Available online at http://scik.org
Adv. Fixed Point Theory, 2024, 14:19
https://doi.org/10.28919/afpt/8547
ISSN: 1927-6303

# APPROXIMATION OF COMMON FIXED POINTS OF A SEQUENCE OF COMMUTING GENERALIZED NONEXPANSIVE MAPPINGS

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**Abstract.** It is the purpose of this paper to discuss the generalizations of nonexpansive mappings. We prove additional properties of these generalizations, particularly focusing on a sequence of commuting mappings satisfying condition $B_{\gamma,\mu}$. We propose iterative algorithms to approximate a common fixed point of a sequence of commuting mappings satisfying condition $B_{\gamma,\mu}$. With some mild assumptions on parameters, the convergence of these algorithms to a common fixed point of a sequence of commuting mappings satisfying condition $B_{\gamma,\mu}$ is also proved. Our results extend and improve many recent results in the literature.

**Keywords:** nonexpansive mapping; condition $B_{\gamma,\mu}$; commuting mappings; fixed point.

**2020 AMS Subject Classification:** 47H05, 47H10, 47J25, 49J40, 54H25, 91B99.

## 1. Introduction

Fixed point theory for nonexpansive mappings is flourished as main area of study after the appearance of four existence theorems in 1965 by Browder[5, 6], Göhde [13] and Kirk [16]. Many important results have been discovered which are related to existence of fixed points for nonexpansive mappings as well as to the structure of the fixed points set and to techniques for

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Received March 16, 2024

approximating fixed points. Many mathematicians are attracted to the field to focus their research on the extension of it because it is a generalization of the celebrated Banach contraction principle[3] with its wide range of applications that pioneered the field of metric fixed point theory. Thousands of citations of late Professor William Arthur Kirk's 1965 article [16] indicates without doubt the wide range of applications and importance of the generalized mapping.

In 1972, Goebel and Kirk[12] introduced another generalization of nonexpansive mappings called asymptotically nonexpansive mappings and proved existence and approximation of fixed points of such self-mappings in Banach spaces. Several authors (see Agarwal et al.[1], Betiuk-Pilarska and Benavides [4], Chidume [7], Dhompongsa et al. [9], Garcia-Falset et al.[10], Goebel and Kirk [11], Khamsi and Khan [14], Lael and Heidarpoor [17], Mishra[18], Mishra et al. [20], Mishra et al. [19], Pant and Shukla [23], Patir et al [21, 22], Sangago[24], Sangago[25], Suzuki [28, 29, 30], Ullah et al. [32] and the references therein) have contributed immensely in this field.

Because of its important linkages with the theory of monotone and accretive operators, fixed point theory for nonexpansive mappings has long been considered as a fundamental part of nonlinear functional analysis. Different new classes of generalized nonexpansive mappings with interesting properties have been developed in this context. Researching of the practical significance of the metric fixed point approach in solving problems of applied sciences such us signal processing, inverse problems, equilibrium problems, game theory in market economy, optimization and so on come to the center stage in recent decades.

In this paper we analyze and generalize some of recent results in the generalization of nonexpansive mappings with particular attention to the mappings introduced by Suzuki [28, 30], and further investigated by Patir et al.[22] and Thakur et al.[31].

Throughout this article, $\mathbb{N}$ and $\mathbb{R}$ stand for the set of natural numbers and the set of all real numbers, respectively. For a sequence $\{x_n\}$ of a normed space $E$ and a point $x$ in $E$, the strong convergence of $\{x_n\}$ to $x$ is denoted by $x_n \longrightarrow x$ and the weak convergence of $\{x_n\}$ to $x$ is denoted by $x_n \rightharpoonup x$.

First we present some basic concepts. Let $X$ be a nonempty set and $G : X \to X$ be a mapping. We say that a point $x \in X$ is said to be a fixed point of $G$ when $Gx = x$. $Fix(G)$ denotes the set of all fixed points of $G$; that is,

$$Fix(G) = \{x \in X : Gx = x\}.$$

**Definition 1.1** ([8, 15])**.** *Let $E$ be a real Banach space. Let $K$ be a nonempty subset of $E$ and $G : K \to K$. We say $G$ **Lipschitz continuous** if there exists a real constant $\lambda \geq 0$ such that*

$$\|Gx - Gy\| \leq \lambda \|x - y\| \text{ for all } x, y \in K. \tag{1.1}$$

*When $\lambda \in [0,1)$ in (1.1), $G$ is called a **contraction** mapping. If $\lambda \leq 1$ in (1.1), $G$ said to be a **nonexpansive** mapping; that is, $G$ satisfies the inequality*

$$\|Gx - Gy\| \leq \|x - y\|, \text{ for all } x, y \in K. \tag{1.2}$$

In 1972 Goebel and Kirk [12] introduced a generalization of nonexpansive mappings as stated in the following definition.

**Definition 1.2** ([12])**.** *Let $E$ be a real Banach space. Let $K$ be a nonempty subset of $E$ and $G : K \to K$. $G$ is said to be an **asymptotically nonexpansive** mapping if there exists a sequence $\{r_n\}$ in $[1, \infty)$ such that $\lim_{n\to\infty} r_n = 1$ and*

$$\|G^n x - G^n y\| \leq r_n \|x - y\|, \text{ for all } x, y \in K. \tag{1.3}$$

Also in 1972 Dotson [8] introduced a generalization of nonexpansive mappings as stated below.

**Definition 1.3** ([8])**.** *Let $E$ be a real Banach space. Let $K$ be a nonempty subset of $E$ and $G : K \to K$. $G$ is said to be a **quasi-nonexpansive** mapping if $Fix(G) \neq \emptyset$, and*

$$\|Gx - z\| \leq \|x - z\| \text{ for all } z \in Fix(G),\ x \in K. \tag{1.4}$$

It follows that a nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive. Taking $r_n = 1$ for each $n$, we see that a nonexpansive mapping is also asymptotically nonexpansive. Existence and approximation of fixed points of these mappings were also proved (see [8], [2], [12], [16], and the references therein).

Recently new classes of generalized nonexpansive mappings were introduced by Suzuki [30] in 2008, Garcia-Falset et al. [10] in 2011 and Patir et al. [22] in 2018 as stated in the following definitions and proved fixed point theorems for their generalizations.

**Definition 1.4** ([30])**.** *Let $K$ be a nonempty subset of the Banach space $E$ and $G : K \longrightarrow K$. Then $G$ is said to satisfy condition (C) if for all $x, y \in K$*

$$\frac{1}{2}\|x - Gx\| \leq \|x - y\| \;\rightarrow\; \|Gx - Gy\| \leq \|x - y\|. \tag{1.5}$$

Suzuki (Propostion 1 and Proposition 2 of [30]) proved that the condition (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness.

**Definition 1.5** ([10])**.** *Let $K$ be a nonempty subset of the Banach space $E$ and $G : K \longrightarrow K$. Then $G$ is said to satisfy condition $(C_\lambda)$, where $\lambda \in (0,1)$, if for all $x, y \in K$*

$$\lambda\|x - Gx\| \leq \|x - y\| \;\rightarrow\; \|Gx - Gy\| \leq \|x - y\|. \tag{1.6}$$

It was shown by Garcia-Falset et al. [10] that Condition (C) is a particular case of Condition $(C_\lambda)$ with $\lambda = \frac{1}{2}$. Hence a nonexpansive self-mapping satisfies the condition $(C_\lambda)$ for each $\lambda \in (0,1)$.

