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COMMUTING AND WEAKLY COMMUTING MAPS IN GENERALIZED RECTANGULAR METRIC SPACES

ASHA RANI, ANKIT*

Department of Mathematics, SRM University Delhi-NCR, Sonipat-131029, Haryana, India

Abstract. We prove a common fixed point theorem using the notion of commuting and weakly commuting maps in generalized rectangular metric spaces. We have also provided an example in support of our results.

Keywords: fixed point theorem; commuting maps; weakly commuting maps; G-metric spaces; rectangular metric spaces; generalized rectangular metric spaces.

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1. Introduction

The dawn of fixed point theory began in 1912 when Brouwer [1] proved a fixed point result for continuous self maps on a closed ball. Over the last few decades fixed point theory has been one of the most interesting research areas in non-linear functional analysis. Banach contaction principle [2] is a fundamental tool of fixed point theory given by Banach in 1922. After which a lot of implications of banach contraction came into existence. Gahler [3, 4] during sixties introduced the notion of 2-metric space as a generalization of usual notion of a metric space (X,d). However, many authors proved that there is no relation between these two functions. In 1992, Dhage [5] came out with the concept of D-metric space. Most of the claims concerning the

*Corresponding author
E-mail address: ankitbhardwaj390@gmail.com
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fundamental topological structure of Dhage's D-metric space were proved invalid by Mustafa and Sims [6] in 2003. To overcome this difficulty, they introduced a more suitable and robust notion of a generalized metric space known as G-metric space. In 2000, Branciari [7] introduced the notion of rectangular metric spaces by replacing triangle inequality in a metric space with a three term expression. Motivated by these generalizations Adewale, Olaleru, Olaoluwa and Akewe [8] in 2021 introduced the notion of generalized rectangular metric spaces which exends a rectangular metric space.

2. Preliminaries

We start with basic definitions and a detailed overview of the essential results developed in the interesting works mentioned above. Mustafa and Sims define G-metric space as follows::

Definition 2.1. (see [6]) Let X be a non-empty set and $G : X \times X \times X \to [0,\infty)$ be a function satisfying the following properties:

1. $G(\xi,\eta,\tau) = 0$ if and only if $\xi = \eta = \tau$,
2. $G(\xi,\xi,\eta) > 0$, $\forall\ \xi,\eta \in X$ with $\xi \neq \eta$,
3. $G(\xi,\xi,\eta) < G(\xi,\eta,\tau)$, $\forall\ \xi,\eta,\tau \in X$ with $\tau \neq \eta$,
4. $G(\xi,\eta,\tau) = G(\xi,\tau,\eta) = G(\eta,\xi,\tau) = ...$ (symmetry in all three variables),
5. $G(\xi,\eta,\tau) \leq G(\xi,\alpha,\alpha) + G(\alpha,\eta,\tau)$, $\forall\ \alpha,\xi,\eta,\tau \in X$.

Then the function G is called a G-metric and the pair (X,G) is called a G-metric space.

The rectangular metric space was defined by Branciari as follows:

Definition 2.2. (see [7]) Let X be a non-empty set and $d : X \times X \to [0,\infty)$ be a function satisfying the following properties:

1. $d(\xi,\eta) = 0$ if and only if $\xi = \eta$ for all $\xi,\eta \in X$,
2. $d(\xi,\eta) = d(\eta,\xi)$, for all $\xi,\eta \in X$,
3. $d(\xi,\eta) \leq d(\xi,\alpha) + d(\alpha,\beta) + d(\beta,\eta)$, for all $\xi,\eta \in X$ and all distinct points $\alpha,\beta \in X - \{\xi,\eta\}$. Then the function d is called a rectangular metric and the pair (X,d) is called a rectangular metric space.

Definition 2.3. (see [8]) Let X be a non-empty set and $G : X \times X \times X \to [0,\infty)$ be a function satisfying the following properties:

1. $G(\xi,\eta,\tau) = 0$ if and only if $\xi = \eta = \tau$,
2. $G(\xi,\xi,\eta) > 0$, $\forall\ \xi,\eta \in X$ with $\xi \neq \eta$,
3. $G(\xi,\eta,\tau) = G(\xi,\tau,\eta) = G(\eta,\xi,\tau) = ...,$
4. $G(\xi,\eta,\tau) \leq G(\xi,\alpha,\alpha)+G(\alpha,\beta,\beta)+G(\beta,\eta,\eta)+G(\eta,\eta,\tau)$, $\forall\ \xi,\eta,\tau \in X$ and all distinct points $\alpha,\beta \in X-\{\xi,\eta,\tau\}$.

Then the function G is called a generalized rectangular metric and the pair (X,G) is called a generalized rectangular metric space.

