphi_b(0),\phi_b(0)),F(\varphi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)})),\phi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)})\}=F(\varphi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)})),\phi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)}).$
As $F\in\mathscr{F}$ and $\varphi_b\in\Psi_b$, we have

$\varphi_b(\mathfrak{d}(\eta,\Xi u))\leq F(\varphi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)}),\phi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+sd(\eta,\Xi u)}))\leq\varphi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)})$
which implies that $\mathfrak{d}(\eta,\Xi u)\leq\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)}<\mathfrak{d}(\eta,\Xi u)$,

a contradiction.

Therefore, $\max\{F(\varphi_b(0),\phi_b(0)),F(\varphi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)})),\phi_b(\frac{\mathfrak{d}(\eta,\Xi u)}{1+s\mathfrak{d}(\eta,\Xi u)})\}=F(\varphi_b(0),\phi_b(0))$.

Since $F\in\mathscr{F}$ and $\varphi_b\in\Psi_b$, we have

$\varphi_b(\mathfrak{d}(\eta,\Xi u))\leq F(\varphi_b(0),\phi_b(0))\leq\varphi_b(0)\implies Bu=\eta$.

Therefore $Bu=\eta=\Upsilon u$. Now, by *Case (ii)*, the conclusion follows.

Case (iv). $\Xi(\mathfrak{S})$ is b-closed.

Since $\eta\in\Xi(\mathfrak{S})\subseteq\Sigma(\mathfrak{S})\ \exists\ u\in\mathfrak{S}\ \ni\ \eta=\Sigma u$.

Suppose $\Xi u\neq\eta$. Now,

$$\text{(2.12)}\qquad\begin{cases}\varphi_b(s\mathfrak{d}(\Lambda u,\Xi\xi_{2n+1}))\leq\max\{F(\varphi_b(\mathfrak{d}(\Sigma u,\Upsilon\xi_{2n+1})),\phi_b(\mathfrak{d}(\Sigma u,\Upsilon\xi_{2n+1}))),\\ \qquad F(\varphi_b(\mathfrak{d}(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\mathfrak{d}(\Lambda u,\Sigma u)}{1+\mathfrak{d}(\Lambda u,\Xi\xi_{2n+1})}),\\ \qquad \phi_b(\mathfrak{d}(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\mathfrak{d}(\Lambda u,\Sigma u)}{1+\mathfrak{d}(\Lambda u,\Xi\xi_{2n+1})}))\}\end{cases}$$

On letting upper limit as $n\to\infty$ in (2.12), using (2.6) and Lemma 1.8, we have

$$\begin{aligned}\varphi_b(s\frac{1}{s}\mathfrak{d}(\Lambda u,\eta))&\leq\varphi_b(s\limsup_{n\to\infty}\mathfrak{d}(\Lambda u,\Xi\xi_{2n+1}))\\&\leq\limsup_{n\to\infty}(\max\{F(\varphi_b(\mathfrak{d}(\Sigma u,\Upsilon\xi_{2n+1})),\phi_b(\mathfrak{d}(\Sigma u,\Upsilon\xi_{2n+1}))),\\&\qquad F(\varphi_b(\mathfrak{d}(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\mathfrak{d}(\Lambda u,\Sigma u)}{1+\mathfrak{d}(\Lambda u,\Xi\xi_{2n+1})}),\\&\qquad \phi_b(\mathfrak{d}(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\mathfrak{d}(\Lambda u,\Sigma u)}{1+\mathfrak{d}(\Lambda u,\Xi\xi_{2n+1})}))\})\\&=F(\varphi_b(0),\phi_b(0))\leq\varphi_b(0).\end{aligned}$$

As $\varphi_b\in\Psi_b$, we have $\mathfrak{d}(\Lambda u,\eta)\leq 0$ which implies that $\Lambda u=\eta$.

Therefore, $\Lambda u=\eta=\Sigma u$. As in *Case (i)*, the conclusion follows. □

Theorem 2.4. *Let* Λ, Ξ, Σ *and* Υ *be self-maps on a b-metric space* $(\mathfrak{S}, \mathfrak{d})$*, satisfy* (2.1) *and* (2.3). *If the pairs* (Λ, Σ) *and* (Ξ, Υ) *are weakly compatible and either one of the set* $(\Sigma(\mathfrak{S}), \mathfrak{d}), (\Upsilon(\mathfrak{S}), \mathfrak{d}), (\Lambda(\mathfrak{S}), \mathfrak{d})$ *(or)* $(\Xi(\mathfrak{S}), \mathfrak{d})$ *is b-complete, then* Λ, Ξ, Σ *and* Υ *have unique common fixed point.*

Proof. By Lemma 2.2, for each $\xi_0 \in \mathfrak{S}$, the sequence $\{\zeta_n\}$ defined by (2.2) is b-Cauchy.
Since $\Sigma(\mathfrak{S})$ is complete, $\exists\ \eta \in \Sigma(\mathfrak{S}) \ni \lim_{n\to\infty} \zeta_n = \eta$.
As $\eta \in \Sigma(\mathfrak{S})$, $\exists u \in \mathfrak{S} \ni \eta = \Sigma u$. Suppose $\Lambda u \neq \eta$. Now,

$$(2.13)\quad \begin{cases} \varphi_b(s\mathfrak{d}(\Lambda u, \Xi\xi_{2n+1})) \leq \max\{F(\varphi_b(\mathfrak{d}(\Sigma u, \Upsilon\xi_{2n+1})), \phi_b(\mathfrak{d}(\Sigma u, \Upsilon\xi_{2n+1}))), \\ \qquad F(\varphi_b(\mathfrak{d}(\Xi\xi_{2n+1}, \Upsilon\xi_{2n+1})\frac{1+\mathfrak{d}(\Lambda u, \Sigma u)}{1+\mathfrak{d}(\Lambda u, \Xi\xi_{2n+1})}), \\ \qquad \phi_b(\mathfrak{d}(\Xi\xi_{2n+1}, \Upsilon\xi_{2n+1})\frac{1+\mathfrak{d}(\Lambda u, \Sigma u)}{1+\mathfrak{d}(\Lambda u, \Xi\xi_{2n+1})}))\} \end{cases}$$

On letting limit superior as $n \to \infty$ in (2.13), using (2.6) and Lemma 1.8, we have

$$\begin{aligned} \varphi_b(s\tfrac{1}{s}\mathfrak{d}(\Lambda u, \eta)) &\leq \varphi_b(s \limsup_{n\to\infty} \mathfrak{d}(\Lambda u, \Xi\xi_{2n+1})) \\ &\leq \limsup_{n\to\infty}(\max\{F(\varphi_b(\mathfrak{d}(\Sigma u, \Upsilon\xi_{2n+1})), \phi_b(\mathfrak{d}(\Sigma u, \Upsilon\xi_{2n+1}))), \\ &\qquad F(\varphi_b(\mathfrak{d}(\Xi\xi_{2n+1}, \Upsilon\xi_{2n+1})\tfrac{1+\mathfrak{d}(\Lambda u, \Sigma u)}{1+\mathfrak{d}(\Lambda u, \Xi\xi_{2n+1})}), \\ &\qquad \phi_b(\mathfrak{d}(\Xi\xi_{2n+1}, \Upsilon\xi_{2n+1})\tfrac{1+\mathfrak{d}(\Lambda u, \Sigma u)}{1+\mathfrak{d}(\Lambda u, \Xi\xi_{2n+1})}))\}) \\ &= F(\varphi_b(0), \phi_b(0)) \leq \varphi_b(0). \end{aligned}$$

Since $\varphi_b \in \Psi_b$, we have $d(\Lambda u, \eta) \leq 0$ which implies that $\Lambda u = \eta$.

Therefore, $\Lambda u = \eta = \Sigma u$.

As (Λ, Σ) is weakly compatible and $\Lambda u = \Sigma u$, so that

$\Lambda\Sigma u = \Sigma\Lambda u$. i.e., $\Lambda\eta = \Sigma\eta$.

Assume $\Lambda\eta \neq \eta$. Now,

$$(2.14)\quad \begin{cases} \varphi_b(s\mathfrak{d}(\Lambda\eta, \Xi\xi_{2n+1})) \leq \max\{F(\varphi_b(\mathfrak{d}(\Sigma\eta, \Upsilon\xi_{2n+1})), \phi_b(\mathfrak{d}(\Sigma\eta, \Upsilon\xi_{2n+1}))), \\ \qquad F(\varphi_b(\mathfrak{d}(\Xi\xi_{2n+1}, \Upsilon\xi_{2n+1})\frac{1+\mathfrak{d}(\Lambda\eta, \Sigma\eta)}{1+\mathfrak{d}(\Lambda\eta, \Xi\xi_{2n+1})}), \\ \qquad \phi_b(\mathfrak{d}(\Xi\xi_{2n+1}, \Upsilon\xi_{2n+1})\frac{1+\mathfrak{d}(\Lambda\eta, \Sigma\eta)}{1+\mathfrak{d}(\Lambda\eta, \Xi\xi_{2n+1})}))\} \end{cases}$$

On letting upper limit as $n \to \infty$ in (2.14), using (2.6) and Lemma 1.8, we have

$$\begin{aligned} \varphi_b(s\frac{1}{s}\mathfrak{d}(\Lambda\eta, \eta)) &\leq \varphi_b(s \limsup_{n\to\infty} \mathfrak{d}(\Lambda\eta, \Xi\xi_{2n+1})) \\ &\leq \limsup \lim_{n\to\infty}(\max\{F(\varphi_b(\mathfrak{d}(\Sigma\eta, \Upsilon\xi_{2n+1})), \phi_b(\mathfrak{d}(\Sigma\eta, \Upsilon\xi_{2n+1}))), \end{aligned}$$

$$F(\varphi_b(\eth(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})}),$$

$$\phi_b(\eth(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})}))\})$$

$$\leq \max\{F(\varphi_b(\limsup_{n\to\infty}(\eth(\Sigma\eta,\Upsilon\xi_{2n+1}))),\phi_b(\limsup_{n\to\infty}(\eth(\Sigma\eta,\Upsilon\xi_{2n+1})))),$$

$$F(\varphi_b(\limsup_{n\to\infty}(\eth(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})})),$$

$$\phi_b(\limsup_{n\to\infty}(\eth(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})})))\}$$

$$\leq \max\{F(\varphi_b(s\eth(\Lambda\eta,\eta)),\phi_b(s\eth(\Lambda\eta,\eta))),F(\varphi_b(0),\phi_b(0))\}.$$

If $\max\{F(\varphi_b(s\eth(\Lambda\eta,\eta)),\phi_b(s\eth(\Lambda\eta,\eta))),F(\varphi_b(0),\phi_b(0))\} = F(\varphi_b(s\eth(\Lambda\eta,\eta)),\phi_b(s\eth(\Lambda\eta,\eta)))$

then we have $\varphi_b(\eth(\Lambda\eta,\eta)) \leq \varphi_b(s\eth(\Lambda\eta,\eta))$.

Since $\varphi_b \in \Psi_b$, we have

$\eth(\Lambda\eta,\eta) \leq s\eth(\Lambda\eta,\eta) \implies (1-s)\eth(\Lambda\eta,\eta) \leq 0 \implies \Lambda\eta = \eta$.

Suppose $\max\{F(\varphi_b(s\eth(\Lambda\eta,\eta)),\phi_b(s\eth(\Lambda\eta,\eta))),F(\varphi_b(0),\phi_b(0))\} = F(\varphi_b(0),\phi_b(0))$.

Then $\varphi_b(\eth(\Lambda\eta,\eta)) \leq F(\varphi_b(0),\phi_b(0)) \leq \varphi_b(0)$.

As $\varphi_b \in \Psi_b$, we get $\eth(\Lambda\eta,\eta) \leq 0 \implies \Lambda\eta = \eta$. Hence, η is a common fixed point of Λ and Σ. According to Lemma 2.1, conclusion follows.

Similarly, we can prove that η is the unique common fixed point of Λ,Ξ,Σ and Υ when either $\Upsilon(\mathfrak{S})$ or $\Lambda(\mathfrak{S})$ or $\Xi(\mathfrak{S})$ is complete. □

Theorem 2.5. *Let Λ,Ξ,Σ and Υ be self-maps on a complete b-metric space $(\mathfrak{S},\eth)$, satisfy* (2.1) *and* (2.3)*. Further suppose that either*

- (*i*) *(Λ,Σ) is reciprocally continuous and compatible pair of maps, and (Ξ,Υ) is a pair of weakly compatible maps (or)*
- (*ii*) *(Ξ,Υ) is reciprocally continuous and compatible pair of maps, and (Λ,Σ) is a pair of weakly compatible maps.*

Then Λ,Ξ,Σ and Υ have a unique common fixed point.

Proof. By Lemma 2.2, for each $\xi_0 \in \mathfrak{S}$, the sequence $\{\zeta_n\}$ defined by (2.2) is b-Cauchy in $\mathfrak{S}$. As $\mathfrak{S}$ complete, then $\exists\, \eta \in \mathfrak{S} \ni \lim_{n\to\infty}\zeta_n = \eta$.

Assume (i) holds.

As (Λ,Σ) is reciprocally continuous, we have

$\lim_{n\to\infty}\Lambda\Sigma\xi_{2n+2}=\Lambda\eta$ and $\lim_{n\to\infty}\Sigma\Lambda\xi_{2n+2}=\Sigma\eta$.