**Definition 1.6** ([22])**.** *Let $K$ be a nonempty subset of the Banach space $E$ and $G : K \longrightarrow K$. Then $G$ is said to satisfy condition $B_{\gamma,\mu}$ if there exists $\gamma \in [0,1]$, $\mu \in [0, \frac{1}{2}]$ with $2\mu \leq \gamma$ such that for all $x, y \in K$*

$$\gamma\|x - Gx\| \leq \|x - y\| + \mu\|y - Gy\| \;\rightarrow\; \|Gx - Gy\| \leq (1-\gamma)\,\|x - y\| + \mu(\|x - Gy\| + \|y - Gx\|). \tag{1.7}$$

Patir et al. [22] constructed examples to justify that their generalization was more general than that of Condition (C) and Condition $(C_\lambda)$. Both authors justified that the inclusions were strict. In case $Fix(T) \neq \emptyset$, each of the conditions (C), $(C_\lambda)$ and $B_{\gamma,\mu}$ implies quazi-nonexpansiveness of self-mapping $G$.

Suzuki [28] stated and proved the following characterization for two commuting nonexpansive mappings.

**Proposition 1.7** ([28]). *Let $K$ be a closed convex subset of the Banach space $E$. Let $G_1, G_2 : K \longrightarrow K$ be commuting nonexpansive mappings (i.e., $G_1 \circ G_2 = G_2 \circ G_1$). Let $\{x_n\}$ be a sequence in $K$ that converges strongly to some $z \in K$. If $\{\alpha_n\}$ is a sequence in $(0, \frac{1}{2})$ converging to $0$ such that*

$$\lim_{n\to\infty} \frac{\|(1-\alpha_n)G_1x_n + \alpha_n G_2 x_n - x_n\|}{\alpha_n} = 0, \tag{1.8}$$

*then $z$ is a common fixed point of $G_1$ and $G_2$.*

In the same article Suzuki [28] extended Proposition 1.7 systematically from two to three and then for a finite family of commuting nonexpansive mappings; and then proved the following fixed point theorem.

**Theorem 1.8** ([28]). *Let $K$ be a compact convex subset of a Banach space $E$. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on $K$. Fix $\lambda \in (0,1)$. Let $\{\alpha_n\}$ be a sequence in $[0, \frac{1}{2}]$ satisfying*

$$\liminf_{n\to\infty} \alpha_n = 0,\ \limsup_{n\to\infty} \alpha_n > 0,\ \lim_{n\to\infty}[\alpha_{n+1} - \alpha_n] = 0.$$

*Define a sequence $\{x_n\}$ in $K$ and*

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \lambda \left(1 - \sum_{k=1}^{n-1} \alpha_n^k \right) T_1 x_n + \lambda \sum_{k=2}^{n} \alpha_n^{k-1} T_k x_n + (1-\lambda)x_n,\ n = 1, 2, ... \end{cases}$$

*Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n : n \in \mathbb{N}\}$.*

**Theorem 1.9** ([22]). *Let $K$ be a nonempty subset of the Banach space $E$. Let $G$ be a self-mapping and satisfies the condition $B_{\gamma,\mu}$ on $K$. For $x_0 \in K$, let a sequence $\{x_n\}$ in $K$ be defined by;*

$$x_{n+1} = \lambda G x_n + (1-\lambda)x_n, \tag{1.9}$$

*where $\lambda \in [\gamma, 1) - \{0\}$and $n \in \mathbb{N} \cup \{0\}$. Then $||Gx_n - x_n|| \to 0$ as $n \to \infty$.*

Motivated by the above results we continue to develop more characterizations of generalizations of nonexpansive mappings. By the help of the new characterizations we prove fixed point theorems for generalized nonexpansive mappings. The main source of inspiration for this article are the works of Suzuki [28, 30], Patir et al.[22] and Thakur et al.[31].

## 2. PRELIMINARIES

We collect here basic concepts and technical lemmas from literature that can be used in the proof of our main results.

Let $K$ be a nonempty closed convex subset of a Banach space $E$ and let $G: K \to K$. A sequence $\{x_n\}$ in $K$ said to be almost fixed point sequence for $G$ if

$$\lim_{n\to\infty} \|Gx_n - x_n\| = 0.$$

**Definition 2.1** ([15]). *A Banach space $E$ is said to be uniformly convex, if for every $\varepsilon, 0 < \varepsilon \leq 2$ there exists a $\delta = \delta(\varepsilon) > 0$ such that*

$$x, y \in B,\ \|x\| \leq 1,\ \|y\| \leq 1,\ \&\ \|x-y\| \geq \varepsilon \text{ implies } \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

One of the main characterization of uniformly convex spaces is the following.

**Lemma 2.2** ([27]). *Let $E$ be a uniformly convex Banach space. Assume that $0 < b \leq t_n \leq c < 1$, $n = 1, 2, 3, \cdots$. Let the sequences $\{x_n\}$ and $\{y_n\}$ in $E$ be such that*

$$\limsup_{n\to\infty} \|x_n\| \leq \nu,\ \limsup_{n\to\infty} \|y_n\| \leq \nu, \text{ and } \lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = \nu, \text{where } \nu \geq 0.$$

*Then*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

**Lemma 2.3** ([1]). *Let $E$ be a uniformly convex Banach space and $K$ be a nonempty closed convex subset of $E$. Let $G: K \to E$ a mapping satisfying the condition $B_{\gamma,\mu}$ on $K$ with $2\mu \leq \gamma,\ \gamma \in [0,1]$ and $\mu \in [0, \frac{1}{2}]$. Then, for any $\varepsilon > 0$, there exists positive number $M(\varepsilon) > 0$ such that $\|x - Gx\| < \varepsilon$ for all $x \in co(\{x_0, x_1\})$, where $x_0, x_1 \in K$ with $\|x_0 - Gx_0\| \leq M(\varepsilon)$ and $\|x_1 - Gx_1\| \leq M(\varepsilon)$.*

**Lemma 2.4.** [1] *Let $E$ be a uniformly convex Banach space and $K$ be a nonempty closed convex bounded subset of $E$. Let $G: K \to E$ a mapping satisfying the condition $B_{\gamma,\mu}$ on $K$ with $2\mu \leq \gamma,\ \gamma \in [0,1]$ and $\mu \in [0, \frac{1}{2}]$. Then, $I - G$ is demiclosed on $K$.*

The following properties of a mapping that satisfies condition $B_{\gamma,\mu}$ were proved in 2018.

**Lemma 2.5** ([22]). *Let $K$ be a nonempty subset of the Banach space $E$. Let $G : K \to K$ satisfy the condition $B_{\gamma,\mu}$ on $K$. Then, for all $x, y \in K$ and for $\theta \in [0,1]$,*

*(i)* $\left\|Gx - G^2x\right\| \leq \|x - Gx\|$,

*(ii) at least one of the following ((a) and (b)) holds:*

*(a)* $\frac{\theta}{2}\|x - Gx\| \leq \|x - y\|$,

*(b)* $\frac{\theta}{2}\left\|Gx - G^2x\right\| \leq \|Gx - y\|$.

*The condition (a) implies*

$$\|Gx - Gy\| \leq (1 - \frac{\theta}{2})\|x - y\| + \mu(\|x - Gy\| + \|y - Gx\|)$$

*and the condition (b) implies*

$$\left\|G^2x - Gy\right\| \leq (1 - \frac{\theta}{2})\|Gx - y\| + \mu(\|Gx - Gy\| + \left\|y - G^2x\right\|).$$

*(iii)*
$$\begin{aligned}\|x - Gy\| \leq& (3 - \theta)\|x - Gx\| + (1 - \frac{\theta}{2})\|x - y\| \\ &+ \mu(2\|x - Gx\| + \|x - Gy\| + \|y - Gx\| + 2||Gx - G^2x||).\end{aligned}$$

**Lemma 2.6** ([22]). *For a nonempty subset $K$ of a Banach space $E$, let $G : K \to E$ be a mapping satisfying $B_{\gamma,\mu}$ condition. If $p$ is a fixed point of $G$ on $K$, then for all $x \in K$,*

$$\|Gx - p\| \leq \|x - p\|.$$

## 3. MOTIVATION OF THE PROBLEM

Recently in 2023 the authors [Sangago et al. [26]] stated and proved the weakest form of Proposition 1.7 for two commuting self-mappings satisfying the condition $B_{\gamma,\mu}$. The proof of the result is also included for sake of completeness because the article has been under review.