Remark 2.4. (see [8]) If $\eta = \tau$ and we set $G(\xi,\eta,\eta) = d(\xi,\eta)$. Definition 2.3 reduces to rectangular metric space [7].

Definition 2.5. (see [8]) Let (X,G) be a generalized rectangular metric space. For $\xi \in X$, $r > 0$, the G-sphere with center ξ and radius r is

$$S_G(\xi,r) = \{\tau \in X : G(\xi,\tau,\tau) < r\}.$$

Definition 2.6. (see [8]) Let (X,G) be a generalized rectangular metric space. The sequence $\{\xi_n\} \subset X$ is G-convergent to τ if it converges to τ in the generalized rectangular metric space.

Definition 2.7. (see [8]) Let (X,G) be a generalized rectangular metric space and $\{\xi_n\}$ be a sequence in X. Then $\{\xi_n\}$ converges to ξ if and only if $G(\xi_n,\xi,\xi) \to 0\ as\ n \to \infty$.

Definition 2.8. (see [8]) Let (X,G) be a generalized rectangular metric space and $\{\xi_n\}$ be a sequence in X. Then $\{\xi_n\}$ is said to be a cauchy sequence if and only if $G(\xi_n,\xi_m,\xi_l) \to 0\ as\ n,m,l \to \infty$.

3. Main Results

There is considerable interest in examining common fixed points for a pair of maps that satisfy the contraction condition in metric space. There are some interesting and elegant results in this direction by various authors. Introduction of commutativity by Jungck [9] in 1976 was the turning point for the ”fixed point arena”. The result was further generalized and extended

in various ways by many authors. It paves the ways to compute the fixed point for pair/pairs of mappings. In particular now we look in the context of common fixed point theorem in generalized rectangular metric spaces. We start with the following contraction conditions:
Let (X,G) be a complete generalized rectangular metric space and T be a mapping from (X,G) into itself and consider the following conditions:

(1) $G(T\xi,T\eta,T\tau) \leq KG(\xi,\eta,\tau)$, for all $\xi,\eta,\tau \in X$, where $0 \leq K < 1$.

Every self mapping T of X satisfying condition (1) is continuous. Now our focus is to generalize the condition (1) for a pair of self maps S and T of X in the following way:

(2) $G(S\xi,S\eta,S\tau) \leq KG(T\xi,T\eta,T\tau)$, for all $\xi,\eta,\tau \in X$, where $0 \leq K < 1$.

It is necessary to add additional assumptions to prove the existence of common fixed points for (2). Most of the theorems followed a similar pattern of maps: (i) contraction, (ii) continuity of functions (either one or both) and (iii) some condition on pair of mappings were given. Condition (ii) can be relaxed in some cases but condition (i) and (iii) are unavoidable.
Now we introduce the concept of commuting and weakly commuting maps in a generalized rectangular metric space.

Definition 3.1. [9] Two self mappings f and g on a metric space (X,d) are said to be commuting if $fg\xi = gf\xi$, $\forall\ \xi \in X$.

Theorem 3.2. *Let (X,G) be a complete generalized rectangular metric space and let f,g be self mappings of X satisfying the following conditions:*

(3.2.1) $f(X) \subseteq g(X)$,

(3.2.2) f or g is continuous,

(3.2.3) $G(f\xi,f\eta,f\tau) \leq KG(g\xi,g\eta,g\tau)$ *for every* $\xi,\eta,\tau \in X$ *and* $0 \leq K < 1$.

Then f and g have a unique common fixed point in X provided f and g commute.

Proof. Let ξ_0 be an arbitrary point in X. By 3.2.1, one can choose a point ξ_1 in X such that $f\xi_0 = g\xi_1$. In general one can choose ξ_{n+1} such that $\eta_n = f\xi_n = g\xi_{n+1}$, n = 0, 1, 2, ... From 3.2.3, take $\xi = \xi_n, \eta = \xi_{n+1}, \tau = \xi_{n+1}$, we have

$$G(f\xi_n, f\xi_{n+1}, f\xi_{n+1}) \leq KG(g\xi_n, g\xi_{n+1}, g\xi_{n+1}) = KG(f\xi_{n-1}, f\xi_n, f\xi_n).$$

Continuing in the same way, we deduce that

$$G(f\xi_n, f\xi_{n+1}, f\xi_{n+1}) \leq K^n G(f\xi_0, f\xi_1, f\xi_1)$$

and hence

$$G(\eta_n, \eta_{n+1}, \eta_{n+1}) \leq K^n G(\eta_0, \eta_1, \eta_1). \tag{1}$$