By compatibility of (Λ,Σ), we get

$\lim_{n\to\infty}\eth(\Lambda\Sigma\xi_{2n+2},\Sigma\Lambda\xi_{2n+2})=0.$

i.e., $\liminf_{n\to\infty}\eth(\Lambda\Sigma\xi_{2n+2},\Sigma\Lambda\xi_{2n+2})=\limsup_{n\to\infty}\eth(\Lambda\Sigma\xi_{2n+2},\Sigma\Lambda\xi_{2n+2})=0.$

According to Lemma 1.8, we have

$\frac{1}{s^2}\eth(\Lambda\eta,\Sigma\eta)\leq\limsup_{n\to\infty}\eth(\Lambda\Sigma\xi_{2n+2},\Sigma\Lambda\xi_{2n+2})=0\implies\Lambda\eta=\Sigma\eta.$

Since $\Lambda(\mathfrak{S})\subseteq\Upsilon(\mathfrak{S})\ \exists\ u\in\mathfrak{S}\ \ni\ \Lambda\eta=\Upsilon u.$

Therefore, $\Lambda\eta=\Sigma\eta=\Upsilon u.$

If $\Lambda\eta\neq\Xi u$, then

$$\begin{aligned}\varphi_b(s\eth(\Lambda\eta,\Xi u))&\leq\max\{F(\varphi_b(\eth(\Sigma\eta,\Upsilon u)),\phi_b(\eth(\Sigma\eta,\Upsilon u))),\\&\qquad F(\varphi_b(\eth(\Xi u,\Upsilon u)\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi u)}),\phi_b(\eth(\Xi u,\Upsilon u)\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi u)}))\}\\&=\max\{F(\varphi_b(0),\phi_b(0)),F(\varphi_b(\frac{\eth(\Xi u,\Lambda\eta)}{1+\eth(\Lambda\eta,\Xi u)}),\phi_b(\frac{\eth(\Xi u,\Lambda\eta)}{1+\eth(\Lambda\eta,\Xi u)}))\}\end{aligned}$$

If $\max\{F(\varphi_b(0),\phi_b(0)),F(\varphi_b(\frac{\eth(\Xi u,\Lambda\eta)}{1+\eth(\Lambda\eta,\Xi u)}),\phi_b(\frac{\eth(\Xi u,\Lambda\eta)}{1+\eth(\Lambda\eta,\Xi u)}))\}=F(\varphi_b(\frac{\eth(\Xi u,\Lambda\eta)}{1+\eth(\Lambda\eta,\Xi u)}),\phi_b(\frac{\eth(\Xi u,\Lambda\eta)}{1+\eth(\Lambda\eta,\Xi u)}))$

then we have $\varphi_b(s\eth(\Lambda\eta,\Xi u))\leq F(\varphi_b(\frac{\eth(\Xi u,\Lambda\eta)}{1+\eth(\Lambda\eta,\Xi u)}),\phi_b(\frac{\eth(\Xi u,\Lambda\eta)}{1+\eth(\Lambda\eta,\Xi u)}))\leq\varphi_b(\frac{\eth(\Xi u,\Lambda\eta)}{1+\eth(\Lambda\eta,\Xi u)}).$

As $\varphi_b\in\Psi_b$, we have $s\eth(\Lambda\eta,\Xi u)\leq\frac{\eth(\Xi u,\Lambda\eta)}{1+\eth(\Lambda\eta,\Xi u)}$,

a contradiction.

Therefore, $\max\{F(\varphi_b(0),\phi_b(0)),F(\varphi_b(\frac{\eth(\Xi u,\Lambda\eta)}{1+\eth(\Lambda\eta,\Xi u)}),\phi_b(\frac{\eth(\Xi u,\Lambda\eta)}{1+\eth(\Lambda\eta,\Xi u)}))\}=F(\varphi_b(0),\phi_b(0))$

which implies that $\varphi_b(s\eth(\Lambda\eta,\Xi u))\leq F(\varphi_b(0),\phi_b(0))\leq\varphi_b(0).$

Since $\varphi_b\in\Psi_b$, we have $\eth(\Lambda\eta,\Xi u)\leq 0$. i.e., $\Lambda\eta=\Xi u\implies\Lambda\eta=\Xi u=\Sigma\eta=\Upsilon u.$

Since (Λ,Σ) is weakly compatible and $\Lambda\eta=\Sigma\eta$, we have $\Lambda\Sigma\eta=\Sigma\Lambda\eta$. i.e ,$\Lambda\Lambda\eta=\Sigma\Lambda\eta$.

If $\Lambda\Lambda\eta\neq\Lambda\eta$, then

$$\begin{aligned}\varphi_b(s\eth(\Lambda\Lambda\eta,\Lambda\eta))&=\varphi_b(s\eth(\Lambda\Lambda\eta,\Xi u))\\&\leq\max\{F(\varphi_b(\eth(\Sigma\Lambda\eta,\Upsilon u)),\phi_b(\eth(\Sigma\Lambda\eta,\Upsilon u))),\\&\qquad F(\varphi_b(\eth(\Xi u,\Upsilon u)\frac{1+\eth(\Lambda\Lambda\eta,\Sigma\Lambda\eta)}{1+\eth(\Lambda\Lambda\eta,\Xi u)}),\phi_b(\eth(\Xi u,\Upsilon u)\frac{1+\eth(\Lambda\Lambda\eta,\Sigma\Lambda\eta)}{1+\eth(\Lambda\Lambda\eta,\Xi u)}))\}\\&=\max\{F(\varphi_b(\eth(\Lambda\Lambda\eta,\Lambda\eta)),\phi_b(\eth(\Lambda\Lambda\eta,\Lambda\eta))),F(\varphi_b(0),\phi_b(0))\}\end{aligned}$$

If $\max\{F(\varphi_b(\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta)),\phi_b(\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta))),F(\varphi_b(0),\phi_b(0))\}$

$=F(\varphi_b(\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta)),\phi_b(\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta)))$ then we have

$\varphi_b(s\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta))\le F(\varphi_b(\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta)),\phi_b(\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta)))\le\varphi_b(\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta))$.

By the property of φ_b, we have $s\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta)\le\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta)$,

a contradiction.

Therefore, $\max\{F(\varphi_b(\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta)),\phi_b(\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta))),F(\varphi_b(0),\phi_b(0))\}=F(\varphi_b(0),\phi_b(0))$

which implies that $\varphi_b(s\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta))\le F(\varphi_b(0),\phi_b(0))\le\varphi_b(0)$.

Since $\varphi_b\in\Psi_b$, we have $\mathfrak{d}(\Lambda\Lambda\eta,\Lambda\eta)\le 0$. i.e., $\Lambda\Lambda\eta=\Lambda\eta$.

Hence, $\Lambda\Lambda\eta=\Sigma\Lambda\eta=\Lambda\eta$, and that $\Lambda\eta$ is a common fixed point of Λ and Σ.

Since (Ξ,Υ) is weakly compatible and $\Xi u=\Upsilon u$, we have $\Xi\Upsilon u=\Upsilon\Xi u$.

Therefore $\Xi\Lambda\eta=\Upsilon\Lambda\eta$. If $\Xi\Lambda\eta\neq\Lambda\eta$, then

$$\begin{aligned}\varphi_b(s\mathfrak{d}(\Xi\Lambda\eta,\Lambda\eta))&=\varphi_b(s\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta))\\&\le\max\{F(\varphi_b(\mathfrak{d}(\Sigma\eta,\Upsilon\Lambda\eta)),\phi_b(\mathfrak{d}(\Sigma\eta,\Upsilon\Lambda\eta))),\\&\qquad F(\varphi_b(\mathfrak{d}(\Xi\Lambda\eta,\Upsilon\Lambda\eta)\frac{1+\mathfrak{d}(\Lambda\eta,\Sigma\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta)}),\phi_b(\mathfrak{d}(\Xi\Lambda\eta,\Upsilon\Lambda\eta)\frac{1+\mathfrak{d}(\Lambda\eta,\Sigma\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta)}))\}\\&=\max\{F(\varphi_b(\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta)),\phi_b(\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta))),F(\varphi_b(0),\phi_b(0))\}\end{aligned}$$

If $\max\{F(\varphi_b(\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta)),\phi_b(\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta))),F(\varphi_b(0),\phi_b(0))\}$

$=F(\varphi_b(\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta)),\phi_b(\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta)))$ then we have

$\varphi_b(s\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta))\le F(\varphi_b(\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta)),\phi_b(\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta)))\le\varphi_b(\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta))$.

By the property of φ_b, we have $s\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta)\le\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta)$,

a contradiction.

Therefore, $\max\{F(\varphi_b(\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta)),\phi_b(\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta))),F(\varphi_b(0),\phi_b(0))\}=F(\varphi_b(0),\phi_b(0))$

which implies that $\varphi_b(s\mathfrak{d}(\Lambda\eta,\Xi\Lambda\eta))\le F(\varphi_b(0),\phi_b(0))\le\varphi_b(0)$.

Since $\varphi_b\in\Psi_b$, we have $d(\Lambda\eta,\Xi\Lambda\eta)\le 0$. i.e., $\Xi\Lambda\eta=\Lambda\eta$. Hence, $\Xi\Lambda\eta=\Lambda\eta$.

Therefore $\Xi\Lambda\eta=\Upsilon\Lambda\eta=\Lambda\eta$. Hence, $\Lambda\Lambda\eta=\Xi\Lambda\eta=\Sigma\Lambda\eta=\Upsilon\Lambda\eta=\Lambda\eta$.

Therefore $\Lambda\eta$ is a common fixed point of Λ,Ξ,Σ and Υ. If $\Lambda\eta\neq\eta$, then

$$\left\{\begin{aligned}\varphi_b(s\mathfrak{d}(\Lambda\eta,\Xi\xi_{2n+1}))\le\max\{&F(\varphi_b(\mathfrak{d}(\Sigma\eta,\Upsilon\xi_{2n+1})),\phi_b(\mathfrak{d}(\Sigma\eta,\Upsilon\xi_{2n+1}))),\\&F(\varphi_b(\mathfrak{d}(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\tfrac{1+\mathfrak{d}(\Lambda\eta,\Sigma\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi\xi_{2n+1})}),\\&\phi_b(\mathfrak{d}(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\tfrac{1+\mathfrak{d}(\Lambda\eta,\Sigma\eta)}{1+\mathfrak{d}(\Lambda\eta,\Xi\xi_{2n+1})}))\}\end{aligned}\right. \tag{2.15}$$

On taking upper limit as $n \to \infty$ in (2.15), using (2.6) and Lemma 1.8, we have

$$\begin{aligned}
\varphi_b(s\frac{1}{s}\eth(\Lambda\eta,\eta)) &\leq \varphi_b(s\limsup_{n\to\infty}\eth(\Lambda\eta,\Xi\xi_{2n+1})) \\
&\leq \limsup\lim_{n\to\infty}(\max\{F(\varphi_b(\eth(\Sigma\eta,\Upsilon\xi_{2n+1})),\phi_b(\eth(\Sigma\eta,\Upsilon\xi_{2n+1}))), \\
&\quad F(\varphi_b(\eth(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})}), \\
&\quad \phi_b(\eth(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})}))\}) \\
&\leq \max\{F(\varphi_b(\limsup_{n\to\infty}(\eth(\Sigma\eta,\Upsilon\xi_{2n+1}))),\phi_b(\limsup_{n\to\infty}(\eth(\Sigma\eta,\Upsilon\xi_{2n+1})))), \\
&\quad F(\varphi_b(\limsup_{n\to\infty}(\eth(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})})), \\
&\quad \phi_b(\limsup_{n\to\infty}(\eth(\Xi\xi_{2n+1},\Upsilon\xi_{2n+1})\frac{1+\eth(\Lambda\eta,\Sigma\eta)}{1+\eth(\Lambda\eta,\Xi\xi_{2n+1})})))\} \\
&\leq \max\{F(\varphi_b(s\eth(\Lambda\eta,\eta)),\phi_b(s\eth(\Lambda\eta,\eta))),F(\varphi_b(0),\phi_b(0))\}.
\end{aligned}$$

If $\max\{F(\varphi_b(s\eth(\Lambda\eta,\eta)),\phi_b(s\eth(\Lambda\eta,\eta))),F(\varphi_b(0),\phi_b(0))\} = F(\varphi_b(s\eth(\Lambda\eta,\eta)),\phi_b(s\eth(\Lambda\eta,\eta)))$ then we have $\varphi_b(\eth(\Lambda\eta,\eta)) \leq \varphi_b(s\eth(\Lambda\eta,\eta))$.

Since $\varphi_b \in \Psi_b$, we have

$\eth(\Lambda\eta,\eta) \leq s\eth(\Lambda\eta,\eta) \implies (1-s)\eth(\Lambda\eta,\eta) \leq 0 \implies \Lambda\eta = \eta$.

Suppose $\max\{F(\varphi_b(s\eth(\Lambda\eta,\eta)),\phi_b(s\eth(\Lambda\eta,\eta))),F(\varphi_b(0),\phi_b(0))\} = F(\varphi_b(0),\phi_b(0))$.

Then $\varphi_b(\eth(\Lambda\eta,\eta)) \leq F(\varphi_b(0),\phi_b(0)) \leq \varphi_b(0)$.

As $\varphi_b \in \Psi_b$, we get $\eth(\Lambda\eta,\eta) \leq 0 \implies \Lambda\eta = \eta$.

Therefore $\Lambda\eta = \Xi\eta = \Sigma\eta = \Upsilon\eta = \eta$.

Similarly, the proof follows under the assumption of (ii). □

Theorem 2.6. *Let $\Lambda,\Xi,\Sigma,\Upsilon : \mathfrak{S} \to \mathfrak{S}$ be self-maps of $\mathfrak{S}$, satisfy* (2.1) *and* (2.3)*. Suppose that one of the pairs (Λ,Σ) and (Ξ,Υ) satisfies the b-(E.A)-property and that one of the subspace $\Lambda(\mathfrak{S}),\Xi(\mathfrak{S}),\Sigma(\mathfrak{S})$ and $\Upsilon(\mathfrak{S})$ is b-closed in $\mathfrak{S}$. Further, if the pairs (Λ,Σ) and (Ξ,Υ) are weakly compatible, then Λ,Ξ,Σ and Υ have a unique common fixed point.*

Proof. Suppose (Λ,Σ) satisfies the b-(E.A)-property. So $\exists$ a sequence $\{\xi_n\} \subseteq \mathfrak{S} \ni$

$$\lim_{n\to\infty}\Lambda\xi_n = \lim_{n\to\infty}\Sigma\xi_n = q \text{ for some } q \in \mathfrak{S} \tag{2.16}$$

As $\Lambda(\mathfrak{S}) \subseteq \Upsilon(\mathfrak{S})$ $\exists$ a sequence $\{\zeta_n\} \subseteq \mathfrak{S} \ni \Lambda\xi_n = \Upsilon\zeta_n$, and hence

$$\lim_{n\to\infty} \Upsilon\zeta_n = q. \tag{2.17}$$

Now, our claim is $\lim_{n\to\infty} \Xi\zeta_n = q$.

