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MIN-PHASE-ISOMETRIES ON THE UNIT SPHERE OF $\mathscr{L}_p$-TYPE SPACES

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Abstract. Let X, Y be two real L_p-spaces ($p > 0$), then a surjective map $f : \mathrm{S}_X \to \mathrm{S}_Y$ satisfies

$$\min\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \min\{\|x+y\|, \|x-y\|\} \quad (x, y \in S_X),$$

if and only if f is a multiplication of a linear isometry and a map with rang $\{-1, 1\}$. It can be regarded as a new Wigner's theorem for real L_p-spaces ($p > 0$).

keywords: L_p-spaces($p > 0$); Wigner's theorem; phase-equivalent; min-phase-isometry.
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1. Introduction

The metric structure of normed space affects the linear structure to some extent and has been a topic of concern for many scholars. The classical Mazur-Ulam [17] states that every surjective isometry between real normed spaces is automatically affine. In 1972, P. Mankiewiz [16] showed that every surjective isometry between the open connected subsets of normed space can be extended to a surjective affine isometry on the whole space. This means that the metric spaces on the unit sphere of a real normed space constrains the linear structure of the whole space. We are interested in whether the sphere can be raised for a particular space. In 1987,

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Tingley [12] first studied isometry on the unit sphere and proposed the following problem: let X and Y be normed spaces with the unit spheres S_X and S_Y, Assume that $T : \mathrm{S}_X \to \mathrm{S}_Y$ is a surjective isometry. Does there exist a linear isometry $\widetilde{T} : \mathrm{S}_X \to \mathrm{S}_Y$ such that $\widetilde{T}|_{S_X} = T$. Subsequently, Wang [14] appears to be the first to sovle the space-specific Tingley problem and have a positive answer.

In addition, Wigner's[2, 5, 6, 13] theorem is associated with linear isometry mappings. The famous wigner's[1, 3, 4, 7, 8, 10, 11]theorem plays an important role in quantum mechanics. It can be described in several ways, one of which is as follows: Let H and K be real or complex Hilbert spaces and let $f : \mathrm{H} \to \mathrm{K}$ be a mapping. Then f satisfies the functional equation

$$|< f(x), f(y) >| = |< x, y >| \quad (x, y \in H) \tag{1.1}$$

if and only if f is phase equivalent to a linear or conjugate linear isometry. Later, G. Maksa[6] and Z. Pales proved the expression of wigner's theorem on real version: let X and Y be two real inner produce spaces.Suppose that $f : \mathrm{X} \to \mathrm{Y}$ is a surjective mapping satisfying

$$\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \{\|x+y\|, \|x-y\|\} \quad (x, y \in X), \tag{1.2}$$

if and only if f is phase equivalent to a surjective linear isometry. And They asked the following question, does it still hold true when X and Y be two normed spaces but not inner product spaces. Recently, there was a positive answer to the above question when X and Y are real atomic $\mathscr{L}_P$ spaces $(P > 0)$.

Since in the following lemma we prove that f is a max-phase-isometry but cannot be phase equivalent to a surjective linear isometry. By Xujian Huang and Dongni[18] Tan explored min-phase-isometries and Wigner's theorem on real normed spaces and Xihong Jin[15] explored for the unit sphere for $\mathscr{L}_P$-type space. In this article we will prove that a surjective map $f : \mathrm{S}_X \to \mathrm{S}_Y$ satisfies if and only if f is phase equivalent to linear isometries and it can be extended to the whole space .