**Lemma 3.1** ([26]). *Let $K$ be a closed convex subset of the Banach space $E$. Let $G_1, G_2 : K \longrightarrow K$ be mappings satisfying the condition $B_{\gamma,\mu}$, where $2\mu < \gamma$, with $G_1 \circ G_2 = G_2 \circ G_1$ on $K$. Let $\{x_n\}$ be a sequence in $K$ that converges strongly to some $z \in K$. If $\{\alpha_n\}$ is a sequence in $(0, \frac{1}{2})$ converging to $0$ such that*

$$\lim_{n\to\infty} \|(1 - \alpha_n)G_1x_n + \alpha_nG_2x_n - x_n\| = 0, \tag{3.1}$$

*then $z$ is a common fixed point of $G_1$ and $G_2$.*

*Proof.* For $n \in \mathbb{N}$ it follows from (i) and (iii) of Lemma 2.5 that

$$\begin{aligned} \|z - G_1x_n\| &\leq (3-\gamma)\|z - G_1z\| + \frac{2-\gamma}{2}\|z - x_n\| \\ &\quad + \mu\left(2\|z - G_1z\| + \|z - G_1x_n\| + \|x_n - G_1z\| + 2\left\|G_1z - G_1^2z\right\|\right) \\ &\leq (3-\gamma+4\mu)\|z - G_1z\| + (1-\gamma)\|z - x_n\| + \mu\|z - G_1x_n\| + \mu\|x_n - G_1z\|. \end{aligned} \tag{3.2}$$

It follows from (3.2) that

$$\|z - G_1x_n\| \leq \frac{3-\gamma+4\mu}{1-\mu}\|z - G_1z\| + \frac{1-\gamma}{1-\mu}\|z - x_n\| + \frac{\mu}{1-\mu}\|x_n - G_1z\| \tag{3.3}$$

Because $\{x_n\}$ is a bounded sequence, it follows from (3.3) that $\{G_1x_n\}$ is a bounded sequence. By similar argument we conclude that $\{G_2x_n\}$ is also a bounded sequence.

For each $n \in \mathbb{N}$ it follows from triangle inequality that

$$\|(1-\alpha_n)G_1x_n + \alpha_nG_2x_n - x_n\| \geq (1-\alpha_n)\|G_1x_n - x_n\| - \alpha_n\|x_n - G_2x_n\|. \tag{3.4}$$

Thus for each $n \in \mathbb{N}$, we obtain from (3.4) that

$$\|G_1x_n - x_n\| \leq \frac{1}{1-\alpha_n}\|(1-\alpha_n)G_1x_n + \alpha_nG_2x_n - x_n\| + \frac{\alpha_n}{1-\alpha_n}\|x_n - G_2x_n\|. \tag{3.5}$$

Using (3.1), the assumption $a_n \to 0$ as $n \to \infty$, and boundedness of the sequence $\{\|x_n - G_2x_n\|\}$, it follows from (3.5) that

$$\lim_{n\to\infty}\|G_1x_n - x_n\| = 0. \tag{3.6}$$

To show that $z$ is a fixed point of $G_1$ we utilize (ii) of Lemma 2.5.

**Case 1**. There exists a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that

$$\frac{\gamma}{2}\|x_{n_k} - G_1x_{n_k}\| \leq \|x_{n_k} - z\| \ \forall k \in \mathbb{N}. \tag{3.7}$$

It follows from Lemma 2.5 (ii)(a) and (3.7) that

$$\begin{aligned} \|G_1z - G_1x_{n_k}\| &\leq (1-\frac{\gamma}{2})\|z - x_{n_k}\| + \mu(\|z - G_1x_{n_k}\| + \|x_{n_k} - G_1z\|) \\ &\leq (1-\frac{\gamma}{2}+\mu)\|z - x_{n_k}\| + 2\mu\|x_{n_k} - G_1x_{n_k}\| + \mu\|G_1x_{n_k} - G_1z\|. \end{aligned} \tag{3.8}$$

We get from (3.8) that

$$\|G_1z-G_1x_{n_k}\| \le \frac{(1-\frac{\gamma}{2}+\mu)}{1-\mu}\|z-x_{n_k}\| + \frac{2\mu}{1-\mu}\|x_{n_k}-G_1x_{n_k}\|. \tag{3.9}$$

It follows from (3.6), (3.9) and convergence of $\{x_n\}$ to $z$ that

$$\lim_{k\to\infty}\|G_1x_{n_k}-G_1z\| = 0. \tag{3.10}$$

For each $k \in \mathbb{N}$ we have

$$\|z-G_1z\| \le \|z-x_{n_k}\| + \|x_{n_k}-G_1x_{n_k}\| + \|G_1x_{n_k}-G_1z\|. \tag{3.11}$$

Using (3.6) and convergence of $\{x_n\}$ to $z$, and letting $k\to\infty$ in (3.11), we get

$$G_1z = z. \tag{3.12}$$

Therefore, $z$ is a fixed point of $G_1$.

**Case 2**. There exists a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that

$$\frac{\gamma}{2}\left\|G_1x_{n_k}-G_1^2x_{n_k}\right\| \le \|G_1x_{n_k}-z\| \;\; \forall k\in\mathbb{N}. \tag{3.13}$$

It follows from Lemma 2.5[(i) & (ii)(b)] and (3.13) that

(3.14)

$$\begin{aligned}\left\|G_1z-G_1^2x_{n_k}\right\| &\le (1-\frac{\gamma}{2})\|z-G_1x_{n_k}\| + \mu\left(\|G_1z-G_1x_{n_k}\| + \left\|z-G_1^2x_{n_k}\right\|\right)\\ &\le (1-\frac{\gamma}{2}+3\mu)\|x_{n_k}-G_1x_{n_k}\| + (1-\frac{\gamma}{2}+\mu)\|x_{n_k}-z\| + \mu\left\|G_1^2x_{n_k}-G_1z\right\|.\end{aligned}$$

We get from (3.14) that

$$\left\|G_1z-G_1^2x_{n_k}\right\| \le \frac{(1-\frac{\gamma}{2}+\mu)}{1-\mu}\|x_{n_k}-G_1x_{n_k}\| + \frac{(1-\frac{\gamma}{2}+\mu)}{1-\mu}\|x_{n_k}-z\|. \tag{3.15}$$

It follows from (3.6), (3.15) and convergence of $\{x_n\}$ to $z$ that

$$\lim_{k\to\infty}\left\|G_1^2x_{n_k}-G_1z\right\| = 0. \tag{3.16}$$

For each $k\in\mathbb{N}$, it follows from repeated application of triangle inequality and Lemma 2.5(i) that

$$\begin{aligned}\|z-G_1z\| &\le \|z-x_{n_k}\| + \|x_{n_k}-G_1x_{n_k}\| + \left\|G_1x_{n_k}-G_1^2x_{n_k}\right\| + \left\|G_1^2x_{n_k}-G_1z\right\|\\ &\le \|z-x_{n_k}\| + 2\|x_{n_k}-G_1x_{n_k}\| + \left\|G_1^2x_{n_k}-G_1z\right\|.\end{aligned} \tag{3.17}$$

Using (3.6), (3.16), and letting $k \to \infty$ in (3.17), we get

$$G_1 z = z.$$

Therefore, $z$ is a fixed point of $G_1$.