From (2.3), we get

$$\begin{aligned} \varphi_b(s(\eth(\Lambda\xi_n, \Xi\zeta_n))) \leq \max\{&F(\varphi_b(\eth(\Sigma\xi_n, \Upsilon\zeta_n)), \phi_b(\eth(\Sigma\xi_n, \Upsilon\zeta_n))), \\ &F(\varphi_b(\eth(\Xi\zeta_n, \Upsilon\zeta_n)\tfrac{1+\eth(\Lambda\xi_n,\Sigma\xi_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}), \phi_b(\eth(\Xi\zeta_n, \Upsilon\zeta_n)\tfrac{1+\eth(\Lambda\xi_n,\Sigma\xi_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}))\} \end{aligned} \tag{2.18}$$

Taking upper limit as $n \to \infty$ in (2.18), and using (2.16) and (2.17), we get

$$\begin{aligned} \limsup_{n\to\infty} \varphi_b(s(\eth(\Lambda\xi_n, \Xi\zeta_n))) &\leq \limsup_{n\to\infty}(\max\{F(\varphi_b(\eth(\Sigma\xi_n, \Upsilon\zeta_n)), \phi_b(\eth(\Sigma\xi_n, \Upsilon\zeta_n))), \\ &\qquad F(\varphi_b(\eth(\Xi\zeta_n, \Upsilon\zeta_n)\tfrac{1+\eth(\Lambda\xi_n,\Sigma\xi_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}), \phi_b(\eth(\Xi\zeta_n, \Upsilon\zeta_n)\tfrac{1+\eth(\Lambda\xi_n,\Sigma\xi_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}))\}) \\ &\leq \max\{F(\varphi_b(0), \phi_b(0)), F(\varphi_b(\limsup_{n\to\infty} \tfrac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}), \phi_b(\limsup_{n\to\infty} \tfrac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}))\}. \end{aligned}$$

If $\max\{F(\varphi_b(0), \phi_b(0)), F(\varphi_b(\limsup_{n\to\infty} \frac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}), \phi_b(\limsup_{n\to\infty} \frac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}))\}$
$= F(\varphi_b(\limsup_{n\to\infty} \frac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}), \phi_b(\limsup_{n\to\infty} \frac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}))$ then
$\limsup_{n\to\infty} \varphi_b(s(\eth(\Lambda\xi_n, \Xi\zeta_n))) \leq F(\varphi_b(\limsup_{n\to\infty} \frac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}), \phi_b(\limsup_{n\to\infty} \frac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}))$.
By the properties of F and φ_b, we have

$$\begin{aligned} \varphi_b(\limsup_{n\to\infty} s(\eth(\Lambda\xi_n, \Xi\zeta_n))) &\leq F(\varphi_b(\limsup_{n\to\infty} \tfrac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}), \phi_b(\limsup_{n\to\infty} \tfrac{\eth(\Lambda\xi_n,\Xi by_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)})) \\ &\leq \varphi_b(\limsup_{n\to\infty} \tfrac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)}) \end{aligned}$$

$\implies \limsup_{n\to\infty} s(\eth(\Lambda\xi_n, \Xi\zeta_n)) \leq \limsup_{n\to\infty} \frac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+\eth(\Lambda\xi_n,\Xi\zeta_n)} \leq \limsup_{n\to\infty}(\eth(\Lambda\xi_n, \Xi\zeta_n))$.
Since $(s-1) \geq 0$, we have $\limsup_{n\to\infty}(\eth(\Lambda\xi_n, \Xi\zeta_n)) \leq 0 \implies \lim_{n\to\infty}(\eth(\Lambda\xi_n, \Xi\zeta_n)) \leq 0$.
If $\max\{F(\varphi_b(0), \phi_b(0)), F(\varphi_b(\limsup_{n\to\infty} \frac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+d(\Lambda\xi_n,\Xi\zeta_n)}), \phi_b(\limsup_{n\to\infty} \frac{\eth(\Lambda\xi_n,\Xi\zeta_n)}{1+d(\Lambda\xi_n,\Xi\zeta_n)}))\}$
$= F(\varphi_b(0), \phi_b(0))$ then

$\limsup_{n\to\infty} \varphi_b(s(\eth(\Lambda\xi_n, \Xi\zeta_n))) \leq F(\varphi_b(0), \phi_b(0))$.
Since $F \in \mathscr{F}$ and $\varphi_b \in \Psi_b$, we have

$\varphi_b(\limsup_{n\to\infty} s(\eth(\Lambda\xi_n, \Xi\zeta_n))) \leq F(\varphi_b(0), \phi_b(0)) \leq \varphi_b(0)$
$\implies \limsup_{n\to\infty} s(\eth(\Lambda\xi_n, \Xi\zeta_n)) \leq 0 \implies \lim_{n\to\infty}(\eth(\Lambda\xi_n, \Xi\zeta_n)) \leq 0.$
Therefore

$$\lim_{n\to\infty} \eth(\Lambda\xi_n, \Xi\zeta_n) = 0. \tag{2.19}$$

We have

$$\mathfrak{d}(q,\Xi\zeta_n) \leq s[\mathfrak{d}(q,\Lambda\xi_n)+\mathfrak{d}(\Lambda\xi_n,\Xi\zeta_n)]. \tag{2.20}$$

Letting limits as $n\to\infty$ in (2.20), and using (2.16) and (2.19), we get

$\lim_{n\to\infty}\mathfrak{d}(q,\Xi\zeta_n) \leq s[\lim_{n\to\infty}\mathfrak{d}(q,\Lambda\xi_n)+\lim_{n\to\infty}\mathfrak{d}(\Lambda\xi_n,\Xi\zeta_n)] = 0 \implies \lim_{n\to\infty}\mathfrak{d}(q,\Xi\zeta_n) = 0.$

Case (i). $\Upsilon(\mathfrak{S})$ is b-closed.

Since $q\in\Upsilon(\mathfrak{S})\ \exists\ r\in\mathfrak{S}\ \ni\ \Upsilon r = q$. Assume $d(\Xi r,q)>0$. From (2.3), we have

$$\begin{cases} \varphi_b(s\mathfrak{d}(\Lambda\xi_{2n+2},\Xi r)) \leq \max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi_{2n+2},\Upsilon r)),\phi_b(\mathfrak{d}(\Sigma\xi_{2n+2},\Upsilon r)), \\ \qquad F(\varphi_b(\mathfrak{d}(\Xi r,\Upsilon r)\frac{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Xi r)}), \\ \qquad \phi_b(\mathfrak{d}(\Xi r,\Upsilon r)\frac{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Xi r)})\} \end{cases} \tag{2.21}$$

On letting upper limit as $n\to\infty$ in (2.21), using (2.6) and Lemma 1.8, we have

$$\begin{aligned} \varphi_b(s\frac{1}{s}\mathfrak{d}(q,\Xi r)) &\leq \limsup_{n\to\infty}\varphi_b(s\mathfrak{d}(\Lambda\xi_{2n+2},\Xi r)) \\ &\leq \limsup_{n\to\infty}(\max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi_{2n+2},\Upsilon r)),\phi_b(\mathfrak{d}(\Sigma\xi_{2n+2},\Upsilon r)), \\ &\qquad F(\varphi_b(\mathfrak{d}(\Xi r,\Upsilon r)\frac{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Sigma\xi_{2n+2})}{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Xi r)})), \\ &\qquad \phi_b(\mathfrak{d}(\Xi r,\Upsilon r)\frac{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Sigma b\xi_{2n+2})}{1+\mathfrak{d}(\Lambda\xi_{2n+2},\Xi r)}))\}) \\ &= \max\{F(\varphi_b(0),\phi_b(0)),F(\varphi_b(\frac{\mathfrak{d}(q,\Xi r)}{1+s\mathfrak{d}(q,\Xi r)})),\phi_b(\frac{\mathfrak{d}(q,\Xi r)}{1+s\mathfrak{d}(q,\Xi r)})\}. \end{aligned}$$

If $\max\{F(\varphi_b(0),\phi_b(0)),F(\varphi_b(\frac{\mathfrak{d}(q,\Xi r)}{1+s\mathfrak{d}(q,\Xi r)})),\phi_b(\frac{\mathfrak{d}(q,\Xi r)}{1+s\mathfrak{d}(q,\Xi r)})\} = F(\varphi_b(\frac{\mathfrak{d}(q,\Xi r)}{1+s\mathfrak{d}(q,\Xi r)}),\phi_b(\frac{\mathfrak{d}(q,\Xi r)}{1+s\mathfrak{d}(q,\Xi r)}))$
then by the properties of F and φ_b we have

$\varphi_b(\mathfrak{d}(q,\Xi r)) \leq F(\varphi_b(\frac{\mathfrak{d}(q,\Xi r)}{1+s\mathfrak{d}(q,\Xi r)})),\phi_b(\frac{\mathfrak{d}(q,\Xi r)}{1+s\mathfrak{d}(q,\Xi r)}) \leq \varphi_b(\frac{\mathfrak{d}(q,\Xi r)}{1+s\mathfrak{d}(q,\Xi r)})$

which implies that $\mathfrak{d}(q,\Xi r) \leq \frac{\mathfrak{d}(q,\Xi r)}{1+s\mathfrak{d}(q,\Xi r)} < \mathfrak{d}(q,\Xi r)$,

a contradiction.

Therefore, $\max\{F(\varphi_b(0),\phi_b(0)),F(\varphi_b(\frac{\mathfrak{d}(q,\Xi r)}{1+s\mathfrak{d}(q,\Xi r)})),\phi_b(\frac{\mathfrak{d}(q,\Xi r)}{1+s\mathfrak{d}(q,\Xi r)})\} = F(\varphi_b(0),\phi_b(0))$.

Since $F\in\mathscr{F}$ and $\varphi_b\in\Psi_b$, we have

$\varphi_b(\mathfrak{d}(q,\Xi r)) \leq F(\varphi_b(0),\phi_b(0)) \leq \varphi_b(0) \implies \mathfrak{d}(q,\Xi r)\leq 0.$

Thus $\Xi r = q$. Since $\Xi(\mathfrak{S})\subseteq\Sigma(\mathfrak{S})$, we have $q\in\Sigma(\mathfrak{S})\ \exists\ \eta\in\mathfrak{S}\ \ni\ \Sigma\eta = q = \Xi r$.

Assume $\Lambda\eta \neq q$. From (2.3), we have

$$\begin{aligned}
\varphi_b(s\eth(\Lambda\eta, q)) &= \varphi_b(s\eth(\Lambda\eta, \Xi r)) \\
&\leq \max\{F(\varphi_b(\eth(\Sigma\eta, \Upsilon r)), \phi_b(\eth(\Sigma\eta, \Upsilon r))), \\
&\qquad F(\varphi_b(\eth(\Xi r, \Upsilon r)\frac{1+\eth(\Lambda\eta, \Sigma\eta)}{1+\eth(\Lambda\eta, \Xi r)}), \phi_b(\eth(\Xi r, \Upsilon r)\frac{1+\eth(\Lambda\eta, \Sigma\eta)}{1+\eth(\Lambda\eta, \Xi r)}))\} \\
&= F(\varphi_b(0), \phi_b(0)).
\end{aligned}$$

Therefore $\varphi_b(s\eth(\Lambda\eta, q)) \leq F(\varphi_b(0), \phi_b(0)) \leq \varphi_b(0)$.

Since $\varphi_b \in \Psi_b$, we have $s\eth(\Lambda\eta, \eta) \leq 0$ implies that $A\eta = q$.

Thus $\Lambda\eta = \Sigma\eta = q$. Since the pairs (Λ, Σ) and (Ξ, Υ) are weakly compatible, we have $\Lambda q = \Sigma q$ and $\Xi q = \Upsilon q$. Assume $\Lambda q \neq q$. From (2.3), we have

$$\begin{aligned}
\varphi_b(s\eth(\Lambda q, q)) &= \varphi_b(s\eth(\Lambda q, \Xi r)) \\
&\leq \max\{F(\varphi_b(\eth(\Sigma q, \Upsilon r)), \phi_b(\eth(\Sigma q, \Upsilon r))), \\
&\qquad F(\varphi_b(\eth(\Xi r, \Upsilon r)\frac{1+\eth(\Lambda q, \Sigma q)}{1+\eth(\Lambda q, \Xi r)}), \phi_b(\eth(\Xi r, \Upsilon r)\frac{1+\eth(\Lambda q, \Sigma q)}{1+\eth(\Lambda q, \Xi r)}))\} \\
&= \max\{F(\varphi_b(\eth(\Lambda q, q)), \phi_b(\eth(\Lambda q, q))), F(\varphi_b(0), \phi_b(0))\}.
\end{aligned}$$

If $\max\{F(\varphi_b(\eth(\Lambda q, q)), \phi_b(\eth(\Lambda q, q))), F(\varphi_b(0), \phi_b(0))\} = F(\varphi_b(\eth(\Lambda q, q)), \phi_b(\eth(\Lambda q, q)))$ then $\varphi_b(s\eth(\Lambda q, q)) \leq F(\varphi_b(\eth(\Lambda q, q)), \phi_b(\eth(\Lambda q, q))) \leq \varphi_b(\eth(\Lambda q, q)) \implies s\eth(\Lambda q, q) \leq \eth(\Lambda q, q)$.