2. Preliminaries

Throughout this section, we consider the spaces all over the real field and denote by $\mathbb{R}$ the set of of reals. This paper mainly discusses the atomic $\mathscr{L}_p$-spaces on $\mathbb{R}$ with $p > 0, p \neq 2$. The spaces X and Y are used to denote such spaces unless otherwise stated. We use S_X and S_Y to denote the unit spheres of X and Y respectively. Moreover, f denotes a mapping from S_X to S_Y. An atomic $\mathscr{L}_p$-space ($p>0$) is linearly isometric to $l_p(\Gamma)$, where Γ is a nonempty index set. The atomic L_p-space is

$$l_p(\Gamma)=\{x=\sum_{\gamma}\xi_\gamma e_\gamma : \|x\|=(\sum_{\gamma}|\xi_\gamma|^p)^{\frac{1}{p}}<\infty,\ \xi_\gamma\in\mathbb{R},\ \gamma\in\Gamma\},$$

where $e_\gamma:\Gamma\to\mathbb{R}$ is the function for which $e_\gamma(\gamma)=1, e_\gamma(\gamma')=0, \forall\gamma'\in\Gamma, \gamma'\neq\gamma$. For every $x=\sum_{\gamma\in\Gamma}\xi_\gamma e_\gamma\in \mathrm{X}$, we denote the support of x by Γ_x, i.e.,

$$\Gamma_x=\{\gamma\in\Gamma:\xi_\gamma\neq 0\}.$$

Then x can be rewritten in the form $x=\sum_{\gamma\in\Gamma_x}\xi_\gamma e_\gamma\in X$. For all $x,y\in l_p(\Gamma)$, if $\Gamma_x\cap\Gamma_y=\emptyset$, then we say that x is orthogonal to y and write $x\perp y$. It should be noted that $l_p(\Gamma)$ for $0<p<1$ is a quasi-normed space but not a normed space.

Definition 2.1. We say a mapping $f\colon S_X\to S_Y$ is a min-phase-isometry which satisfies

$$\min\{\|f(x)+f(y)\|,\|f(x)-f(y)\|\}=\min\{\|x+y\|,\|x-y\|\}\quad (x,y\in \mathrm{S}_X). \tag{2.1}$$

3. Main Results

Lemma 3.1. *For any two real numbers ξ and η,*

$$|\xi+\eta|^p+|\xi-\eta|^p=2(|\xi|^p+|\eta|^p)\Leftrightarrow \xi\cdot\eta=0,\quad p>0,\ p\neq 2.$$

By this lemma, one can conclude the following result whose proof is obvious, and thus omitted.

Lemma 3.2. [1] *Let x,y be two elements in $l_p(\Gamma)$, where $p>1$ and $p\neq 2$. Then, it exists two situations:*

- $\|x+y\|^p+\|x-y\|^p\geq 2(\|x\|^p+\|y\|^p)$ *for all* $p>2$;
- $\|x+y\|^p+\|x-y\|^p\leq 2(\|x\|^p+\|y\|^p)$ *for all* $1<p<2$;

The equal sign holds if and only if $x\perp y$.

Lemma 3.3. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta), 1 < p < 2$. Suppose that $f\colon \mathrm{S}_X \to \mathrm{S}_Y$ is a surjective min-phase-isometry. Then for any $x, y \in \mathrm{S}_X$, we have*

$$x \perp y \Leftrightarrow f(x) \perp f(y).$$

Proof. Let $x, y \in S_X$ with $x = \sum_{\gamma\in\Gamma} \xi_\gamma e_\gamma, y = \sum_{\gamma\in\Gamma} \eta_\gamma e_\gamma$ and $x \perp y$. Since f is a min-phase-isometry, we have

$$\min\{\|f(x)+f(y)\|^p, \|f(x)-f(y)\|^p\} = \min\{\|x+y\|^p, \|x-y\|^p\} = 2.$$

Thus, $\|f(x)+f(y)\|^p + \|f(x)-f(y)\|^p \geq 4$. By Lemma 3.2, we know $\|f(x)+f(y)\|^p + \|f(x)-f(y)\|^p \leq 4$. So we have $\|f(x)+f(y)\|^p + \|f(x)-f(y)\|^p = 4$. In conclusion, $f(x) \perp f(y)$. The proof is complete. □