We note that

$$(G_1 \circ G_2)z = (G_2 \circ G_1)z = G_2 z. \tag{3.18}$$

For each $n \in \mathbb{N}$ we have

$$\begin{aligned} \|G_2 z - x_n\| &\leq \|G_2 z - (1-\alpha_n)G_1 x_n - \alpha_n G_2)x_n\| + \|(1-\alpha_n)G_1 x_n + \alpha_n G_2)x_n - x_n\| \\ &\leq (1-\alpha_n)\|G_2 z - G_1 x_n\| + \alpha_n \|G_2 z - G_2 x_n\| + \|(1-\alpha_n)G_1 x_n + \alpha_n G_2)x_n - x_n\|. \end{aligned} \tag{3.19}$$

Because $\gamma\|G_2 z - G_1(G_2 z)\| = 0 \leq \|G_2 z - x_n\| + \mu \|x_n - G_1 x_n\|$, it follows from $B_{\gamma,\mu}$ condition that

$$\begin{aligned} \|G_2 z - G_1 x_n\| &= \|G_1(G_2 z) - G_1 x_n\| \\ &\leq (1-\gamma)\|G_2 z - x_n\| + \mu\left[\|G_2 z - G_1 x_n\| + \|x_n - G_2 z\|\right] \\ &\leq (1-\gamma+\mu)\|G_2 z - x_n\| + \mu\|G_2 z - G_1 x_n\|. \end{aligned} \tag{3.20}$$

It follows from (3.20) that

$$\|G_2 z - G_1 x_n\| \leq \left(\frac{1-\gamma+\mu}{1-\mu}\right)\|G_2 z - x_n\|. \tag{3.21}$$

We get from (3.19) and (3.21) that

$$\left[1-(1-\alpha_n)\left(\frac{1-\gamma+\mu}{1-\mu}\right)\right]\|G_2 z - x_n\| \leq \alpha_n \|G_2 z - G_2 x_n\| + \|(1-\alpha_n)G_1 x_n + \alpha_n G_2)x_n - x_n\|. \tag{3.22}$$

Since $\|G_2 z - G_2 x_n\|$ is a bounded sequence, letting $n \to \infty$ in (3.22) we get

$$\left(\frac{\gamma-2\mu}{1-\mu}\right)\|G_2 z - z\| \leq 0. \tag{3.23}$$

Because $\dfrac{\gamma-2\mu}{1-\mu} > 0$, we have $G_2 z = z$. Hence $z$ is a common fixed point of $G_1$ and $G_2$. □

Let $K$ be a nonempty closed convex subset of a given Banach space $E$. Let $\mu \in [0, \frac{1}{2}]$ and $\gamma \in [0,1]$ such that $2\mu \leq \gamma$. Let $G_1, G_2 : K \to K$ be commutating mappings (that is; $G_1 \circ G_2 = G_2 \circ G_1,$) satisfying the condition $B_{\gamma,\mu}$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $\left(0, \frac{1}{2}\right)$ and $\lambda \in (\gamma, 1)$. Let us define a sequence $\{x_n\}$ in $K$ by the iteration

$$\begin{cases} x_0 \in K \\ y_n = (1-\alpha_n)G_1x_n + \alpha_n G_2 x_n \\ x_{n+1} = \lambda y_n + (1-\lambda)x_n, \ n = 0,1,2,\cdots \end{cases} \tag{3.24}$$

**Lemma 3.2.** *If $F = Fix(G_1) \cap Fix(G_2) \neq \emptyset$, then the sequence $\{x_n\}$ defined in (3.24) is bounded.*

*Proof.* Let $p \in F$. Then it follows from Lemma 2.6 that for each $x \in K$

$$\|G_1x - p\| \leq \|x - p\| \ \& \ \|G_2x - p\| \leq \|x - p\|.$$

Thus for each $n = 0,1,2,\cdots$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \lambda \|y_n - p\| + (1-\lambda)\|x_n - p\| \\ &\leq \lambda [(1-\alpha_n)\|G_1x_n - p\| + \alpha_n \|G_2x_n - p\|] + (1-\lambda)\|x_n - p\| \\ &\leq \lambda [(1-\alpha_n)\|x_n - p\| + \alpha_n \|x_n - p\|] + (1-\lambda)\|x_n - p\| \\ &\leq \lambda \|x_n - p\| + (1-\lambda)\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

Therefore, $\{\|x_n - p\|\}$ is a decreasing sequence, and so that $\{x_n\}$ is a bounded sequence. This completes the proof. □

Motivated and inspired by the above results we further investigate and generalize Lemma 3.1 to countably many family of mappings satisfying the condition $B_{\gamma,\mu}$. We also prove the convergence of some iterative algorithms to common fixed points of such mappings with mild assumptions on the parameters.

**Methodology**: Well developed analytic as well as fixed point theoretical methods to prove our results are implemented. Mainly the key existing methods in the literature to prove our results are taken from [22, 26, 27, 28] and references therein.

## 4. MAIN RESULTS

It is mainly the purpose of this section to state and prove the generalizations we make on the main results of Suzuki [28] for countably many commuting mappings satisfying $B_{\gamma,\mu}$. Moreover, we prove the convergence of an iterative algorithm to a common fixed point of these mappings.

### 4.1. Proofs of Technical Lemmas.

Let us start with three mappings satisfying the condition $B_{\gamma,\mu}$.

**Lemma 4.1.** *Let $K$ be a nonempty closed convex subset of a Banach space $E$. Let $G_1, G_2$ and $G_3$ be commuting mappings satisfying the condition $B_{\gamma,\mu}$, where $2\mu < \gamma$, on $K$. Let $\{\alpha_n\}$ be a sequence in $(0, \frac{1}{2})$ converging to $0$, and let $\{x_n\}$ be a sequence in $K$ that converges strongly to some $z \in K$ and satisfies*

$$\lim_{n\to\infty} \left\|(1-\alpha_n-\alpha_n^2)G_1x_n + \alpha_n G_2 x_n + \alpha_n^2 G_3 x_n - x_n\right\| = 0. \tag{4.1}$$

*Then $z$ is a common fixed point of $G_1, G_2$ and $G_3$.*

*Proof.* For each $n \in \mathbb{N}$, it follows from (i) and (iii) of Lemma 2.5 that

$$\begin{aligned} \|z - G_1x_n\| &\le (3-\gamma)\|z - G_1z\| + \frac{2-\gamma}{2}\|z - x_n\| \\ &\quad + \mu\left(2\|z - G_1z\| + \|z - G_1x_n\| + \|x_n - G_1z\| + 2\left\|G_1z - G_1^2z\right\|\right) \\ &\le (3-\gamma+4\mu)\|z - G_1z\| + (1-\gamma)\|z - x_n\| + \mu\|z - G_1x_n\| + \mu\|x_n - G_1z\|. \end{aligned} \tag{4.2}$$

It follows from (4.2) that

$$\|z - G_1x_n\| \le \frac{3-\gamma+4\mu}{1-\mu}\|z - G_1z\| + \frac{1-\gamma}{1-\mu}\|z - x_n\| + \frac{\mu}{1-\mu}\|x_n - G_1z\| \tag{4.3}$$

Because $\{x_n\}$ is a bounded sequence, it follows from (4.3) that $\{G_1x_n\}$ is a bounded sequence. By similar argument we conclude that both sequences $\{G_2x_n\}$ and $\{G_3x_n\}$ are also bounded.

For each $n \in \mathbb{N}$, we get by applying the triangle inequality that

$$\|(1-\alpha_n)G_1x_n + \alpha_n G_2x_n - x_n\| \le \left\|(1-\alpha_n-\alpha_n^2)G_1x_n + \alpha_n G_2x_n + \alpha_n^2 G_3x_n - x_n\right\| \tag{4.4}$$

$$+\alpha_n^2 \left\| G_1x_n - G_3x_n \right\|.$$

It follows from (4.4), $\alpha_n \to 0$, and boundedness of the sequences $\{G_1x_n\}$, $\{G_2x_n\}$ and $\{G_3x_n\}$ that

$$\lim_{n\to\infty} \|(1-\alpha_n)G_1x_n + \alpha_n G_2x_n - x_n\| = 0. \tag{4.5}$$

Thus it follows from Lemma 3.1 that $z$ is a common fixed point of $G_1$ and $G_2$.