As $(s-1) \geq 0$, we have $\eth(\Lambda q, q) \leq 0$.

If $\max\{F(\varphi_b(\eth(\Lambda q, q)), \phi_b(\eth(\Lambda q, q))), F(\varphi_b(0), \phi_b(0))\} = F(\varphi_b(0), \phi_b(0))$

then by properties of F and φ_b, we get

$\varphi_b(s\eth(\Lambda q, q)) \leq F(\varphi_b(0), \phi_b(0)) \leq \varphi_b(0)$ which implies that $\eth(\Lambda q, q) \leq 0$.

Thus, $\Lambda q = q$. Hence $\Lambda q = \Sigma q = q$. By Lemma 2.1, the proof follows.

Case (ii). Assume $\Lambda(\mathfrak{S})$ is b-closed.

Since $q \in \Lambda(\mathfrak{S})$ and $\Lambda(\mathfrak{S}) \subseteq \Upsilon(\mathfrak{S})$ $\exists\, r \in \mathfrak{S} \ni q = \Upsilon r$. From **Case (i)** the proof follows.

Case (iii). Suppose $\Sigma(\mathfrak{S})$ is b-closed.

As similar in **Case (i)**, the conclusion follows.

Case (iv). Suppose $\Xi(\mathfrak{S})$ is b-closed.

We follow as in **Case (ii)**.

For the case of (Ξ,Υ) satisfies the b-(E.A)-property, we follow the argument similar to the case (Λ,Σ) satisfies the b-(E.A)-property. □

3. Corollaries and Examples

We deduce a few corollaries from the primary findings and offer examples to back up our findings in this section.

Corollary 3.1. *Let $\{\Lambda_n\}_{n=1}^{\infty},\Sigma$ and Υ be self-maps on a complete b-metric space $(\mathfrak{S},\mathfrak{d})$ satisfying $\Lambda_1\subseteq\Sigma(\mathfrak{S})$ and $\Lambda_1\subseteq\Upsilon(\mathfrak{S})$. Assume that there exist $F\in\mathscr{F},\varphi_b\in\Psi_b,\phi_b\in\Phi_b\ \ni$*

(3.1)

$$\varphi_b(s\mathfrak{d}(\Lambda_1\xi,\Lambda_j\zeta))\leq\max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta))),$$
$$F(\varphi_b(\mathfrak{d}(\Lambda_j\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda_1\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda_1\xi,\Lambda_j\xi)}),\phi_b(\mathfrak{d}(\Lambda_j\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda_1\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda_1\xi,\Lambda_j\xi)}))\}$$

$\forall\ \xi,\zeta\in\mathfrak{S}$ and $j\in\mathbb{N}$. If the pairs (Λ_1,Σ) and (Λ_1,Υ) are weakly compatible and one of the range sets $\Lambda_1(\mathfrak{S}),\Sigma(\mathfrak{S})$ and $\Upsilon(\mathfrak{S})$ is b-closed, then $\{\Lambda_n\}_{n=1}^{\infty},\Sigma$ and Υ have a unique common fixed point in $\mathfrak{S}$.

Proof. From the hypothesis of Λ_1,Σ and Υ, the existence of common fixed point follows by taking $\Lambda=\Xi=\Lambda_1$ in Theorem 2.3.

Therefore $\Lambda_1\eta=\Sigma\eta=\Upsilon\eta=\eta$ (say).

Let $j\in\mathbb{N}$ with $j\neq 1$.

Now,

$$\begin{aligned}\varphi_b(s\mathfrak{d}(\eta,\Lambda_j\eta))&=\varphi_b(s\mathfrak{d}(\Lambda_1\eta,\Lambda_j\eta))\\&\leq\max\{F(\varphi_b(\mathfrak{d}(\Sigma\eta,\Upsilon\eta)),\phi_b(\mathfrak{d}(\Sigma\eta,\Upsilon\eta))),\\&\quad F(\varphi_b(\mathfrak{d}(\Lambda_j\eta,\Upsilon\eta)\frac{1+\mathfrak{d}(\Lambda_1\eta,\Sigma\eta)}{1+\mathfrak{d}(\Lambda_1\eta,\Lambda_j\eta)}),\phi_b(\mathfrak{d}(\Lambda_j\eta,\Upsilon\eta)\frac{1+\mathfrak{d}(\Lambda_1\eta,\Sigma\eta)}{1+\mathfrak{d}(\Lambda_1\eta,\Lambda_j\eta)}))\}\\&=\max\{F(\varphi_b(0),\phi_b(0)),F(\varphi_b(\frac{\mathfrak{d}(\Lambda_j\eta,\eta)}{1+\mathfrak{d}(\eta,\Lambda_j\eta)}),\phi_b(\frac{\mathfrak{d}(\Lambda_j\eta,\eta)}{1+\mathfrak{d}(\eta,\Lambda_j\eta)}))\}.\end{aligned}$$

Since $F\in\mathscr{F}$ and $\varphi_b\in\Psi_b$, we have $d(\eta,\Lambda_j\eta)\leq 0\implies\Lambda_j\eta=\eta$ for $j=1,2,3,\ldots$ and uniqueness of common fixed point follows from (3.1). □

Corollary 3.2. *Let Λ, Ξ, Σ and Υ be self-maps on a complete b-metric space $(\mathfrak{S}, \mathfrak{d})$ and satisfy* (2.1) *and the inequality*

$$\varphi_b(s\mathfrak{d}(\Lambda\xi, \Xi\zeta)) \leq F(\varphi_b(\mathfrak{d}(\Sigma\xi, \Upsilon\zeta)), \phi_b(\mathfrak{d}(\Sigma\xi, \Upsilon\zeta)))$$

for all $\xi, \zeta \in \mathfrak{S}$. If the pairs (Λ, Σ) and (Ξ, Υ) are weakly compatible and one of the range sets $\Sigma(\mathfrak{S}), \Upsilon(\mathfrak{S}), \Lambda(\mathfrak{S})$ and $\Xi(\mathfrak{S})$ is closed, then for any $\xi_0 \in \mathfrak{S}$, the sequence $\{\zeta_n\}$ defined by (2.2) *is b-Cauchy in $\mathfrak{S}$ and $\lim_{n\to\infty} \zeta_n = \eta$(say), $\eta \in \mathfrak{S}$ and η is the unique common fixed point of Λ, Ξ, Σ and Υ.*

Corollary 3.3. *Let Λ, Ξ, Σ and Υ be self-maps on a complete b-metric space $(\mathfrak{S}, \mathfrak{d})$ and satisfy* (2.1) *and the inequality*

$$\varphi_b(s\mathfrak{d}(\Lambda\xi, \Xi\zeta)) \leq F(\varphi_b(\mathfrak{d}(\Xi\zeta, \Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}), \phi_b(\mathfrak{d}(\Xi\zeta, \Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}))$$

for all $\xi, \zeta \in \mathfrak{S}$. If the pairs (Λ, Σ) and (Ξ, Υ) are weakly compatible and one of the range sets $\Sigma(\mathfrak{S}), \Upsilon(\mathfrak{S}), \Lambda(\mathfrak{S})$ and $\Xi(\mathfrak{S})$ is closed, then for any $\xi_0 \in \mathfrak{S}$, the sequence $\{\zeta_n\}$ defined by (2.2) *is Cauchy in $\mathfrak{S}$ and $\lim_{n\to\infty} \zeta_n = \eta$(say), $\eta \in \mathfrak{S}$ and η is the unique common fixed point of Λ, Ξ, Σ and Υ*

From Theorem 2.6, the following corollaries follows.

Corollary 3.4. *Let $(\mathfrak{S}, \mathfrak{d})$ be a b-metric space with coefficient $s \geq 1$. Let $\Lambda, \Xi, \Sigma, \Upsilon : \mathfrak{S} \to \mathfrak{S}$ be self-maps of $\mathfrak{S}$ and satisfy* (2.1) *and the inequality*

$$\varphi_b(s\mathfrak{d}(\Lambda\xi, \Xi\zeta)) \leq F(\varphi_b(\mathfrak{d}(\Sigma\xi, \Upsilon\zeta)), \phi_b(\mathfrak{d}(\Sigma\xi, \Upsilon\zeta)))$$

for all $\xi, \zeta \in \mathfrak{S}$. Suppose that one of the pairs (Λ, Σ) and (Ξ, Υ) satisfies the b-(E.A)-property and that one of the subspace $\Lambda(\mathfrak{S}), \Xi(\mathfrak{S}), \Sigma(\mathfrak{S})$ and $\Upsilon(\mathfrak{S})$ is b-closed in $\mathfrak{S}$. Then the pairs (Λ, Σ) and (Ξ, Υ) have a point of coincidence in $\mathfrak{S}$. Moreover, if the pairs (Λ, Σ) and (Ξ, Υ) are weakly compatible, then Λ, Ξ, Σ and Υ have a unique common fixed point in $\mathfrak{S}$.

Corollary 3.5. *Let $(\mathfrak{S}, \mathfrak{d})$ be a b-metric space with coefficient $s \geq 1$. Let $\Lambda, \Xi, \Sigma, \Upsilon : \mathfrak{S} \to \mathfrak{S}$ be self-maps of $\mathfrak{S}$ and satisfy* (2.1) *and the inequality*

$$\varphi_b(s\mathfrak{d}(\Lambda\xi, \Xi\zeta)) \leq F\left(\varphi_b\left(\mathfrak{d}(\Xi\zeta, \Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}\right), \phi_b\left(\mathfrak{d}(\Xi\zeta, \Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi by)}\right)\right)$$

for all $\xi,\zeta \in \mathfrak{S}$. Suppose that one of the pairs (Λ,Σ) and (Ξ,Υ) satisfies the b-(E.A)-property and that one of the subspace $\Lambda(\mathfrak{S}),\Xi(\mathfrak{S}),\Sigma(\mathfrak{S})$ and $\Upsilon(\mathfrak{S})$ is b-closed in $\mathfrak{S}$. Then the pairs (Λ,Σ) and (Ξ,Υ) have a point of coincidence in $\mathfrak{S}$. Moreover, if the pairs (Λ,Σ) and (Ξ,Υ) are weakly compatible, then Λ,Ξ,Σ and Υ have a unique common fixed point in $\mathfrak{S}$.

By choosing $\Lambda = \Xi = f_b$ and $\Sigma = \Upsilon = g_b$ in Theorem 2.3, we have the following.

Corollary 3.6. *Let $(\mathfrak{S},\mathfrak{d})$ be a b-metric space and let f_b and g_b be self-maps of $\mathfrak{S}$. Assume that there exist $\varphi_b \in \Psi_b,\ \phi_b \in \Phi_b$, and $F \in \mathscr{F}$ such that*

$$\varphi_b(s\mathfrak{d}(f_b\xi,f_b\zeta)) \leq \max\{F(\varphi_b(\mathfrak{d}(g_b\xi,g_b\zeta)),\phi_b(\mathfrak{d}(g_b\xi,g_b\zeta))),$$
$$F\left(\varphi_b\left(\mathfrak{d}(g_b\zeta,f_b\zeta)\frac{1+\mathfrak{d}(g_b\xi,f_b\xi)}{1+\mathfrak{d}(g_b\xi,g_b\zeta)}\right),\phi_b\left(\mathfrak{d}(g_b\zeta,f_b\zeta)\frac{1+\mathfrak{d}(g_b\xi,f_b\xi)}{1+\mathfrak{d}(g_b\xi,g_b\zeta)}\right)\right)\} \text{ for all } \xi,\zeta \in \mathfrak{S}.$$

If $f_b(\mathfrak{S}) \subseteq g_b(\mathfrak{S})$, the pair (f_b,g_b) is weakly compatible and $g_b(\mathfrak{S})$ is a complete subspace of $\mathfrak{S}$ then f_b and g_b have a unique fixed point in $\mathfrak{S}$.

Corollary 3.7. *Let $(\mathfrak{S},\mathfrak{d})$ be a b-metric space and let f_b and g_b be self-maps of $\mathfrak{S}$. Assume that there exist $\varphi_b \in \Psi_b,\ \phi_b \in \Phi_b$, and $F \in \mathscr{F}$ such that*

$$\varphi_b(s\mathfrak{d}(f_b\xi,f_b\zeta)) \leq F\left(\varphi_b(\mathfrak{d}(g_b\xi,g_b\zeta)),\phi_b(\mathfrak{d}(g_b\xi,g_b\zeta))\right) \text{ for all } \xi,\zeta \in \mathfrak{S}.$$

If $f_b(\mathfrak{S}) \subseteq g_b(\mathfrak{S})$, the pair (f_b,g_b) is weakly compatible and $g_b(\mathfrak{S})$ is a complete subspace of $\mathfrak{S}$ then f_b and g_b have a unique fixed point in $\mathfrak{S}$.

Corollary 3.8. *Let $(\mathfrak{S},\mathfrak{d})$ be a b- metric space and let f_b and g_b be self-maps of $\mathfrak{S}$. Assume that there exist $\varphi_b \in \Psi_b,\ \phi_b \in \Phi_b$, and $F \in \mathscr{F}$ such that*

$$\varphi_b(s\mathfrak{d}(f_b\xi,f_b\zeta)) \leq F\left(\varphi_b\left(\mathfrak{d}(g_b\zeta,f_b\zeta)\frac{1+\mathfrak{d}(g_b\xi,f_b\xi)}{1+\mathfrak{d}(g_b\xi,g_b\zeta)}\right),\phi_b\left(\mathfrak{d}(g_b\zeta,f_b\zeta)\frac{1+\mathfrak{d}(g_b\xi,f_b\xi)}{1+\mathfrak{d}(g_b\xi,g_b\zeta)}\right)\right) \text{ for all } \xi,\zeta \in \mathfrak{S}.$$

If $f_b(\mathfrak{S}) \subseteq g_b(\mathfrak{S})$, the pair (f_b,g_b) is weakly compatible and $g_b(\mathfrak{S})$ is a complete subspace of $\mathfrak{S}$ then f_b and g_b have a unique fixed point in $\mathfrak{S}$.