Theorem 3.4. *Let X and Y be inner spaces. Suppose that $f\colon \mathrm{S}_X \to \mathrm{S}_Y$ is a min-phase-isometry. Then there exists a function $\varepsilon : \mathrm{S}_X \to \{-1, 1\}$ such that εf is an isometry.*

Proof. For any $x, y \in \mathrm{S}_X$, we have

$$< x, y >= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2).$$

Since $x \perp y$, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) = 4.$$

Therefore

$$< x, y >= \frac{1}{4}(4 - 2\|x-y\|^2) = 1 - \frac{1}{2}\|x-y\|^2,$$

or

$$< x, y >= \frac{1}{4}(2\|x+y\|^2 - 4) = \frac{1}{2}\|x+y\|^2 - 1.$$

Since $f(x) \perp f(y)$, we have

$$\|f(x)+f(y)\|^2 + \|f(x)-f(y)\|^2 = 2(\|f(x)\|^2 + \|f(y)\|^2) = 4.$$

Therefore

$$< f(x), f(y) >= \frac{1}{4}(4 - 2\|f(x)-f(y)\|^2) = 1 - \frac{1}{2}\|f(x)-f(y)\|^2,$$

or

$$< f(x), f(y) >= \frac{1}{4}(2\|f(x)+f(y)\|^2-4) = \frac{1}{2}\|f(x)+f(y)\|^2-1.$$

Since

$$\min\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \min\{\|x+y\|, \|x-y\|\},$$

In conclusion, $|< f(x), f(y) >| = |< x, y >|$.

The proof is complete. $\square$

Lemma 3.5. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$, $p > 0, p \neq 2$. Suppose that $f\colon \mathrm{S}_X \to \mathrm{S}_Y$ is a surjective min-phase-isometry. Then*

(a). $f(-x) = -f(x)$ for all $x \in \mathrm{S}_X$.

(b). f is injective for all $x \in \mathrm{S}_X$.

(c). there is a bijection $\sigma : \Gamma \to \Delta$ such that $f(-e_\gamma) \in \{e_{\sigma(\gamma)}, -e_{\sigma(\gamma)}\}$.

Proof. Since f is surjective, for each $x \in \mathrm{S}_X$, there is $y \in \mathrm{S}_X$ such that $f(y) = -f(x)$. It implies that

$$\begin{aligned}\min\{\|y+x\|, \|x-y\|\} &= \min\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\}\\ &= \min\{0, \|2f(x)\|\} = 0.\end{aligned}$$

So $x+y=0$ or $x-y=0$, which implies that $y \in \{x, -x\}$. Since $f(x) \in S_Y$, then $y = -x$. Therefore, it is an odd mapping.

Let $f(x_1) = f(x_2)$, we have

$$\begin{aligned}\min\{\|x_1+x_2\|, \|x_1-x_2\|\} &= \min\{\|f(x_1)+f(x_2)\|, \|f(x_1)-f(x_2\|\}\\ &= \min\{0, \|2f(x_1)\|\} = 0.\end{aligned}$$

So $x_1+x_2=0$ or $x_1-x_2=0$, which implies that $x_1 \in \{x_2, -x_2\}$. Since $f(-x) = -f(x)$, then $x_1 = x_2$. Hence, f is injective.

Let $\gamma \in \Gamma$ and denote by $\Delta_{f(e_\gamma)}$ the support of $f(e_\gamma)$. For any $\delta \in \Delta_{f(e_\gamma)}$, we can find $x \in \mathrm{S}_X$ such that $f(x) = e_\delta$. For any $\gamma' \in \Gamma$ with $\gamma' \neq \gamma$, by Lemma 3.3

$$f(e_\gamma) \perp f(e_{\gamma'}) \Rightarrow f(x) \perp f(e_{\gamma'}) \Rightarrow x \perp e_{\gamma'}.$$