For each $n \in \mathbb{N}$, it follows from simple triangle inequality that

$$\|(1-\alpha_n)G_1x_n + \alpha_n G_3x_n - x_n\| \leq \left\|(1-\alpha_n-\alpha_n^2)G_1x_n + \alpha_n G_2x_n + \alpha_n^2 G_3x_n - x_n\right\| \tag{4.6}$$
$$+\alpha_n \left\|\alpha_n G_1x_n - G_2x_n + (1-\alpha_n)G_3x_n\right\|.$$

It follows from (4.6), $\alpha_n \to 0$, and boundedness of the sequences $\{G_1x_n\}$, $\{G_2x_n\}$ and $\{G_3x_n\}$ that

$$\lim_{n\to\infty} \|(1-\alpha_n)G_1x_n + \alpha_n G_3x_n - x_n\| = 0. \tag{4.7}$$

Thus it follows from (4.7) and Lemma 3.1 that $z$ is a common fixed point of $G_1$ and $G_3$. Hence

$$z \in Fix(G_1) \cap Fix(G_2) \cap Fix(G_3).$$

This completes the proof. □

Now we prove for finitely many commuting self mappings satisfying the condition $B_{\gamma,\mu}$ in the following proposition.

**Lemma 4.2.** *Let $K$ be a nonempty convex closed subset of a uniformly convex Banach space $E$. Let $G_1, G_2, \cdots, G_m$, $m \in \mathbb{N}$ be commuting self mappings satisfying the condition $B_{\gamma,\mu}$ on $K$ with $\mu \in [0, \frac{1}{2}], \gamma \in [0,1]$ such that $2\mu < \gamma$. Let $\{x_n\}$ be a sequence in $K$ converging strongly to some $z \in K$. If $\{\alpha_n\}$ is a sequence in $(0, \frac{1}{2})$ converging to $0$ such that*

$$\lim_{n\to\infty} \left\| \Big(1 - \sum_{i=1}^{m-1} \alpha_n^i\Big) G_1x_n + \sum_{i=2}^{m} \alpha_n^{i-1} G_ix_n - x_n \right\| = 0, \tag{4.8}$$

*then $z$ is a common fixed point of $G_1, G_2, \cdots, G_m$.*

*Proof.* It follows from the proof of Lemma 4.1 that the sequences $\{G_1x_n\},\ \{G_2x_n\},\cdots,\{G_mx_n\}$ are bounded. Let $j\in\{2,3,...,m\}$. Then For each $n\in\mathbb{N}$, it follows from triangle inequality that

$$\left\|(1-\alpha_n)G_1x_n+\alpha_nG_jx_n-x_n\right\|\leq\left\|\left(1-\sum_{i=1}^{m-1}\alpha_n^i\right)G_1x_n+\sum_{i=2}^{m}\alpha_n^{i-1}G_ix_n-x_n\right\| + \alpha_n\left\|\sum_{i=2}^{m-1}\alpha_n^{i-1}G_1x_n-\sum_{i=2}^{m}\alpha_n^{i-1}G_ix_n+G_jx_n\right\|. \tag{4.9}$$

It follows from (4.9), $\alpha_n\to 0$, and boundedness of the sequences $\{G_1x_n\},\ \{G_2x_n\},\cdots,\{G_mx_n\}$ that

$$\lim_{n\to\infty}\left\|(1-\alpha_n)G_1x_n+\alpha_nG_jx_n-x_n\right\|=0. \tag{4.10}$$

Thus it follows from (4.10) and Lemma 3.1 that $z$ is a common fixed point of $G_1$ and $G_j$. Therefore,

$$z\in\bigcap_{i=1}^{m}Fix(G_i).$$

This completes the proof. □

Now we prove for infinitely many commuting self mappings satisfying the condition $B_{\gamma,\mu}$ in the following Lemma.

**Lemma 4.3.** *Let $K$ be a nonempty bounded convex closed subset of a uniformly convex Banach space $E$. Let $\{G_m\}_{m=1}^{\infty}$ be a sequence of commuting self mappings satisfying the condition $B_{\gamma,\mu}$ on $K$ with $\mu\in[0,\frac{1}{2}],\gamma\in[0,1]$ such that $2\mu<\gamma$. Let $\{x_n\}$ be a sequence in $K$ converging strongly to some $z\in K$. If $\{\alpha_n\}$ is a sequence in $(0,\frac{1}{2})$ converging to $0$ such that*

$$\lim_{n\to\infty}\left\|\left(1-\sum_{m=1}^{\infty}\alpha_n^m\right)G_1x_n+\sum_{m=2}^{\infty}\alpha_n^mG_mx_n\ -x_n\right\|=0, \tag{4.11}$$

*then $z$ is a common fixed point of the sequence $\{G_m\}_{m=1}^{\infty}$.*

*Proof.* Put $\delta=\sup_{x,y\in K}\|x-y\|$. For each $k=2,3,4,\cdots$, it follows from simple triangle inequality that

$$\|(1-\alpha_n)G_1x_n+\alpha_nG_kx_n-x_n\|\leq\left\|\left(1-\sum_{m=1}^{\infty}\alpha_n^m\right)G_1x_n+\sum_{m=2}^{\infty}\alpha_n^mG_mx_n-x_n\right\|$$

$$
\begin{aligned}
&+\left\|\sum_{m=2}^{\infty}\alpha_n^m G_1x_n-\sum_{m=2}^{\infty}\alpha_n^m G_m x_n+\alpha_n G_k x_n\right\| \\
&\leq\left\|\left(1-\sum_{m=1}^{\infty}\alpha_n^m\right)G_1x_n+\sum_{m=2}^{\infty}\alpha_n^m G_m x_n-x_n\right\| \\
&+\sum_{m=2}^{\infty}\alpha_n^m\left\|G_1x_n-G_m x_n\right\|+\alpha_n\left\|G_k x_n\right\| \\
&\leq\left\|\left(1-\sum_{m=1}^{\infty}\alpha_n^m\right)G_1x_n+\sum_{m=2}^{\infty}\alpha_n^m G_m x_n-x_n\right\| \\
&+\alpha_n\delta\left(1+\sum_{m=1}^{\infty}\alpha_n^m\right) \\
&=\left\|\left(1-\sum_{m=1}^{\infty}\alpha_n^m\right)G_1x_n+\sum_{m=2}^{\infty}\alpha_n^m G_m x_n-x_n\right\|+\frac{\alpha_n}{1-\alpha_n}\delta \qquad (4.12)
\end{aligned}
$$

By letting $n\to\infty$ we get from (4.11) and (4.12) that

$$
\lim_{n\to\infty}\|(1-\alpha_n)G_1x_n+\alpha_n G_k x_n\ -x_n\|=0. \qquad (4.13)
$$

Thus it follows from (4.13) and Lemma 3.1 that $z$ is a common fixed point of $G_1$ and $G_k$. Therefore,

$$
z\in\bigcap_{k=1}^{\infty}Fix(G_k).
$$

This completes the proof. □

**Remark 4.4.** *The boundedness condition imposed on $K$ in Lemma 4.3 is equivalent to stating*

$$
\bigcap_{k=1}^{\infty}Fix(G_k)\neq\emptyset.
$$

## 4.2. Fixed Point Theorems.

By using the technical lemmas proved above we propose iterative algorithms and prove their convergence to a common fixed point of a given family of self mappings.