Corollary 3.9. *Let $(\mathfrak{S},\mathfrak{d})$ be a complete b- metric space and let f_b and g_b be self-maps of $\mathfrak{S}$. Assume that there exist $\varphi_b \in \Psi_b$, and $\lambda \in (0,1)$ such that*

$$\varphi_b(s\mathfrak{d}(f_b\xi,g_b\zeta)) \leq \lambda\varphi_b\left(\mathfrak{d}(\zeta,g_b\zeta)\frac{1+\mathfrak{d}(\xi,f_b\xi)}{1+\mathfrak{d}(\xi,\zeta)}\right) \text{ for all } \xi,\zeta \in \mathfrak{S}.$$

If either f_b or g_b is b-continuous then f_b and g_b have a unique fixed point in $\mathfrak{S}$.

Corollary 3.10. *Let $(\mathfrak{S},\mathfrak{d})$ be a b- metric space and let f_b and g_b be self-maps of $\mathfrak{S}$. Assume that there exist $\varphi_b \in \Psi_b$, and $\lambda \in (0,1)$ such that*

$$\varphi_b(s\mathfrak{d}(f_b\xi,f_b\zeta)) \leq \lambda\varphi_b\left(\mathfrak{d}(g_b\zeta,f_b\zeta)\frac{1+\mathfrak{d}(g_b\xi,f_b\xi)}{1+\mathfrak{d}(g_b\xi,g_b\zeta)}\right) \text{ for all } \xi,\zeta\in\mathfrak{S}.$$

If $f_b(\mathfrak{S})\subseteq g_b(\mathfrak{S})$, the pair (f_b,g_b) is weakly compatible and $g_b(\mathfrak{S})$ is a complete subspace of $\mathfrak{S}$ then f_b and g_b have a unique fixed point in $\mathfrak{S}$.

Corollary 3.11. *Let $(\mathfrak{S},\mathfrak{d})$ be a complete b- metric space with a parameter $\mathfrak{s}\geq 1$ and let f_b be self-map of $\mathfrak{S}$. Assume that there exist $\varphi_b\in\Psi_b$, and $\lambda\in(0,1)$ such that*

$$\varphi_b(s\mathfrak{d}(f_b\xi,f_b\zeta)) \leq \lambda\varphi_b\left(\mathfrak{d}(\zeta,f_b\zeta)\frac{1+\mathfrak{d}(\xi,f_b\xi)}{1+\mathfrak{d}(\xi,\zeta)}\right) \text{ for all } \xi,\zeta\in\mathfrak{S}.$$

Then f_b has a unique fixed point in $\mathfrak{S}$.

The following is an example in support of Theorem 2.3.

Example 3.12. Let $\mathfrak{S}=[0,1]$. We define $\mathfrak{d}:\mathfrak{S}\times\mathfrak{S}\to\mathbb{R}^+$ by

$$\mathfrak{d}(\xi,\zeta)=\begin{cases} 0 & \text{if } \xi=\zeta \\ (\xi+\zeta)^2 & \text{if } \xi\neq\zeta, \end{cases} \quad \text{for all } \xi,\zeta\in\mathfrak{S}.$$

Then $(\mathfrak{S},\mathfrak{d})$ is a complete b-metric space with coefficient $s=2$.

We define $\Lambda,\Xi,\Sigma,\Upsilon$ on $\mathfrak{S}$ by

$$\Lambda(\xi)=\begin{cases} \frac{1}{2} & \text{if } \xi\in[0,\frac{2}{3}) \\ 0 & \text{if } \xi\in[\frac{2}{3},1), \end{cases} \quad \Xi(\xi)=\begin{cases} \frac{1}{2} & \text{if } \xi\in[0,\frac{2}{3}) \\ \frac{1}{5} & \text{if } \xi\in[\frac{2}{3},1), \end{cases}$$

$$\Sigma(\xi)=\begin{cases} \frac{1}{5} & \text{if } \xi=0 \\ \frac{1}{3}+\frac{\xi}{3} & \text{if } \xi\in(0,\frac{2}{3}) \\ 1 & \text{if } \xi\in[\frac{2}{3},1) \end{cases} \quad \text{and} \quad \Upsilon(\xi)=\begin{cases} \frac{1}{3}+\frac{\xi}{3} & \text{if } \xi\in[0,\frac{2}{3}) \\ \xi-\frac{2}{3} & \text{if } \xi\in[\frac{2}{3},1). \end{cases}$$

We define $F:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ by $F(s,t)=\frac{99}{100}s,\ \varphi_b,\ \phi_b:\mathbb{R}^+\to\mathbb{R}^+$ by

$\varphi_b(t)=\frac{3}{4}t,\ \phi_b(t)=\frac{t}{3}$ for all $t\geq 0$. Clearly $\Lambda(\mathfrak{S})\subseteq\Upsilon(\mathfrak{S})$ and $\Xi(\mathfrak{S})\subseteq\Sigma(\mathfrak{S})$.

The pairs (Λ,Σ) and (Ξ,Υ) are weakly compatible. Without loss of generality we assume $\xi\geq\zeta$

Case (i). $\xi,\zeta\in[0,\frac{2}{3})$.

Here $\mathfrak{d}(\Lambda\xi,\Xi\zeta)=0,\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)=(\frac{7}{12}+\frac{\xi}{3}+\frac{\zeta}{2})^2$.

Clearly the inequality (2.3) holds in this case.

Case (ii). $\xi,\zeta\in[\frac{2}{3},1)$.

Here $\mathfrak{d}(\Lambda\xi,\Xi\zeta)=(\frac{1}{5})^2,\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)=(\frac{1}{3}+\zeta)^2$.

$$\begin{aligned}\varphi_b(s\mathfrak{d}(\Lambda\xi,\Xi\zeta))=\frac{3}{50}\leq\frac{297}{400}(\frac{1}{3}+\zeta)^2 &= F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)))\\ &\leq \max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta))),\\ &F(\varphi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}),\phi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}))\}.\end{aligned}$$

Case (iii). $\xi \in [\frac{2}{3},1), \zeta \in [0,\frac{2}{3})$.

Here $\mathfrak{d}(\Lambda\xi,\Xi\zeta) = (\frac{1}{2})^2, \mathfrak{d}(\Sigma\xi,\Upsilon\zeta) = (\frac{5}{4}+\frac{\zeta}{2})^2$.

$$\begin{aligned}\varphi_b(s\mathfrak{d}(\Lambda\xi,\Xi\zeta)) = \tfrac{3}{8} \leq \tfrac{297}{400}(\tfrac{5}{4}+\tfrac{\zeta}{2})^2 = F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)))\\ \leq \max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta))),\\ F(\varphi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}),\phi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}))\}.\end{aligned}$$

Therefore Λ,Ξ,Σ and Υ satisfy all the hypotheses of Theorem 2.3 and $\frac{1}{2}$ is the unique common fixed point in $\mathfrak{S}$.

The following is an illustration of Theorem 2.5.

Example 3.13. Let $\mathfrak{S} = [0,10]$. We define $\mathfrak{d} : \mathfrak{S}\times\mathfrak{S} \to \mathbb{R}^+$ by

$$\mathfrak{d}(\xi,\zeta) = \begin{cases} 0 & \text{if } \xi = \zeta \\ (\xi+\zeta)^2 & \text{if } \xi \neq \zeta, \end{cases} \quad \text{for all } \xi,\zeta \in \mathfrak{S}.$$

Then $(\mathfrak{S},\mathfrak{d})$ is a complete b-metric space with coefficient $s = 2$.

We define self-maps $\Lambda,\Xi,\Sigma,\Upsilon$ on $\mathfrak{S}$ by

$$\Lambda(\xi) = \begin{cases} \frac{\xi^2}{8} & \text{if } \xi \in [0,1] \\ \xi-\frac{2}{3} & \text{if } \xi \in (1,10], \end{cases} \quad \Xi(\xi) = \begin{cases} \frac{\xi^2}{4} & \text{if } \xi \in [0,1] \\ \frac{\xi-1}{2} & \text{if } \xi \in (1,10], \end{cases}$$

$$\Sigma(\xi) = \begin{cases} \frac{\xi}{2} & \text{if } \xi \in [0,1) \\ \xi-\frac{1}{2} & \text{if } \xi \in [1,10] \end{cases} \quad \text{and } \Upsilon(\xi) = \xi.$$

We define $F : \mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}$ by $F(s,t) = \frac{99}{100}s,\ \varphi_b,\ \phi_b : \mathbb{R}^+ \to \mathbb{R}^+$ by

$\varphi_b(t) = \frac{3}{4}t,\ \phi_b(t) = \frac{t}{3}$ for all $t \geq 0$.

Clearly $\Lambda(\mathfrak{S}) = [0,\frac{1}{8}]\cup(\frac{1}{3},\frac{28}{3}) \subseteq [0,10] = \Upsilon(\mathfrak{S})$ and $\Xi(\xi) = [0,\frac{4}{5}) \subseteq [0,\frac{19}{2}] = \Sigma(\mathfrak{S})$.

Let $\{\xi_n\} = \{\frac{1}{2n}\} \subseteq [0,10]$ for $n \geq 2$.

Then $\lim\limits_{n\to\infty}\Lambda\xi_n = \lim\limits_{n\to\infty}\Sigma\xi_n = 0$ and $\lim\limits_{n\to\infty}\Xi\xi_n = \lim\limits_{n\to\infty}\Upsilon\xi_n = 0$.

Now $\lim\limits_{n\to\infty}\Lambda\Sigma\xi_n = 0 = \Lambda(0), \lim\limits_{n\to\infty}\Sigma\Lambda\xi_n = 0 = \Sigma(0)$ and

$\lim\limits_{n\to\infty}\Xi\Upsilon\xi_n = 0 = \Xi(0), \lim\limits_{n\to\infty}\Upsilon\Xi\xi_n = 0 = \Upsilon(0)$.

Therefore the pairs (Λ,Σ) and (Ξ,Υ) are reciprocally continuous.

Clearly the pairs (Λ,Σ) and (Ξ,Υ) are weakly compatible.

Without loss of generality we assume $\xi \geq \zeta$

Case (i). $\xi,\zeta \in [0,1)$. Here $\mathfrak{d}(\Lambda\xi,\Xi\zeta) = (\frac{\xi^2}{8}+\frac{\zeta^2}{4})^2, \mathfrak{d}(\Sigma\xi,\Upsilon\zeta) = (\frac{\xi}{2}+\zeta)^2$.

$$\begin{aligned}\varphi_b(s\mathfrak{d}(\Lambda\xi,\Xi\zeta)) &= \tfrac{3}{32}(\tfrac{\xi^2}{2}+\zeta^2)^2 \le \tfrac{297}{400}(\tfrac{\xi}{2}+\zeta)^2\\ &= F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)))\\ &\le \max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(d(\Sigma\xi,\Upsilon\zeta))),\\ &F(\varphi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}),\phi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}))\}.\end{aligned}$$

Case (ii). $\xi,\zeta\in(1,10]$. Here $\mathfrak{d}(\Lambda\xi,\Xi\zeta)=(\xi+\frac{\zeta}{2}-\frac{7}{6})^2, \mathfrak{d}(\Sigma\xi,\Upsilon\zeta)=(\xi+\zeta-\frac{1}{2})^2$.

$$\begin{aligned}\varphi_b(s\mathfrak{d}(\Lambda\xi,\Xi\zeta)) &= \tfrac{3}{2}(\xi+\tfrac{\zeta}{2}-\tfrac{7}{6})^2 \le \tfrac{297}{400}(\xi+\zeta-\tfrac{1}{2})^2\\ &= F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)))\\ &\le \max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta))),\\ &F(\varphi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}),\phi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}))\}.\end{aligned}$$

Case (iii). $\xi=1,\zeta\in[0,1)$. Here $d(\Lambda\xi,\Xi\zeta)=(\frac{1}{8}+\frac{\zeta^2}{4})^2, \mathfrak{d}(\Sigma\xi,\Upsilon\zeta)=(\frac{1}{2}+\zeta)^2$.

$$\begin{aligned}\varphi_b(s\mathfrak{d}(\Lambda\xi,\Xi\zeta)) &= \tfrac{3}{32}(\tfrac{1}{2}+\zeta^2)^2 \le \tfrac{297}{400}(\tfrac{1}{2}+\zeta)^2\\ &= F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)))\\ &\le \max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta))),\\ &F(\varphi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}),\phi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}))\}.\end{aligned}$$

Case (iv). $\xi\in(1,10],\zeta\in[0,1)$. Here $\mathfrak{d}(\Lambda\xi,\Xi\zeta)=(\xi+\frac{\zeta^2}{4}-\frac{2}{3})^2, \mathfrak{d}(\Sigma\xi,\Upsilon\zeta)=(\xi+\zeta-\frac{1}{2})^2$.

$$\begin{aligned}\varphi_b(s\mathfrak{d}(\Lambda\xi,\Xi\zeta)) &= \tfrac{3}{2}(\xi+\tfrac{\zeta^2}{4}-\tfrac{2}{3})^2 \le \tfrac{297}{400}(\xi+\zeta-\tfrac{1}{2})^2\\ &= F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)))\\ &\le \max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta))),\\ &F(\varphi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}),\phi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}))\}.\end{aligned}$$

Therefore Λ,Ξ,Σ and Υ satisfy all the hypotheses of Theorem 2.5 and 0 is the unique common fixed point in $\mathfrak{S}$.

The following is an illustration to support Theorem 2.6.