This means $x \in \{e_\gamma, -e_\gamma\}$, and $\{f(e_\gamma), f(-e_\gamma)\} \in \{e_\delta, -e_\delta\}$. So $\Delta_{f(e_\gamma)}$ is a singleton. Now we define an injective mapping $\sigma : \Gamma \to \Delta$ by $\sigma(\gamma) = \delta$. We will show that σ is a surjective mapping. Suppose it is true, there is a $\delta_0 \in \Delta$ such that $\delta_0 \notin \sigma(\Gamma)$. As f is surjective, there exists $y \in S_X$ satisfying $f(y) = e_{\delta_0}$. By Lemma 3.3 again,

$$f(y) \perp f(e_\gamma) \Rightarrow y \perp e_\gamma, \ \forall \gamma \in \Gamma.$$

So $y = 0$, which is a contradiction. □

Lemma 3.6. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$, $1 < p < 2$. Suppose that $f \colon S_X \to S_Y$ is a surjective min-phase-isometry. As Lemma 3.5, let $\sigma : \Gamma \to \Delta$ be the bijection. Then for any element $x = \sum_{\gamma \in \Gamma} \xi_\gamma e_\gamma \in S_X$, we have $f(x) = \sum_{\gamma \in \Gamma} \eta_\gamma f(e_\gamma)$, where $|\xi_\gamma| = |\eta_\gamma|$ for any $\gamma \in \Gamma$.*

Proof. We can assume that $x = \sum_{\gamma \in \Gamma} \xi_\gamma e_\gamma \in S_X$, $\sum_{\gamma \in \Gamma} |\xi_\gamma|^p = 1$. It is easy to see that $f(x) \perp e_{\gamma'}$ for each $\gamma' \in \Gamma \backslash \Gamma_x$. We can write $f(x) = \sum_{\gamma \in \Gamma} \eta_\gamma f(e_\gamma)$, $\sum_{\gamma \in \Gamma} |\eta_\gamma|^p = 1$. For any $\gamma \in \Gamma_x$, we have

$$\begin{aligned}
&\min\{\|f(x) + f(e_\gamma)\|^p, \|f(x) - f(e_\gamma)\|^p\} \\
=&\min\{\|x + e_\gamma\|^p, \|x - e_\gamma\|^p\} \\
=&\min\{(1 - |\xi_\gamma|^p + |\xi_\gamma + 1|^p), (1 - |\xi_\gamma|^p + |\xi_\gamma - 1|^p)\} \\
=&1 - |\xi_\gamma|^p + ||\xi_\gamma| - 1|^p
\end{aligned}$$

On the other hand, $f(e_\gamma) = \pm e_{\sigma(\gamma)}$, we have

$$\begin{aligned}
&\min\{\|f(x) + f(e_\gamma)\|^p, \|f(x) - f(e_\gamma)\|^p\} \\
=&\min\{(1 - |\eta_\gamma|^p + |\eta_\gamma + 1|^p), (1 - |\eta_\gamma|^p + |\eta_\gamma - 1|^p)\} \\
=&1 - |\eta_\gamma|^p + ||\eta_\gamma| - 1|^p
\end{aligned}$$

A short calculation shows that

$$||\xi_\gamma| - 1|^p - |\xi_\gamma|^p = ||\eta_\gamma| - 1|^p - |\eta_\gamma|^p$$

Since the function $\varphi(t) = (1 - t)^p - t^p$ is strictly increasing (decreasing) on $[0, 1]$ for $p > 1$. Thus $|\xi_\gamma| = |\eta_\gamma|$ for any $\gamma \in \Gamma_x$. □

Lemma 3.7. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$, $1 < p < 2$. Suppose that $f\colon \mathrm{S}_X \to \mathrm{S}_Y$ is a surjective min-phase-isometry. Then for all nonzero orthogonal vectors x, y in S_X,and $a, b \in \mathbb{R}$,there exist two real numbers α and β with absolute value 1 such that*

$$f(ax+by) = \alpha a f(x) + \beta b f(y) where \quad ax+by \in S_X \text{ and } \alpha, \beta \in \{-1, 1\}.$$