Let $K$ be a nonempty closed convex subset of a given Banach space $E$. Let $\mu\in[0,\frac{1}{2}]$ and $\gamma\in[0,1]$ such that $2\mu\leq\gamma$. Let $G_1,G_2$ and $G_3$ be commuting self mappings satisfying the condition $B_{\gamma,\mu}$ on $K$. Let $\{\alpha_n\}$ be a sequence in $(0,\frac{1}{2})$ and $\lambda\in(\gamma,1)$. Let us define a sequence

$\{x_n\}$ in $K$ by the iteration

$$\begin{cases} x_0 \in K \\ y_n = (1-\alpha_n-\alpha_n^2)G_1x_n + \alpha_n G_2 x_n + \alpha_n^2 G_3 x_n \\ x_{n+1} = \lambda y_n + (1-\lambda)x_n, n = 0,1,2,\cdots. \end{cases} \tag{4.14}$$

**Lemma 4.5.** *If $F = Fix(G_1) \cap Fix(G_2) \cap Fix(G_3) \neq \emptyset$, then the sequence $\{x_n\}$ defined by* (4.14) *is bounded.*

*Proof.* Let $p \in F$. Then it follows from Lemma 2.6 that for each $x \in K$,

$$\|G_i x - p\| \leq \|x - p\|, \ i = 1,2,3 \tag{4.15}$$

For each $n = 0,1,2,\cdots$ applying (4.15) yields

$$\begin{aligned} \|x_{n+1}-p\| &\leq \lambda \|y_n - p\| + (1-\lambda)\|x_n - p\| \\ &\leq \lambda \left[(1-\alpha_n-\alpha_n^2)\|G_1x_n - p\| + \alpha_n \|G_2x_n - p\| + \alpha_n^2 \|G_3x_n - p\|\right] + (1-\lambda)\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned}$$

Therefore, $\{\|x_n - p\|\}$ is a decreasing sequence, and hence $\{x_n\}$ is a bounded sequence. □

**Lemma 4.6.** *Let $E$ be a uniformly convex Banach space and $K$ be a non-empty closed convex subset of $E$. Let $G_1, G_2$ and $G_3$ be commuting self mappings satisfying the condition $B_{\gamma,\mu}$, where $\gamma \in [0,1], \mu \in [0,\frac{1}{2}]$ such that $2\mu < \gamma$ on $K$. Suppose that $F = \bigcap_{i=1}^{3} Fix(G_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,\frac{1}{2})$ converging to $0$ and $\lambda \in (\gamma,1)$. Then, for each $x_0 \in K$, the sequence $\{x_n\}$ defined by* (4.14) *satisfies*

$$\lim_{n\to\infty} \left\|(1-\alpha_n-\alpha_n^2)G_1x_n + \alpha_n G_2x_n + \alpha_n^2 G_3 x_n - x_n\right\| = 0.$$

*Proof.* Let $p \in F$. Put $w_n = y_n - p$ and $z_n = x_n - p$. It follows from Lemma 4.5 that for some $\nu \geq 0$,

$$\lim_{n\to\infty} \|z_n\| = \lim_{n\to\infty} \|x_n - p\| = \nu. \tag{4.16}$$

$$\limsup_{n\to\infty} \|w_n\| = \limsup_{n\to\infty} \|y_n - p\| \leq \nu. \tag{4.17}$$

$$\lim_{n\to\infty}\|(1-\lambda)w_n+\lambda z_n\| = \lim_{n\to\infty}\|x_{n+1}-p\| = \nu \tag{4.18}$$

It follows from (4.16), (4.17), (4.18) and Lemma 2.2 that

$$\lim_{n\to\infty}\|w_n - z_n\| = 0,$$

and so that

$$\lim_{n\to\infty}\left\|(1-\alpha_n-\alpha_n^2)G_1x_n+\alpha_nG_2x_n+\alpha_n^2G_3x_n-x_n\right\| = \lim_{n\to\infty}\|w_n-z_n\| = 0.$$

This completes the proof. □

For a nonempty compact convex subset $K$ of a uniformly convex Banach space $E$, we have the following fixed point result.

**Theorem 4.7.** *Let $K$ be a nonempty compact convex subset of a uniformly convex Banach space $E$. Let $G_1, G_2$ and $G_3$ be commuting self mappings satisfying the condition $B_{\gamma,\mu}$, where $\gamma \in [0,1], \mu \in [0,\frac{1}{2}]$ such that $2\mu < \gamma$ on $K$ and $F = \bigcap_{i=1}^{3} Fix(G_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,\frac{1}{2})$ converging to $0$ and $\lambda \in (\gamma, 1)$. For $x_0 \in K$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (4.14) converges strongly to a common fixed point of $G_1, G_2$ and $G_3$.*

*Proof.* It follows from Lemma 4.5 that the sequence $\{x_n\}$ is bounded. Since $K$ is compact, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that for some $p \in K$

$$\lim_{j\to\infty} x_{n_j} = p. \tag{4.19}$$

It follows from Lemma 4.6 that

$$\lim_{j\to\infty}\left\|(1-\alpha_{n_j}-\alpha_{n_j}^2)G_1x_{n_j}+\alpha_{n_j}G_2x_{n_j}+\alpha_{n_j}^2G_3x_{n_j}-x_{n_j}\right\| = 0. \tag{4.20}$$

It follows from Lemma 4.1 that $p$ is a common fixed point of $G_1, G_2$ and $G_3$; that is, $p \in F$. The proof of Lemma 4.5, (4.19) and (4.20) give us that

$$\lim_{n\to\infty}\|x_n - p\| = \lim_{j\to\infty}\left\|x_{n_j}-p\right\| = 0.$$

Therefore, $\{x_n\}$ converges strongly to a common fixed point of $G_1, G_2$ and $G_3$. □

Let $K$ be a non-empty closed convex subset of a given Banach space $E$. Let $\gamma \in [0,1], \mu \in [0,\frac{1}{2}]$ such that $2\mu < \gamma$. Let $G_1, G_2, \cdots, G_m$, $m \in \mathbb{N}$ be a finitely many commuting self mappings satisfying the condition $B_{\gamma,\mu}$ on $K$. Let $\{\alpha_n\}$ be a sequence in $(0,\frac{1}{2})$ and $\lambda \in (\gamma, 1)$. Define a sequence $\{x_n\}$ in $K$ by

$$\begin{cases} x_0 \in K \\ y_n = \Big(1 - \sum_{k=1}^{m-1} \alpha_n^k\Big) G_1 x_n + \sum_{k=2}^{m} \alpha_n^{k-1} G_k x_n \\ x_{n+1} = \lambda y_n + (1-\lambda) x_n, \; n = 0,1,2,\cdots. \end{cases} \tag{4.21}$$

**Lemma 4.8.** *If $F = \bigcap_{i=1}^{m} Fix(G_i) \neq \emptyset$, then the sequence $\{x_n\}$ defined by (4.21) is bounded.*

*Proof.* Let $p \in F$. Then it follows from Lemma 2.2 that for each $x \in K$,

$$\|G_i x_n - p\| \leq \|x_n - p\| \text{ for } i = 1,2,\cdots,m \tag{4.22}$$

By using (4.22) and repeated application of triangle inequality, for each $n = 0,1,2,\cdots$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \lambda \|y_n - p\| + (1-\lambda) \|x_n - p\| \\ &\leq \lambda \Big[\Big(1 - \sum_{k=1}^{m-1} \alpha_n^k\Big) \|G_1 x_n - p\| + \sum_{k=2}^{m} \alpha_n^{k-1} \|G_k x_n - p\|\Big] + (1-\lambda) \|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{4.23}$$

Therefore, $\{\|x_n - p\|\}$ is a decreasing sequence and so that $\{x_n\}$ is a bounded sequence. □