Example 3.14. Let $\mathfrak{S}=[0,1]$ and let $\mathfrak{d}:\mathfrak{S}\times\mathfrak{S}\to\mathbb{R}^+$ defined by

$$\mathfrak{d}(\xi,\zeta)=\begin{cases}0 & \text{if } \xi=\zeta,\\ \frac{11}{15}+\frac{\xi}{23} & \text{if } \xi,\zeta\in(0,\frac{2}{3}),\\ \frac{4}{5}+\frac{\xi+\zeta}{10} & \text{if } \xi,\zeta\in[\frac{2}{3},1],\\ \frac{12}{25} & \text{otherwise.}\end{cases}$$

Then clearly $\mathfrak{d}$ is b-metric with coefficient $s=\frac{52}{49}$.

We specify $\Lambda,\Xi,\Sigma,\Upsilon:\mathfrak{S}\to\mathfrak{S}$ by

$\Lambda(\xi)=\frac{2}{3}$ if $\xi\in[0,1]$, $\Xi(\xi)=\begin{cases}\frac{1}{2} & \text{if } \xi\in[0,\frac{2}{3})\\ 1-\frac{\xi}{2} & \text{if } \xi\in[\frac{2}{3},1],\end{cases}$ $\Sigma(\xi)=\begin{cases}\frac{2}{3}+\frac{\xi}{9} & \text{if } \xi\in[0,\frac{2}{3})\\ \frac{2+5\xi}{8} & \text{if } \xi\in[\frac{2}{3},1],\end{cases}$

and $\Upsilon(\xi)=\begin{cases}\frac{3}{4}+\frac{\sqrt{\xi}}{5} & \text{if } \xi\in[0,\frac{2}{3})\\ \xi & \text{if } \xi\in[\frac{2}{3},1].\end{cases}$

Clearly $\Lambda(\mathfrak{S})\subseteq\Upsilon(\mathfrak{S})$ and $\Xi(\mathfrak{S})\subseteq\Sigma(\mathfrak{S})$. $\Lambda(\mathfrak{S})=\{\frac{2}{3}\}$ is b-closed.

We take $\{\xi_n\}$ with $\{\xi_n\}=\frac{2}{3}+\frac{1}{n}, n\geq 4$ with

$\lim_{n\to\infty}\Lambda\xi_n=\lim_{n\to\infty}\Sigma\xi_n=\frac{2}{3}$, hence (Λ,Σ) satisfies the b-(E.A)-property.

Clearly the pairs (Λ,Σ) and (Ξ,Υ) are weakly compatible.

We define $F:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ by $F(s,t)=\frac{99}{100}s,\ \varphi_b,\ \phi_b:\mathbb{R}^+\to\mathbb{R}^+$ by

$\varphi_b(t)=\frac{3}{4}t,\ \phi_b(t)=\frac{t}{3}$ for all $t\geq 0$.

Case (i). $\xi,\zeta\in[0,\frac{2}{3})$.

$\mathfrak{d}(\Lambda\xi,\Xi\zeta)=\frac{12}{25}, \mathfrak{d}(\Sigma\xi,\Upsilon\zeta)=\frac{4}{5}+\frac{\xi+\zeta}{10}$.

We now consider

$$\begin{aligned}\varphi_b(s\mathfrak{d}(\Lambda\xi,\Xi\zeta))&=\frac{468}{1225}\leq\frac{297}{400}(\frac{4}{5}+\frac{\xi+\zeta}{10})\\ &\leq F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)))\\ &\leq\max\{F(\varphi_b(d(\Sigma\xi,\Upsilon\zeta)),\phi_b(d(\Sigma\xi,\Upsilon\zeta))),\\ &F(\varphi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}),\phi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+d(\Lambda\xi,\Sigma\xi)}{1+d(\Lambda\xi,\Xi\zeta)}))\}.\end{aligned}$$

Case (ii). $\xi,\zeta\in(\frac{2}{3},1]$.

$\mathfrak{d}(\Lambda\xi,\Xi\zeta)=\frac{12}{25}, \mathfrak{d}(\Sigma\xi,\Upsilon\zeta)=\frac{4}{5}+\frac{\xi+\zeta}{10}$.

We now consider

$$\begin{aligned}\varphi_b(s\mathfrak{d}(\Lambda\xi,\Xi\zeta))&=\frac{468}{1225}\leq\frac{297}{400}(\frac{4}{5}+\frac{\xi+\zeta}{10})\\ &\leq F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)))\\ &\leq\max\{F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta))),\\ &F(\varphi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}),\phi_b(\mathfrak{d}(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\mathfrak{d}(\Lambda\xi,\Sigma\xi)}{1+\mathfrak{d}(\Lambda\xi,\Xi\zeta)}))\}.\end{aligned}$$

Case (iii). $\xi\in(\frac{2}{3},1], \zeta\in[0,\frac{2}{3})$.

$\mathfrak{d}(\Lambda\xi,\Xi\zeta)=\frac{12}{25}, \mathfrak{d}(\Sigma\xi,\Upsilon\zeta)=\frac{4}{5}+\frac{\xi+\zeta}{10}$.

We now consider

$$\begin{aligned}\varphi_b(s\mathfrak{d}(\Lambda\xi,\Xi\zeta))&=\frac{468}{1225}\leq\frac{297}{400}(\frac{4}{5}+\frac{\xi+\zeta}{10})\\ &\leq F(\varphi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)),\phi_b(\mathfrak{d}(\Sigma\xi,\Upsilon\zeta)))\end{aligned}$$

$$\leq \max\{F(\varphi_b(\eth(\Sigma\xi,\Upsilon\zeta)),\phi_b(\eth(\Sigma\xi,\Upsilon\zeta))), F(\varphi_b(\eth(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\eth(\Lambda\xi,\Sigma\xi)}{1+\eth(\Lambda\xi,\Xi\zeta)}),\phi_b(\eth(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\eth(\Lambda\xi,\Sigma\xi)}{1+\eth(\Lambda\xi,\Xi\zeta)}))\}.$$

Case (iv). $\xi=\frac{2}{3},\zeta\in[0,\frac{2}{3})$.

$\eth(\Lambda\xi,\Xi\zeta)=\frac{12}{25},\eth(\Sigma\xi,\Upsilon\zeta)=\frac{4}{5}+\frac{\xi+\zeta}{10}$.

We now consider

$$\begin{aligned}\varphi_b(s\eth(\Lambda\xi,\Xi\zeta))&=\tfrac{468}{1225}\leq\tfrac{297}{400}(\tfrac{4}{5}+\tfrac{\xi+\zeta}{10})\\&\leq F(\varphi_b(\eth(\Sigma\xi,\Upsilon\zeta)),\phi_b(\eth(\Sigma\xi,\Upsilon\zeta)))\\&\leq \max\{F(\varphi_b(\eth(\Sigma\xi,\Upsilon\zeta)),\phi_b(\eth(\Sigma\xi,\Upsilon\zeta))),\\&\quad F(\varphi_b(\eth(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\eth(\Lambda\xi,\Sigma\xi)}{1+\eth(\Lambda\xi,\Xi\zeta)}),\phi_b(\eth(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\eth(\Lambda\xi,\Sigma\xi)}{1+\eth(\Lambda\xi,\Xi\zeta)}))\}.\end{aligned}$$

Case (v). $\zeta\in(\frac{2}{3},1],\xi\in[0,\frac{2}{3})$.

$\eth(\Lambda\xi,\Xi\zeta)=\frac{12}{25},\eth(\Sigma\xi,\Upsilon\zeta)=\frac{4}{5}+\frac{\xi+\zeta}{10}$.

We now consider

$$\begin{aligned}\varphi_b(s\eth(\Lambda\xi,\Xi\zeta))&=\tfrac{468}{1225}\leq\tfrac{297}{400}(\tfrac{4}{5}+\tfrac{\xi+\zeta}{10})\\&\leq F(\varphi_b(\eth(\Sigma\xi,\Upsilon\zeta)),\phi(\eth(\Sigma\xi,\Upsilon\zeta)))\\&\leq \max\{F(\varphi_b(\eth(\Sigma\xi,\Upsilon\zeta)),\phi(\eth(\Sigma\xi,\Upsilon\zeta))),\\&\quad F(\varphi_b(\eth(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\eth(\Lambda\xi,\Sigma\xi)}{1+\eth(\Lambda\xi,\Xi\zeta)}),\phi(d(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\eth(\Lambda\xi,\Sigma\xi)}{1+\eth(\Lambda\xi,\Xi\zeta)}))\}.\end{aligned}$$

Case (vi). $\zeta=\frac{2}{3},\xi\in[0,\frac{2}{3})$.

$d(\Lambda\xi,\Xi\zeta)=\frac{12}{25},\eth(\Sigma\xi,\Upsilon\zeta)=\frac{4}{5}+\frac{\xi+\zeta}{10}$.

We now consider

$$\begin{aligned}\varphi_b(s\eth(\Lambda\xi,\Xi\zeta))&=\tfrac{468}{1225}\leq\tfrac{297}{400}(\tfrac{4}{5}+\tfrac{\xi+\zeta}{10})\\&\leq F(\varphi_b(\eth(\Sigma\xi,\Upsilon\zeta)),\phi(\eth(\Sigma\xi,\Upsilon\zeta)))\\&\leq \max\{F(\varphi_b(\eth(\Sigma\xi,\Upsilon\zeta)),\phi(\eth(\Sigma\xi,\Upsilon\zeta))),\\&\quad F(\varphi_b(\eth(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\eth(\Lambda\xi,\Sigma\xi)}{1+\eth(\Lambda\xi,\Xi\zeta)}),\phi(\eth(\Xi\zeta,\Upsilon\zeta)\tfrac{1+\eth(\Lambda\xi,\Sigma\xi)}{1+\eth(\Lambda\xi,\Xi\zeta)}))\}.\end{aligned}$$

As a result, Λ,Ξ,Σ and Υ fulfill all of Theorem 2.6's hypotheses, and $\frac{2}{3}$ is the only common fixed point.

4. Application to Nonlinear Integral Equations

Let $\Omega=C[a,b]$ be a set of real valued continuous functions on $[a,b]$,where $[a,b]$ is closed and bounded integral in $\mathbb{R}$. we define $d:\Omega\times\Omega\to\mathbb{R}^{+}$ by $d(\xi,\eta)=\max_{t\in[a,b]}|\xi(t)-\eta(t)|^{p}$, where $p>1$ a real number, for all $\xi,\eta\in\Omega$. Therefore (Ω,d) is a complete b-metric space with

$s = 2^{p-1}$. Many author's studied unique solution of a system of nonlinear Integral equations [12, 13, 14] .In this section, we establish the existence of unique common solution of a system of two nonlinear integral equations of Fredholm type defined by

$$\begin{cases} \xi(t) = f(t) + \mu \int_a^b \mathscr{D}_1(t,r,\xi(r))dr, \\ \zeta(t) = f(t) + \mu \int_a^b \mathscr{D}_2(t,r,\zeta(r))dr \end{cases} \tag{4.1}$$

where $\xi \in C[a,b]$ is the unknown function, $\mu \in \mathbb{R}, t, r \in [a,b], \mathscr{D}_1, \mathscr{D}_2 : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ are continuous functions.

Let $\mathscr{F}_1, \mathscr{F}_2 : \Omega \to \Omega$ be two mappings defined by

$$\begin{cases} \mathscr{F}_1(\xi(t)) = f(t) + \mu \int_a^b \mathscr{D}_1(t,r,\xi(r))dr, \\ \mathscr{F}_2(\xi(t)) = f(t) + \mu \int_a^b \mathscr{D}_2(t,r,\xi(r))dr \end{cases} \tag{4.2}$$

Assume the following:

- (i) there exists a continuous function $\gamma : [a,b] \times [a,b] \to \mathbb{R}^+$, such that $\max\limits_{r\in[a,b]} \int_a^b \gamma(t,r)dr \leq 1$;
- (ii) there exists a constant $K \in (0,1)$ such that for all $t, r \in [a,b]$ and $\xi, \zeta \in \mathbb{R}$, the following condition is satisfied:
  $|\mathscr{D}_1(t,r,\xi(r)) - \mathscr{D}_2(t,r,\eta(r)|^p \leq \frac{K}{(b-a)^{p-1}2^{3p-3}} \gamma(t,r)|\eta(r) - \mathscr{F}_2\eta(r)|^p \left[\frac{1+|\xi(r)-\mathscr{F}_1\xi(r)|^p}{1+|\xi(r)-\eta(r)|^p}\right]$
- (iii) $|\mu| \leq 1$.

Theorem 4.1. *Let $\mathscr{F}_1, \mathscr{F}_2 : \Omega \to \Omega$ be defined by* (4.2) *for which the conditions (i), (ii) and (iii) hold. Then, the system of nonlinear integral equations* (4.1) *has a unique solution in* Ω.