Proof. Let x and y be nonzero orthogonal vectors in S_X such that $x = \sum_{\gamma\in\Gamma_x} \xi_\gamma e_\gamma$ and $y = \sum_{\gamma\in\Gamma_y} \eta_\gamma e_\gamma$, and $0 \neq \lambda \in \mathbb{R}$. By Lemma3.6, we can know

$$f(x) = \sum_{\gamma\in\Gamma_x} \xi'_\gamma f(e_\gamma), \quad f(y) = \sum_{\gamma\in\Gamma_y} \eta'_\gamma f(e_\gamma),$$
$$f(ax+by) = a\sum_{\gamma\in\Gamma_x} \xi''_\gamma f(e_\gamma) + b\sum_{\gamma\in\Gamma_y} \eta''_\gamma f(e_\gamma),$$

where $|\xi'_\gamma| = |\xi''_\gamma| = |\xi_\gamma|$ and $|\eta'_\gamma| = |\eta''_\gamma| = |\eta_\gamma|$ for any $\gamma \in \Gamma_x \cup \Gamma_y$. Since f is a min-phase-isometry,

$$\begin{aligned}
&(1-|a|)^p + |b|^p \\
=&\min\{(a+1)^p + |b|^p, (1-a)^p + |b|^p\} \\
=&\min\{\|(ax+by)+x\|^p, \|(ax+by)-x\|^p\} \\
=&\min\{\|f(ax+by)+f(x)\|^p, \|f(ax+by)-f(x)\|^p\} \\
=&\min\{\sum_{\gamma\in\Gamma_x} |a\xi''_\gamma + \xi'_\gamma|^p + |b|^p, \sum_{\gamma\in\Gamma_x} |a\xi''_\gamma - \xi'_\gamma|^p + |b|^p\}.
\end{aligned}$$

We can obtain

$$(1-|a|)^p = \sum_{\gamma\in\Gamma_x} |a\xi''_\gamma + \xi'_\gamma|^p,$$

or

$$(1-|a|)^p = \sum_{\gamma\in\Gamma_x} |a\xi''_\gamma - \xi'_\gamma|^p.$$

Then

$$\sum_{\gamma\in\Gamma_x} |a\xi''_\gamma \pm \xi'_\gamma|^p \geq \sum_{\gamma\in\Gamma_x} (|\xi'_\gamma| - |a\xi''_\gamma|)^p = (1-|a|)^p.$$

Due to strict convexity, it follows that $\xi''_\gamma + \xi'_\gamma = 0$ for all $\gamma \in \Gamma_x$, or $\xi''_\gamma - \xi'_\gamma = 0$ for all $\gamma \in \Gamma_x$. This implies that $\sum_{\gamma\in\Gamma_x} \xi''_\gamma f(e_\gamma) \in \{f(x), -f(x)\}$. In the same way,

$\sum_{\gamma\in\Gamma_y}\eta''{}_\gamma f(e_\gamma) \in \{f(y),-f(y)\}$. In conclusion, $f(ax+by) \in \{af(x)+bf(y), af(x)-bf(y), -af(x)+bf(y), -af(x)-bf(y)\}$ The proof is complete. □

Theorem 3.8. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$, $1<p<2$. Suppose that $f\colon \mathrm{S}_X \to \mathrm{S}_Y$ is a surjective min-phase-isometry, Then f is phase equivalent to an isometry.*

Proof. Fix $\gamma_0 \in \Gamma$, and let $Z := \{x \in X : x \perp e_{\gamma_0}\}, W := \{w \in Y : w \perp f(e_{\gamma_0})\}$. Then $\mathrm{S}_X = \{\frac{z+\lambda e_{\gamma_0}}{\|z+\lambda e_{\gamma_0}\|} : z \in \mathrm{S}_Z, \lambda \in \mathbb{R}\} \cup \{\pm e_{\gamma_0}\}$. For each $\lambda \in \mathbb{R}$, put $a_\lambda = \frac{1}{\|z+\lambda e_{\gamma_0}\|}, b_\lambda = \frac{\lambda}{\|z+\lambda e_{\gamma_0}\|}$. By Lemma 3.7, we can write

$$f(a_\lambda z+b_\lambda e_{\gamma_0}) = \alpha(z,\lambda)a_\lambda f(z)+\beta(z,\lambda)b_\lambda f(e_{\gamma_0}) \quad \alpha(z,\lambda),\beta(z,\lambda) \in \{-1,1\}$$

for any $z \in S_Z$.