**Lemma 4.9.** *Let $E$ be a uniformly convex Banach space and $K$ a non-empty closed convex subset of $E$. Let $G_1, G_2, \cdots, G_m : K \longrightarrow K$ be commuting mappings satisfying the condition $B_{\gamma,\mu}$ on $K$, $\mu \in [0,\frac{1}{2}]$, $\gamma \in [0,1]$ and $2\mu < \gamma$. Suppose that $F = \bigcap_{i=1}^{m} Fix(G_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,\frac{1}{2})$ converging to $0$ and $\lambda \in (\gamma,1)$. Then, for each $x_0 \in K$, the sequence $\{x_n\}$ defined by (4.21) satisfies*

$$\lim_{n\to\infty} \left\| \left(1 - \sum_{k=1}^{m-1} \alpha_n^k\right) G_1 x_n + \sum_{k=2}^{m} \alpha_n^{k-1} G_k x_n - x_n \right\| = 0. \tag{4.24}$$

*Proof.* Let $p \in F$. Put $w_n = y_n - p$ and $z_n = x_n - p$. Then by Lemma 4.8

$$\lim_{n\to\infty} \|z_n\| = \lim_{n\to\infty} \|x_n - p\| = \nu \tag{4.25}$$

for some nonnegative real number $\nu$. We note that by Lemma 2.6

$$\limsup_{n\to\infty} \|w_n\| = \limsup_{n\to\infty} \|y_n - p\| \le \lim_{n\to\infty} \|x_n - p\| = \nu. \tag{4.26}$$

Moreover,

$$\lim_{n\to\infty} \|(1-\lambda)w_n + \lambda z_n\| = \lim_{n\to\infty} \|(1-\lambda)(y_n - p) + \lambda(x_n - p)\| = \lim_{n\to\infty} \|x_{n+1} - p\| = \nu. \tag{4.27}$$

It follows from (4.25), (4.26), (4.27) and Lemma 2.2 that

$$\lim_{n\to\infty} \|w_n - z_n\| = 0,$$

and so that (4.24) holds. □

Now we are in a position to state and prove a fixed point theorem.

**Theorem 4.10.** *Let $K$ be a nonempty compact convex subset of a uniformly convex Banach space $E$ and $m \in \mathbb{N}$. Let $G_1, G_2, \cdots, G_m$ be commuting self mappings on $K$ satisfying the condition $B_{\gamma,\mu}$, where $\gamma \in [0,1]$, $\mu \in [0, \frac{1}{2}]$ such that $2\mu < \gamma$. Assume that $F = \bigcap_{i=1}^{m} Fix(G_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, \frac{1}{2})$ converging to $0$ and $\lambda \in (\gamma, 1)$. Then for each $x_0 \in K$ the sequence $\{x_n\}_{n=0}^{\infty}$ in $K$ defined by (4.21) converges strongly to a point $p \in F$.*

*Proof.* It follows from Lemma 4.8 that the sequence $\{x_n\}$ is bounded. By the compactness of $K$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that for some $p \in K$

$$\lim_{j\to\infty} x_{n_j} = p. \tag{4.28}$$

By Lemma 4.9 we have

$$\lim_{j\to\infty} \left\| (1 - \sum_{k=1}^{m-1} \alpha_{n_j}^k) G_1 x_{n_j} + \sum_{k=2}^{m} \alpha_{n_j}^{k-1} G_k x_{n_j} - x_{n_j} \right\| = 0. \tag{4.29}$$

The hypotheses of Lemma 4.2 are all fulfilled as indicated in (4.28) and (4.29). Therefore, $p$ is a common fixed point of $G_1, G_2, \cdots, G_m$; that is, $p \in F$. The proof of Lemma 4.8 and (4.28) imply that

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{j\to\infty} \left\|x_{n_j} - p\right\| = 0. \tag{4.30}$$

By (4.30) the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $p \in F$. This completes the proof. □

Let us discuss the infinite cases. Let $K$ be a nonempty bounded convex closed subset of the Banach space $E$. Let $\gamma \in [0,1], \mu \in [0, \frac{1}{2}]$ such that $2\mu < \gamma$. Let $\{G_k\}_{k=1}^{\infty}$ be a sequence of commuting self mappings satisfying the condition $B_{\gamma,\mu}$ on $K$. Let $\{\alpha_n\}$ be a sequence in $(0, \frac{1}{2})$ and $\lambda \in (\gamma, 1)$. Define a sequence $\{x_n\}$ in $K$ by

$$\begin{cases} x_0 \in K \\ y_n = \left(1 - \sum_{k=1}^{\infty} \alpha_n^k\right) G_1 x_n + \sum_{k=2}^{\infty} \alpha_n^k G_k x_n \\ x_{n+1} = \lambda y_n + (1-\lambda) x_n,\ n = 0, 1, 2, \cdots. \end{cases} \tag{4.31}$$

**Lemma 4.11.** *Let $E$ be a uniformly convex Banach space and $K$ a nonempty bounded convex closed subset of $E$. Let $\{G_k\}_{k=1}^{\infty}$ be a sequence of commuting mappings satisfying the condition $B_{\gamma,\mu}$ on $K$, $\mu \in [0, \frac{1}{2}]$, $\gamma \in [0,1]$ and $2\mu < \gamma$. Assume that $F = \bigcap_{k=1}^{\infty} Fix(G_k) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, \frac{1}{2})$ converging to $0$ and $\lambda \in (\gamma, 1)$. Then, for each $x_0 \in K$, the sequence $\{x_n\}$ defined by (4.31) satisfies*

$$\lim_{n\to\infty} \left\| \left(1 - \sum_{k=1}^{\infty} \alpha_n^k\right) G_1 x_n + \sum_{k=2}^{\infty} \alpha_n^{k-1} G_k x_n - x_n \right\| = 0. \tag{4.32}$$

*Proof.* Let $p \in F$. It follows from Lemma 2.2 that for each $n = 0, 1, 2, \cdots$

$$\begin{aligned} \|x_{n+1} - p\| &\le \lambda \|y_n - p\| + (1-\lambda) \|x_n - p\| \\ &\le \lambda \left(1 - \sum_{k=1}^{\infty} \alpha_n^k\right) \|G_1 x_n - p\| + \lambda \sum_{k=2}^{\infty} \alpha_n^{k-1} \|G_k x_n - p\| + (1-\lambda) \|x_n - p\| \\ &\le \|x_n - p\|. \end{aligned}$$

Thus $\{\|x_n - p\|\}_{n=1}^{\infty}$ is a decreasing sequence and so for some $\nu \geq 0$

$$\lim_{n\to\infty} \|x_n - p\| = \nu. \tag{4.33}$$

Put $w_n = y_n - p$ and $z_n = x_n - p$. By referring to Lemma 2.6 and $(4.33)$ we get

$$\begin{aligned} \limsup_{n\to\infty} \|y_n - p\| &\leq \limsup_{n\to\infty} \left[ \left(1 - \sum_{k=1}^{\infty} \alpha_n^k\right) \|G_1 x_n - p\| + \sum_{k=2}^{\infty} \alpha_n^{k-1} \|G_k x_n - p\| \right] \\ &\leq \limsup_{n\to\infty} \|x_n - p\| \\ &= \nu. \end{aligned} \tag{4.34}$$

Moreover,

$$\limsup_{n\to\infty} \|(1-\lambda)w_n + \lambda z_n\| \leq \limsup_{n\to\infty} [(1-\lambda)\|y_n - p\| + \lambda \|x_n - p\|] \leq \nu. \tag{4.35}$$

It follows from $(4.33)$, $(4.34)$, $(4.35)$ and Lemma 2.2 that

$$\lim_{n\to\infty} \|w_n - z_n\| = 0,$$

and so that $(4.32)$ holds. □

Let us state and prove a fixed point theorem.