Proof. Let $\xi, \eta \in \Omega$ and let $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$ using Holder's inequality and from the conditions $(i), (ii)$ and (iii), for all t, we have

$$\begin{aligned} d(\mathscr{F}_1\xi, \mathscr{F}_2\eta) &= \max_{t\in[a,b]} |\mathscr{F}_1\xi(t) - \mathscr{F}_2\eta(t)|^p \\ &= |\mu|^p \max_{t\in[a,b]} \left| \int_a^b \mathscr{D}_1(t,r,\xi(r) - \int_a^b \mathscr{D}_2(t,r,\eta(r)dr \right|^p \\ &= |\mu|^p \max_{t\in[a,b]} \left| \int_a^b (\mathscr{D}_1(t,r,\xi(r) - \mathscr{D}_2(t,r,\eta(r))dr \right|^p \end{aligned}$$

$$\leq \left[|\mu|^p \max_{t\in[a,b]} \left(\int_a^b 1^p dr \right)^{\frac{1}{q}} \left(\int_a^b |(\mathscr{D}_1(t,r,\xi(r) - \mathscr{D}_2(t,r,\eta(r))|^p dr \right)^{\frac{1}{p}} \right]^p$$
$$\leq (b-a)^{\frac{p}{q}} \max_{t\in[a,b]} \left(\int_a^b |(\mathscr{D}_1(t,r,\xi(r) - \mathscr{D}_2(t,r,\eta(r))|^p dr \right)$$
$$= (b-a)^{p-1} \max_{t\in[a,b]} \left(\int_a^b |(\mathscr{D}_1(t,r,\xi(r) - \mathscr{D}_2(t,r,\eta(r))|^p dr \right)$$
$$\leq (b-a)^{p-1} \max_{t\in[a,b]} \int_a^b \frac{K}{(b-a)^{p-1}2^{3p-3}} \gamma(t,r)|\eta(r) - \mathscr{F}_2\eta(r)|^p \left[\frac{1+|\xi(r)-\mathscr{F}_1\xi(r)|^p}{1+|\xi(r)-\eta(r)|^p} \right]$$

which implies that

$$sd(\mathscr{F}_1\xi, \mathscr{F}_2\eta) \leq \frac{K}{s^2} d(\eta, \mathscr{F}_2\eta) \left[\frac{1+d(\xi,\mathscr{F}_1\xi)}{1+d(\xi,\eta)} \right].$$
$$\leq \lambda d(\eta, \mathscr{F}_2\eta) \left[\frac{1+d(\xi,\mathscr{F}_1\xi)}{1+d(\xi,\eta)} \right].$$

where $\lambda = \frac{K}{s^2} \in (0,1)$.

Therefore, by taking $\varphi_b(t) = t$, all the conditions of Corollary 3.9 are satisfied, and hence $\mathscr{F}_1, \mathscr{F}_2$ have a unique common solution of the system of nonlinear integral equations (4.1). □

5. Applications to Dynamic Programming

In this section, we assume that $\mathscr{X}_1$ and $\mathscr{X}_2$ be two Banach spaces; $\mathscr{D} \subseteq \mathscr{X}_1$ is the decision space; $\mathscr{S} \subseteq \mathscr{X}_2$ is the state space; $\Omega(\mathscr{S})$ is the Banach space of all bounded real valued functions on $\mathscr{S}$ with b-metric defined by;

$d(\xi,\zeta) = \sup_{t\in\mathscr{S}} \mid \xi(t) - \zeta(t) \mid^p$, for all $\xi, \zeta \in \Omega(\mathscr{S})$ with coefficient $s = 2^{p-1}$ and the norm is defined as $\|\mathscr{F}\| = \sup\{\mid \mathscr{F}(t) \mid : t \in \mathscr{S}\}$, where $\mathscr{F} \in \Omega(\mathscr{S})$.

It is clear that $\Omega(\mathscr{S}, d)$ is a complete b-metric space. The basic form of the functional equation in dynamic programming is given by Bellman and Lee [6] as follows;

$f(\xi) = opt_{\zeta\in\mathscr{D}} H(\xi, \zeta, f(T(\xi,\zeta))), \xi \in \mathscr{S}$, where ξ and ζ denotes the state and decision vectors, respectively. T denotes the transformation of the process, $f(\xi)$ denotes the optimal return function with the initial state ξ and opt represents Sup of Inf. We consider the system of functional equations

$$f_1(\nu_s) = opt_{\nu_d\in\mathscr{D}} \eta_1(\nu_s, \nu_d)) + \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) \forall \nu_s \in \mathscr{S},$$
$$f_2(\nu_s) = opt_{\nu_d\in\mathscr{D}} \eta_2(\nu_s, \nu_d)) + \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \forall \nu_s \in \mathscr{S} \tag{5.1}$$

where ν_s is a state vector, ν_d is a decision vector,ρ_1, ρ_2 represents the transformations of the process, and $f_1(\nu_s), f_2(\nu_s)$ denotes the optimal return functions with initial state ν_s.

Let $\mathscr{F}_1, \mathscr{F}_2 : \Omega(\mathscr{S}) \to \Omega(\mathscr{S})$ be two mappings defined by;

$$\begin{aligned} \mathscr{F}_1 f(\nu_s) &= opt_{\nu_d \in \mathscr{D}} \eta_1(\nu_s,\nu_d)) + \xi_1(\nu_s,\nu_d, f(\rho_1(\nu_s,\nu_d))), \forall \nu_s \in \mathscr{S} \\ \mathscr{F}_2 f(\nu_s) &= opt_{\nu_d \in \mathscr{D}} \eta_2(\nu_s,\nu_d)) + \xi_2(\nu_s,\nu_d, f(\rho_2(\nu_s,\nu_d))), \forall \nu_s \in \mathscr{S} \end{aligned} \tag{5.2}$$

Assume the following:

($\mathscr{D}_a$) $\mathscr{F}_1(\Omega(\mathscr{S})) \subseteq \mathscr{F}_2(\Omega(\mathscr{S}))$ and $\xi_2(\Omega(s))$ is closed subspace of $\Omega(\mathscr{S})$,

($\mathscr{D}_b$) for all $(\nu_s,\nu_d,f,g) \in \mathscr{S} \times \mathscr{D} \times \Omega(\mathscr{S}) \times \Omega(\mathscr{S})$ and there exists $0 < L < 1$, we have;

$$\begin{aligned} &\mid \xi_1(\nu_s,\nu_d, f(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d, g(\rho_2(\nu_s,\nu_d))) \mid + \mid \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) \mid \\ &\leq \left[\frac{L}{2^{3p-3}} \mid \mathscr{F}_2 g - \mathscr{F}_1 g \mid^p \left(\frac{1+\mathscr{F}_2 g - \mathscr{F}_1 g|^p}{1+\mathscr{F}_2 g - \mathscr{F}_1 g|^p} \right) \right]^{\frac{1}{p}} \end{aligned}$$

($\mathscr{D}_c$) ρ_i, ξ_i are bounded $i = 1,2$.

Theorem 5.1. *Let $\mathscr{F}_1, \mathscr{F}_2 : \Omega(\mathscr{S}) \to \Omega(\mathscr{S})$ be defined by* (5.2) *for which the conditions $\mathscr{D}_a - \mathscr{D}_c$ hold. Then, the system of functional equations given by* (5.1) *has a unique bounded common solution in* $\Omega(S)$.

Proof. Let $\nu_s \in \mathscr{S}, f, g \in \Omega(\mathscr{S})$ and $\varepsilon > 0$.
Since ρ_i, ξ_i are bounded for $i = 1,2$ there exists $M \geq 0$ such that

$$\sup\{\| \rho_1(\nu_s,\nu_d) \|, \| \rho_2(\nu_s,\nu_d) \|, \| \xi_1(\nu_s,\nu_d,t) \|, \| \xi_2(\nu_s,\nu_d,t) \| : (\nu_s,\nu_d,t) \in \mathscr{S} \times \mathscr{D} \times \mathbb{R}\} \leq M. \tag{5.3}$$

From the inequalities (5.2) and (5.3), we conclude that $\mathscr{F}_1, \mathscr{F}_2$ are self mappings are $\Omega(\mathscr{S})$
First assume that

$$opt_{\nu_d \in \mathscr{D}} = \inf_{\nu_d \in \mathscr{D}}.$$

From the inequality (5.2), we can find $\nu_d \in \mathscr{D}$ and $(\nu_s, f, g) \in \mathscr{S} \times \Omega(\mathscr{S}) \times \Omega(\mathscr{S})$ such that

$$\mathscr{F}_1 f(\nu_s) > \xi_1(\nu_s,\nu_d, f(\rho_1(\nu_s,\nu_d)) + \eta_1(\nu_s,\nu_d) - \varepsilon \tag{5.4}$$

$$\mathscr{F}_1 g(\nu_s) > \xi_2(\nu_s,\nu_d, g(\rho_2(\nu_s,\nu_d)) + \eta_2(\nu_s,\nu_d) - \varepsilon \tag{5.5}$$

$$\mathscr{F}_1 f(\nu_s) > \xi_1(\nu_s,\nu_d, f(\rho_1(\nu_s,\nu_d)) + \eta_1(\nu_s,\nu_d) \tag{5.6}$$

$$\mathscr{F}_1 g(\nu_s) > \xi_2(\nu_s,\nu_d, g(\rho_2(\nu_s,\nu_d)) + \eta_2(\nu_s,\nu_d) \tag{5.7}$$

By using the inequalities (5.4) and (5.7), we get that

$$\begin{aligned}\mathscr{F}_1 f(\nu_s)-\mathscr{F}_1 g(\nu_s) &> \xi_1(\nu_s,\nu_d,f(\rho_1(\nu_s,\nu_d)))-\xi_2(\nu_s,\nu_d,g(\rho_2(\nu_s,\nu_d)))\\ &\quad +\eta_1(\nu_s,\nu_d)-\eta_2(\nu_s,\nu_d)-\varepsilon\\ &\geq -\{\mid \xi_1(\nu_s,\nu_d,f(\rho_1(\nu_s,\nu_d)))\xi_2(\nu_s,\nu_d,f(\rho_2(\nu_s,\nu_d)))\mid\\ &\quad +\mid \eta_1(\nu_s,\nu_d)-\eta_2(\nu_s,\nu_d)\mid +\varepsilon\}\end{aligned} \tag{5.8}$$

Also, from (5.5) and (5.6), we have

$$\begin{aligned}\mathscr{F}_1 f(\nu_s)-\mathscr{F}_1 g(\nu_s) &\leq \xi_1(\nu_s,\nu_d,f(\rho_1(\nu_s,\nu_d)))-\xi_2(\nu_s,\nu_d,g(\rho_2(\nu_s,\nu_d)))\\ &\quad +\eta_1(\nu_s,\nu_d)-\eta_2(\nu_s,\nu_d)+\varepsilon\\ &\leq\mid \xi_1(\nu_s,\nu_d,f(\rho_1(\nu_s,\nu_d)))\xi_2(\nu_s,\nu_d,f(\rho_2(\nu_s,\nu_d)))\mid\\ &\quad +\mid \eta_1(\nu_s,\nu_d)-\eta_2(\nu_s,\nu_d)\mid +\varepsilon\end{aligned} \tag{5.9}$$

By using (5.8) and (5.9), we get that

$$\begin{aligned}\mid \mathscr{F}_1 f(\nu_s)-\mathscr{F}_1 g(\nu_s)\mid &< \xi_1(\nu_s,\nu_d,f(\rho_1(\nu_s,\nu_d)))-\xi_2(\nu_s,\nu_d,g(\rho_2(\nu_s,\nu_d)))\\ &\quad +\eta_1(\nu_s,\nu_d)-\eta_2(\nu_s,\nu_d)+\varepsilon\\ &\leq \xi_1(\nu_s,\nu_d,f(\rho_1(\nu_s,\nu_d)))-\xi_2(\nu_s,\nu_d,g(\rho_2(\nu_s,\nu_d)))\\ &\quad +\eta_1(\nu_s,\nu_d)-\eta_2(\nu_s,\nu_d)+\varepsilon\end{aligned}$$

Now, we support that

$$opt_{\nu_d\in\mathscr{D}}=\inf_{\nu_d\in\mathscr{D}}.$$

Again, using the inequality (5.2), we can find $\nu_d\in\mathscr{D}$ and $(\nu_s,f,g)\in\mathscr{S}\times\Omega(\mathscr{S})\times\Omega(\mathscr{S})$ such that

$$\mathscr{F}_1 f(\nu_s)<\xi_1(\nu_s,\nu_d,f(\rho_1(\nu_s,\nu_d)))+\eta_1(\nu_s,\nu_d)+\varepsilon \tag{5.10}$$

$$\mathscr{F}_1 g(\nu_s)<\xi_2(\nu_s,\nu_d,g(\rho_2(\nu_s,\nu_d)))+\eta_2(\nu_s,\nu_d)+\varepsilon \tag{5.11}$$

$$\mathscr{F}_1 f(\nu_s)<\xi_1(\nu_s,\nu_d,f(\rho_1(\nu_s,\nu_d)))+\eta_1(\nu_s,\nu_d) \tag{5.12}$$

$$\mathscr{F}_1 g(\nu_s)<\xi_2(\nu_s,\nu_d,g(\rho_2(\nu_s,\nu_d)))+\eta_2(\nu_s,\nu_d) \tag{5.13}$$

Using the inequalities (5.10) and (5.13), we have

(5.14)

$$
\begin{aligned}
\mathscr{F}_1 f(\nu_s) - \mathscr{F}_1 g(\nu_s) &< \xi_1(\nu_s,\nu_d, f(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d, g(\rho_2(\nu_s,\nu_d))) \\
&\quad + \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) + \varepsilon \\
&\leq \mid \xi_1(\nu_s,\nu_d, f(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d, g(\rho_2(\nu_s,\nu_d))) \mid + \mid \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) \mid + \varepsilon
\end{aligned}
$$

Also, from the inequalities (5.11) and (5.12), we get that

(5.15)

$$
\begin{aligned}
\mathscr{F}_1 f(\nu_s) - \mathscr{F}_1 g(\nu_s) &\geq \xi_1(\nu_s,\nu_d, f(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d, g(\rho_2(\nu_s,\nu_d))) \\
&\quad + \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) - \varepsilon \\
&\geq -\{\mid \xi_1(\nu_s,\nu_d, f(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d, g(\rho_2(\nu_s,\nu_d))) \mid + \mid \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) + \varepsilon\}
\end{aligned}
$$

From (5.14) and (5.15), we have

(5.16)

$$
\begin{aligned}
\mid \mathscr{F}_1 f(\nu_s) - \mathscr{F}_1 g(\nu_s) \mid &< \xi_1(\nu_s,\nu_d, f(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d, g(\rho_2(\nu_s,\nu_d))) \\
&\quad + \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) - \varepsilon \\
&\leq \mid \xi_1(\nu_s,\nu_d, f(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d, g(\rho_2(\nu_s,\nu_d))) \mid + \mid \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) + \varepsilon
\end{aligned}
$$