Define a mapping $g : \mathrm{S}_X \to \mathrm{S}_Y$ as follows:

$$g(e_{\gamma_0}) = f(e_{\gamma_0}), \quad g(-e_{\gamma_0}) = -f(e_{\gamma_0}), \quad g(z) = \alpha(z,1)\beta(z,1)f(z)$$

$$g(a_\lambda z+b_\lambda e_{\gamma_0}) = \alpha(z,\lambda)\beta(z,\lambda)a_\lambda f(z)+b_\lambda f(e_{\gamma_0})$$

for all $z \in S_Z$ and $0 \neq \lambda \in \mathbb{R}$. Then g is a min-phase-isometry, which is phase equivalent to f. Since $f(S_Z) = \mathrm{S}_W$, by Lemma 3.6 we can know $g(S_Z) \subset \mathrm{S}_W$. Next, we will show that $g : \mathrm{S}_Z \to \mathrm{S}_W$ is a surjective isometry. Let $z \in S_Z$ and $0 \neq \lambda \in \mathbb{R}$. Take $a_1 = b_1 = \frac{1}{\|z+e_{\gamma_0}\|} = \frac{1}{2^p}$. Since g is a min-phase-isometry, we have

$$\begin{aligned}
&\min\{|a_1+a_\lambda|^p+|a_1+b_\lambda|^p, |a_1-a_\lambda|^p+|a_1-b_\lambda|^p\} \\
&= \min\{\|(a_1z+a_1e_{\gamma_0})+(a_\lambda z+b_\lambda e_{\gamma_0})\|^p, \|(a_1z+a_1e_{\gamma_0})-(a_\lambda z+b_\lambda e_{\gamma_0})\|^p\} \\
&= \min\{\|g(a_1z+a_1e_{\gamma_0})+g(a_\lambda z+b_\lambda e_{\gamma_0})\|^p, \|g(a_1z+a_1e_{\gamma_0})-g(a_\lambda z+b_\lambda e_{\gamma_0})\|^p\} \\
&= \min\{|a_1\alpha(z,1)\beta(z,1)+a_\lambda\alpha(z,\lambda)\beta(z,\lambda)|^p+|a_1+b_\lambda|^p, \\
&\qquad |a_1\alpha(z,1)\beta(z,1)-a_\lambda\alpha(z,\lambda)\beta(z,\lambda)|^p+|a_1-b_\lambda|^p\}.
\end{aligned}$$

If $\alpha(z,1)\beta(z,1) = -\alpha(z,\lambda)\beta(z,\lambda)$, Then we deduce that

$$\min\{|a_1-a_\lambda|^p+|a_1+b_\lambda|^p, |a_1+a_\lambda|^p+|a_1-b_\lambda|^p\}.$$

But

$$\min\{|a_1-a_\lambda|^p+|a_1+b_\lambda|^p,|a_1+a_\lambda|^p+|a_1-b_\lambda|^p\}$$
$$\neq\min\{|a_1+a_\lambda|^p+|a_1+b_\lambda|^p,|a_1-a_\lambda|^p+|a_1-b_\lambda|^p\}.$$