**Theorem 4.12.** *Let $K$ be a nonempty compact convex subset of a uniformly convex Banach space $E$ and $m \in \mathbb{N}$. Let $\{G_k\}_{k=1}^{\infty}$ be a sequence of commuting self mappings on $K$ satisfying the condition $B_{\gamma,\mu}$, where $\gamma \in [0,1]$, $\mu \in [0, \frac{1}{2}]$ such that $2\mu < \gamma$. Assume that $F = \bigcap_{k=1}^{\infty} Fix(G_k) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, \frac{1}{2})$ converging to $0$ and $\lambda \in (\gamma, 1)$. Then for each $x_0 \in K$ the sequence $\{x_n\}_{n=0}^{\infty}$ in $K$ defined by $(4.31)$ converges strongly to a point $p \in F$.*

*Proof.* By the compactness of $K$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that for some $p \in K$

$$\lim_{j\to\infty} x_{n_j} = p. \tag{4.36}$$

By Lemma 4.11 we have

$$\lim_{j\to\infty} \left\| \left(1 - \sum_{k=1}^{\infty} \alpha_{n_j}^k\right) G_1 x_{n_j} + \sum_{k=2}^{\infty} \alpha_{n_j}^{k-1} G_k x_{n_j} - x_{n_j} \right\| = 0. \tag{4.37}$$

The hypotheses of Lemma 4.3 are satisfied by (4.36) and (4.37). Therefore, $p$ is a common fixed point of $G_1, G_2, \cdots$; that is, $p \in F$. The proof of Lemma 4.11 and (4.36) imply that

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{j\to\infty} \left\|x_{n_j} - p\right\| = 0. \tag{4.38}$$

By (4.38) the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $p \in F$. This completes the proof. □

## 5. CONCLUSION

In this paper, we have extended the Suzuki's[28] approach of fixed point searching for nonexpansive mappings to mappings satisfying the condition $B_{\gamma,\mu}$ introduced by Patir et al. [22]. In other words, we brought together both the Suziki's strong nonexpansive mappings and the Patir et al. [22] condition $B_{\gamma,\mu}$ under the same iteration process. Under the resulted iteration process, we proved the approximation of a common fixed point of a sequence of mappings satisfying the weaker condition $B_{\gamma,\mu}$.

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

## REFERENCES


[1] D.R. Sahu, D. O'Regan, R.P. Agarwal, Fixed point theory for Lipschitzian-type mappings with applications, Springer, New York, 2009. https://doi.org/10.1007/978-0-387-75818-3.

[2] Y. Alber, C. Chidume, H. Zegeye, Approximating fixed points of total asymptotically nonexpansive mappings, Fixed Point Theory Appl. 2006 (2006), 10673. https://doi.org/10.1155/FPTA/2006/10673.

[3] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, Fund. Math. 3 (1922), 133–181.

[4] A. Betiuk-Pilarska, T.D. Benavides, The fixed point property for some generalized nonexpansive mappings and renormings, J. Math. Anal. Appl. 429 (2015), 800–813. https://doi.org/10.1016/j.jmaa.2015.04.043.

[5] F.E. Browder, Fixed-point theorems for noncompact mappings in Hilbert space, Proc. Natl. Acad. Sci. U.S.A. 53 (1965), 1272–1276. https://doi.org/10.1073/pnas.53.6.1272.

[6] F.E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Natl. Acad. Sci. U.S.A. 54 (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041.

[7] C.E. Chidume, Applicable functional analysis, TETFUND Book Project, Ibadan University Press, Ibadan, (2014).

[8] W.G. Dotson Jr, Fixed points of quasi-nonexpansive mappings, J. Aust. Math. Soc. 13 (1972), 167–170. https://doi.org/10.1017/s144678870001123x.

[9] S. Dhompongsa, W. Inthakon, A. Kaewkhao, Edelstein's method and fixed point theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. 350 (2009), 12–17. https://doi.org/10.1016/j.jmaa.2008.08.045.

[10] J. García-Falset, E. Llorens-Fuster, T. Suzuki, Fixed point theory for a class of generalized nonexpansive mappings, J. Math. Anal. Appl. 375 (2011), 185–195. https://doi.org/10.1016/j.jmaa.2010.08.069.

[11] K. Goebel, W.A. Kirk, Iteration processes for nonexpansive mappings, Contemp. Math 21 (1983), 115–123.

[12] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972), 171–174.

[13] D. Göhde, Zum prinzip def kontraktiven abbildung, Math. Nachr. 30 (1965), 251–258.

[14] M.A. Khamsi, A.R. Khan, On monotone nonexpansive mappings in $L_1([0,1])$, Fixed Point Theory Appl. 2015 (2015), 94. https://doi.org/10.1186/s13663-015-0346-x.

[15] M.A. Khamsi, W.A. Kirk, An introduction to metric spaces and fixed point theory, John Wiley & Sons, Inc. (2001).

[16] W.A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Mon. 72 (1965), 1004–1006.

[17] F. Lael, Z. Heidarpour, Fixed point theorems for a class of generalized nonexpansive mappings, Fixed Point Theory Appl. 2016 (2016), 82. https://doi.org/10.1186/s13663-016-0571-y.

[18] V.N. Mishra, Some problems on approximations of functions in Banach spaces, PhD Thesis, Department of Mathematics, IIT Roorkee, (2007).

[19] L.N. Mishra, S.K. Tiwari, V.N. Mishra, Fixed point theorems for generalized weakly S-contractive mappings in partial metric spaces, J. Appl. Anal. Comput. 5 (2015), 600–612. https://doi.org/10.11948/2015047.

[20] L.N. Mishra, S.K. Tiwari, V.N. Mishra, et al. Unique fixed point theorems for generalized contractive mappings in partial metric spaces, J. Funct. Spaces 2015 (2015), 960827. https://doi.org/10.1155/2015/960827.

[21] B. Patir, N. Goswami and L.N. Mishra, Fixed point theorems in fuzzy metric spaces for mappings with some contractive type conditions, Korean J. Math. 26 (2018), 307–326.

[22] B. Patir, N. Goswami, V.N. Mishra, Some results on fixed point theory for a class of generalized nonexpansive mappings, Fixed Point Theory Appl. 2018 (2018), 19. https://doi.org/10.1186/s13663-018-0644-1.

[23] R. Pant, R. Shukla, Approximating fixed points of generalized $\alpha$-nonexpansive mappings in Banach spaces, Numer. Funct. Anal. Optim. 38 (2017), 248–266. https://doi.org/10.1080/01630563.2016.1276075.

[24] M.G. Sangago, Weak convergence of iterations for nonexpansive mappings, Int. J. Math. Comput. 6 (2010), 69–76.

[25] M.G. Sangago, A study on convergence of iterative algorithms for nonexpansive mappings in Banach spaces, PhD Thesis, Andhra University, (2010). http://hdl.handle.net/10603/371320.

[26] M.G. Sangago, G.A. Tadesse, R. Tshelametse, et al. Approximation of common fixed points of two commuting generalized nonexpansive mappings, under review.

[27] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Aust. Math. Soc. 43 (1991), 153–159. https://doi.org/10.1017/s0004972700028884.

[28] T. Suzuki, Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces, Fixed Point Theory Appl. 2005 (2005), 685918. https://doi.org/10.1155/fpta.2005.103.

[29] T. Suzuki, Fixed point theorems for more generalized contractions in complete metric spaces, Demonstr. Math. 40 (2007), 219–228. https://doi.org/10.1515/dema-2007-0123.

[30] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. 340 (2008), 1088–1095.

[31] D. Thakur, B.S. Thakur, M. Postolache, Convergence theorems for generalized nonexpansive mappings in uniformly convex Banach spaces, Fixed Point Theory Appl. 2015 (2015), 144. https://doi.org/10.1186/s13663-015-0397-z.

[32] K. Ullah, J. Ahmad, M. Arshad, Z. Ma, Approximation of fixed points for enriched suzuki nonexpansive operators with an application in Hilbert spaces, Axioms. 11 (2021), 14. https://doi.org/10.3390/axioms11010014.