On taking $\varepsilon \to 0$ in (5.16), we obtain that

$$
\begin{aligned}
\mid \mathscr{F}_1 f(\nu_s) - \mathscr{F}_1 g(\nu_s) \mid &\leq \mid \xi_1(\nu_s,\nu_d, f(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d, g(\rho_2(\nu_s,\nu_d))) \mid \\
&\quad + \mid \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) \mid
\end{aligned}
$$

From the condition $(\mathscr{D}_b)$, we have

$$
\begin{aligned}
\mid \mathscr{F}_1 f(\nu_s) - \mathscr{F}_1 g(\nu_s) \mid &\leq \mid \xi_1(\nu_s,\nu_d, f(\rho_1(\nu_s,\nu_d))) - \xi_2(\nu_s,\nu_d, g(\rho_2(\nu_s,\nu_d))) \mid \\
&\quad + \mid \eta_1(\nu_s,\nu_d) - \eta_2(\nu_s,\nu_d) \mid \\
&\leq \left[\tfrac{L}{2^{3p-3}} \mid \mathscr{F}_1 g(\nu_s) - \mathscr{F}_1 g(\nu_s) \mid^p \left(\tfrac{1+|\mathscr{F}_2 g(\nu_s) - \mathscr{F}_1 f(\nu_s)|^p}{1+|\mathscr{F}_2 f(\nu_s) - \mathscr{F}_1 g(\nu_s)|^p} \right) \right]^{\frac{1}{p}} \\
&\leq \left[\tfrac{L}{2^{3p-3}} \sup_{\nu_s \in \mathscr{S}} \mid \mathscr{F}_1 g(\nu_s) - \mathscr{F}_1 g(\nu_s) \mid^p \left(\tfrac{1+|\mathscr{F}_2 g(\nu_s) - \mathscr{F}_1 f(\nu_s)|^p}{1+|\mathscr{F}_2 f(\nu_s) - \mathscr{F}_1 g(\nu_s)|^p} \right) \right]^{\frac{1}{p}}
\end{aligned}
$$

which implies that

$$
\mid \mathscr{F}_1 f - \mathscr{F}_1 g \mid^p \leq \frac{L}{2^{3p-3}} \mid \mathscr{F}_2 g - \mathscr{F}_1 g \mid^p \left(\frac{1+ \mid \mathscr{F}_2 g - \mathscr{F}_1 f \mid^p}{1+ \mid \mathscr{F}_2 f - \mathscr{F}_2 g \mid^p} \right).
$$

Now, for all $f, g \in \Omega(\mathscr{S})$, we have

$$
sd(\mathscr{F}_1 f, \mathscr{F}_1 g) \leq \lambda d(\mathscr{F}_1 g, \mathscr{F}_2 g) \left(\frac{1+d(\mathscr{F}_1 f, \mathscr{F}_2 g)}{1+d(\mathscr{F}_2 f, \mathscr{F}_2 g)} \right),
$$

where $\lambda = \frac{L}{2^{2p-2}} < 1$.

By taking $\varphi_b(t) = t$, it is easy to see that, Theorem 5.1 satisfies all the hypotheses of Corollary 3.10. Therefore, from Corollary 3.10, we conclude that there exists a unique common fixed point of $\mathscr{F}_1$ and $\mathscr{F}_2$ in $\Omega(\mathscr{S})$ which gives us, the system (5.1) of functional equations has a unique bounded common solution. □

6. Application to Nonlinear Fractional Differential Equations

In this part, we use Corollary 3.11 to prove the existence of a solution to the nonlinear fractional differential equation [9]. Let us first review the definition of Caputo fractional derivative. The Caputo fractional derivative with order $\sigma > 0$ (denoted by $\mathfrak{D}_c^{\sigma}$) is defined as follows:

$$\mathfrak{D}_c^{\sigma} g(t) = \frac{1}{\Gamma(m-\sigma)} \int_0^t (t-\tau)^{m-\sigma-1} g^m(\tau) d\tau,$$

where $\sigma \in [m-1, m)$ with $m = [\sigma] + 1 \in \mathbb{N}, [\sigma]$, the integral part of σ and $g : [0,\infty) \to \mathbb{R}$ is continuous. $\Omega = \mathfrak{C}([0,1],\mathbb{R})$, signifies the set of all functions with continuity from $[0,1]$ into $\mathbb{R}$. We now discuss a nonlinear fractional equation that has unique solutions:

$$\mathfrak{D}_c^{\sigma} \xi(t) = \mathfrak{F}(t, \xi(t)) \tag{6.1}$$

with $\xi(0) = 0, \xi(1) = \int_0^{\rho} \xi(\tau) d\tau$
where $\xi, \zeta \in \Omega, t, \rho \in (0,1), \sigma(1,2]$ and $\mathfrak{F} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.
Here we note that, $\xi \in \Omega$ is a solution of (6.1) iff $\xi \in \Omega$ is a solution of the integral equations:

$$\begin{aligned}\xi(t) = {} & \tfrac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} g(\tau, \xi(\tau)) d\tau - \tfrac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^1 (1-\tau)^{\sigma-1} g(\tau, \xi(\tau)) d\tau \\ & + \tfrac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^{\rho} \left(\int_0^{\tau} (\tau - r)^{\sigma-1} g(r, \xi(r)) dr \right) d\tau.\end{aligned}$$

We define the operator $\mathfrak{F} : \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ by

$$\begin{aligned}\mathfrak{F}(\xi)(t) = {} & \tfrac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} g(\tau, \xi(\tau)) d\tau - \tfrac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^1 (1-\tau)^{\sigma-1} g(\tau, \xi(\tau)) d\tau \\ & + \tfrac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^{\rho} \left(\int_0^{\tau} (\tau - r)^{\sigma-1} g(r, \xi(r)) dr \right) d\tau,\end{aligned}$$

where $\mathfrak{K} = \{\xi \in \Omega : \xi(t) \geq 0, \ \forall\, t \in [0,1]\}$ is a b-metric space with b-metric defined as $\mathfrak{d}(\xi, \zeta) = \sup_{t \in [0,1]} |\xi(t) - \zeta(t)|^{\wp}, \ \forall\, \xi, \zeta \in \mathfrak{K}$ with coefficient $\mathfrak{s} = 2^{\wp - 1}$.

Theorem 6.1. *Let $\mathfrak{F} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a mapping. We suppose that the following circumstances exist:*

($\mathfrak{K}_1$) *$\mathfrak{F}$ is a continuous mapping,*

($\mathfrak{K}_2$) *there exists $0 < L < 1 \ni$*

$$|g(t,\eta(t)) - g(t,\xi(t))| \leq \Gamma(\sigma+1)\left(\frac{L}{2^{3\wp-1}}|\eta - \mathfrak{F}\eta|^{\wp}\left(\frac{1+|\xi-\mathfrak{F}\xi|^{\wp}}{1+|\xi-\eta|^{\wp}}\right)\right)^{\frac{1}{\wp}}$$

$\forall\ \xi,\eta \in \mathbb{R}, \xi,\eta \geq 0$ and $\forall\ t \in [0,1]$. Then the system of fractional differential equations (6.1) *has a unique solution.*

Proof. From condition ($\mathfrak{K}_2$), for all $\xi,\eta \in \mathfrak{K}$ and $t \in [0,1]$, we have

$$\begin{aligned}
|\mathfrak{F}(\eta)(t) - \mathfrak{F}(\xi)(t)| = {} & |\tfrac{1}{\Gamma(\sigma)}\int_0^t (t-\tau)^{\sigma-1} g(\tau,\eta(\tau))d\tau \\
& - \tfrac{2t}{(2-\rho^2)\Gamma(\sigma)}\int_0^1 (1-\tau)^{\sigma-1} g(\tau,\eta(\tau))d\tau \\
& + \tfrac{2t}{(2-\rho^2)\Gamma(\sigma)}\int_0^\rho \left(\int_0^\tau (\tau-r)^{\sigma-1} g(r,\eta(r))dr\right)d\tau \\
& - \tfrac{1}{\Gamma(\sigma)}\int_0^t (t-\tau)^{\sigma-1} g(\tau,\xi(\tau))d\tau \\
& + \tfrac{2t}{(2-\rho^2)\Gamma(\sigma)}\int_0^1 (1-\tau)^{\sigma-1} g(\tau,\xi(\tau))d\tau \\
& - \tfrac{2t}{(2-\rho^2)\Gamma(\sigma)}\int_0^\rho \left(\int_0^\tau (\tau-r)^{\sigma-1} g(r,\xi(r))dr\right)d\tau| \\
\leq {} & \tfrac{1}{\Gamma(\sigma)}\int_0^t (t-\tau)^{\sigma-1}|g(\tau,\eta(\tau)) - g(\tau,\xi(\tau))|d\tau \\
& + \tfrac{2t}{(2-\rho^2)\Gamma(\sigma)}\int_0^1 (1-\tau)^{\sigma-1}|g(\tau,\eta(\tau)) - g(\tau,\xi(\tau))|d\tau \\
& + \tfrac{2t}{(2-\rho^2)\Gamma(\sigma)}\int_0^\rho \left(\int_0^\tau (\tau-r)^{\sigma-1}|g(r,\eta(r)) - g(r,\xi(r))|dr\right)d\tau \\
\leq {} & \tfrac{1}{\Gamma(\sigma)}\int_0^t (t-\tau)^{\sigma-1}\Gamma(\sigma+1)\left(L\frac{|\eta-\mathfrak{F}\eta|^{\wp}\left(\frac{1+|\xi-\mathfrak{F}\xi|^{\wp}}{1+|\xi-\eta|^{\wp}}\right)}{2^{3\wp-1}}\right)^{\frac{1}{\wp}}d\tau \\
& + \tfrac{2t}{(2-\rho^2)\Gamma(\sigma)}\int_0^1 (1-\tau)^{\sigma-1}\Gamma(\sigma+1)\left(L\frac{|\eta-\mathfrak{F}\eta|^{\wp}\left(\frac{1+|\xi-\mathfrak{F}\xi|^{\wp}}{1+|\xi-\eta|^{\wp}}\right)}{2^{3\wp-1}}\right)^{\frac{1}{\wp}}d\tau \\
& + \tfrac{2t}{(2-\rho^2)\Gamma(\sigma)}\int_0^\rho \left(\int_0^\tau (\tau-r)^{\sigma-1}\Gamma(\sigma+1)\left(L\frac{|\eta-\mathfrak{F}\eta|^{\wp}\left(\frac{1+|\xi-\mathfrak{F}\xi|^{\wp}}{1+|\xi-\eta|^{\wp}}\right)}{2^{3\wp-1}}\right)^{\frac{1}{\wp}}dr\right)d\tau
\end{aligned}$$

$$\leq \frac{\Gamma(\sigma+1)}{\Gamma(\sigma)}\left(L\frac{|\eta-\mathfrak{F}\eta|^{\wp}\left(\frac{1+|\xi-\mathfrak{F}\xi|^{\wp}}{1+|\xi-\eta|^{\wp}}\right)}{2^{3\wp-1}}\right)^{\frac{1}{\wp}}$$

$$\left[\int_0^t (t-\tau)^{\sigma-1}d\tau + \frac{2t}{(2-\rho^2)}\int_0^1 (1-\tau)^{\sigma-1}d\tau + \frac{2t}{(2-\rho^2)}\int_0^\rho \left(\int_0^\tau (\tau-r)^{\sigma-1}dr\right)d\tau\right]$$

$$= \left(\Gamma(\sigma+1)\left(L\frac{|\eta-\mathfrak{F}\eta|^{\wp}\left(\frac{1+|\xi-\mathfrak{F}\xi|^{\wp}}{1+|\xi-\eta|^{\wp}}\right)}{2^{3\wp-1}}\right)^{\frac{1}{\wp}}\right)\frac{1}{\Gamma(\sigma)}\left[\frac{t^\sigma}{\sigma} + \frac{2t}{(2-\rho^2)\sigma} + \frac{2t}{(2-\rho^2)\sigma}\frac{\rho^{\sigma+1}}{\sigma+1}\right]$$

$$\leq \left(\Gamma(\sigma+1)\left(L\frac{|\eta-\mathfrak{F}\eta|^{\wp}\left(\frac{1+|\xi-\mathfrak{F}\xi|^{\wp}}{1+|\xi-\eta|^{\wp}}\right)}{2^{3\wp-1}}\right)^{\frac{1}{\wp}}\right)\frac{1}{\Gamma(\sigma+1)}\left[\sup_{t\in(0,1)}\{t^\sigma + \frac{2t}{(2-\rho^2)} + \frac{2t}{(2-\rho^2)}\frac{\rho^{\sigma+1}}{\sigma+1}\}\right]$$

$$\leq \left(L\frac{|\eta-\mathfrak{F}\eta|^{\wp}\left(\frac{1+|\xi-\mathfrak{F}\xi|^{\wp}}{1+|\xi-\eta|^{\wp}}\right)}{2^{\wp-1}}\right)^{\frac{1}{\wp}}$$

$\Longrightarrow$

$$(2^{\wp-1})\sup_{t\in[0,1]}|\mathfrak{F}(\eta)(t)-\mathfrak{F}(\xi)(t)|^{\wp} \leq L\sup_{t\in[0,1]}\left(|\eta-\mathfrak{F}\eta|^{\wp}\left(\frac{1+|\xi-\mathfrak{F}\xi|^{\wp}}{1+|\xi-\eta|^{\wp}}\right)\right)$$

Now, let us define $\varphi_b(t)=t$, we have

$$s\mathfrak{d}(\mathfrak{F}(\xi),\mathfrak{F}(\eta)) \leq L\varphi_b\left(\mathfrak{d}(\eta,\mathfrak{F}\eta)\frac{1+\mathfrak{d}(\xi,\mathfrak{F}\xi)}{1+\mathfrak{d}(\xi,\eta)}\right).$$

Therefore, from Corollary 3.11, (6.1) has a unique solution. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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