It is contradiction. So we can obtain $\alpha(z,1)\beta(z,1)=\alpha(z,\lambda)\beta(z,\lambda)$, and

$$g(a_\lambda z+b_\lambda e_{\gamma_0})=a_\lambda g(z)+b_\lambda g(e_{\gamma_0}) \tag{3.1}$$

for all $z\in S_Z$ and $\lambda\in\mathbb{R}$. Let $z_1,z_2\in S_Z$ and $2\lambda>\|z_1+z_2\|$. By (3.1), we can obtain

$$\begin{aligned}
&\frac{1}{1+\lambda^p}\|g(z_1)-g(z_2)\|^p\\
=&\frac{1}{1+\lambda^p}\min\{\|g(z_1)+g(z_2)\|^p+(2\lambda)^p,\|g(z_1)-g(z_2)\|^p\}\\
=&\min\{\|g(\frac{z_1+\lambda e_{\gamma_0}}{\|z_1+\lambda e_{\gamma_0}\|})+g(\frac{z_2+\lambda e_{\gamma_0}}{\|z_2+\lambda e_{\gamma_0}\|})\|^p,\|g(\frac{z_1+\lambda e_{\gamma_0}}{\|z_1+\lambda e_{\gamma_0}\|})-g(\frac{z_2+\lambda e_{\gamma_0}}{\|z_2+\lambda e_{\gamma_0}\|})\|^p\}\\
=&\min\{\|\frac{z_1+\lambda e_{\gamma_0}}{\|z_1+\lambda e_{\gamma_0}\|}+\frac{z_2+\lambda e_{\gamma_0}}{\|z_2+\lambda e_{\gamma_0}\|}\|^p,\|\frac{z_1+\lambda e_{\gamma_0}}{\|z_1+\lambda e_{\gamma_0}\|}-\frac{z_2+\lambda e_{\gamma_0}}{\|z_2+\lambda e_{\gamma_0}\|}\|^p\}\\
=&\frac{1}{1+\lambda^p}\min\{\|z_1+z_2\|^p+(2\lambda)^p,\|z_1-z_2\|^p\}\\
=&\frac{1}{1+\lambda^p}\|z_1-z_2\|^p
\end{aligned}$$

The implies that $\|g(z_1)-g(z_2)\|^p=\|z_1-z_2\|^p$ for any $z_1,z_2\in S_Z$. On the other hand, we know $a_1=b_1=\frac{1}{\|z+e_{\gamma_0}\|}=\frac{1}{2^p}$, so

$$\begin{aligned}
&\frac{1}{2}\|g(z)+g(-z)\|^p\\
=&\frac{1}{2}\min\{\|g(z)+g(-z)\|^p,\|g(z)-g(-z)\|^p+2^p\}\\
=&\min\{\|g(a_1z+a_1e_{\gamma_0})+g(-a_1z-a_1e_{\gamma_0})\|^p,\|g(a_1z+a_1e_{\gamma_0})+g(-a_1z-a_1e_{\gamma_0})\|^p\}\\
=&\min\{\|a_1z+a_1e_{\gamma_0}+(-a_1z-a_1e_{\gamma_0})\|^p,\|a_1z+a_1e_{\gamma_0}+(-a_1z-a_1e_{\gamma_0})\|^p\}\\
=&\frac{1}{2}\{0,2^p\}\\
=&0
\end{aligned}$$

for any $z \in S_Z$. It is implies $g(-z) = -g(z)$ for any $z \in S_Z$. Since g is phase equivalent to f, we see that $g : \mathrm{S}_Z \to \mathrm{S}_W$ is a surjective isometry. In conclusion, $g : \mathrm{S}_X \to \mathrm{S}_W$ is a isometry. The proof is complete. □

Theorem 3.9. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$, $1 < p < 2$. Suppose that $f\colon \mathrm{S}_X \to \mathrm{S}_Y$ is a min-phase-isometry,Then f is a phase-isometry.*

Corollary 3.10. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$, $1 < p < 2$. Suppose that $f\colon \mathrm{S}_X \to \mathrm{S}_Y$ is a min-phase-isometry, Its positive homogenous extension is a phase-isometry which is phase equivalent to a linear isometry.*

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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