Setting $T_n = G(\eta_n, \eta_{n+1}, \eta_{n+1})$, we have

$$T_n \leq K^n \, T_0 \; \forall \; n \in \mathbb{N}. \tag{2}$$

By repeated use of (2) in definition 2.3 and all distinct points $\eta_{n+1}, \eta_{n+2}, ..., \eta_{m-1}$ with $m > n$, we have the following for all odd $m-n$:

$$\begin{aligned}
G(\eta_n, \eta_m, \eta_m) &\leq G(\eta_n, \eta_{n+1}, \eta_{n+1}) + G(\eta_{n+1}, \eta_{n+2}, \eta_{n+2}) + G(\eta_{n+2}, \eta_m, \eta_m) \\
&\leq T_n + T_{n+1} + G(\eta_{n+2}, \eta_m, \eta_m) \\
&\leq T_n + T_{n+1} + T_{n+2} + T_{n+3} + G(\eta_{n+4}, \eta_m, \eta_m) \\
&\leq \sum_{i=n}^{n+3} T_i + G(\eta_{n+4}, \eta_m, \eta_m) \\
&\leq \sum_{i=n}^{m-1} T_i \leq \sum_{i=n}^{\infty} T_i. \qquad (3)
\end{aligned}$$

Similarly, if m - n $\geq$ 4 is even, we have

$$G(\eta_n, \eta_m, \eta_m) \leq \sum_{i=n}^{m-3} T_i + G(\eta_{m-2}, \eta_m, \eta_m). \tag{4}$$

From (2) and (3), we have

$$\begin{aligned}
G(\eta_n, \eta_m, \eta_m) &\leq K^n T_0 + K^{n+1} T_0 + K^{n+2} T_0 + ... + K^{m-2} T_0 + K^{m-1} T_0 \\
&\leq K^n [1 + K + K^2 + K^3 + ... + K^{m-n-1}] T_0 \\
&\leq \frac{K^n}{(1-K)} T_0.
\end{aligned}$$

From (2) and (4), we have

$$\begin{aligned}
G(\eta_n, \eta_m, \eta_m) &\leq K^n (1-K)^{-1} T_0 + G(\eta_{m-2}, \eta_m, \eta_m) \\
&\leq K^n (1-K)^{-1} T_0 + K^{m-2} G(\eta_0, \eta_2, \eta_2).
\end{aligned}$$

Taking the limit of $G(\eta_n,\eta_m,\eta_m)$ *as* $n,m\to\infty$, we have

$$\lim_{n,m\to\infty} G(\eta_n,\eta_m,\eta_m)=0. \tag{5}$$

For $n,m,l\in\mathbb{N}$ with $n>m>l$,

(6)

$$G(\eta_n,\eta_m,\eta_l)\leq G(\eta_n,\eta_{n-1},\eta_{n-1})+G(\eta_{n-1},\eta_{n-2},\eta_{n-2})+G(\eta_{n-2},\eta_m,\eta_m)+G(\eta_m,\eta_m,\eta_l).$$

Taking the limit of $G(\eta_n,\eta_m,\eta_l)$ as $n,m,l\to\infty$, we have

$$\lim_{n,m,l\to\infty} G(\eta_n,\eta_m,\eta_l)=0. \tag{7}$$

So $\{\eta_n\}$ is a G-cauchy sequence. By completeness of (X,G), there exist $\tau\in X$ such that

$$\lim_{n\to\infty}\eta_n=\lim_{n\to\infty} f\xi_n=\lim_{n\to\infty} g\xi_n=\tau.$$

Since f or g is continuous, for definiteness one can assume that g is continuous, therefore there exist $\tau\in X$ such that

$$\lim_{n\to\infty} gg\xi_n=\lim_{n\to\infty} gf\xi_n=g\tau.$$

Further, we have since f and g are commuting maps, therefore by definition, we get

$$\lim_{n\to\infty} gf\xi_n=\lim_{n\to\infty} fg\xi_n=\lim_{n\to\infty} gg\xi_n=g\tau.$$

From (3.2.3), take $\xi=g\xi_n,\eta=\xi_n,\tau=\xi_n$, we have

$$G(fg\xi_n,f\xi_n,f\xi_n)\leq KG(gg\xi_n,g\xi_n,g\xi_n).$$

Proceeding limits as $n\to\infty$, we have $g\tau=\tau$. We now prove that $f\tau=\tau$. Again from (3.2.3) setting $\xi=\xi_n,\eta=\tau,\tau=\tau$, we have

$$G(f\xi_n,f\tau,f\tau)\leq KG(g\xi_n,g\tau,g\tau).$$

Taking limit as n $\to\infty$, we have $f\tau=\tau$. Therefore, we have $f\tau=g\tau=\tau$. Thus τ is a common fixed point of f and g.

Uniqueness: We assume $\tau_1(\neq\tau)$ be another common fixed point of f and g. Then $G(\tau,\tau_1,\tau_1)>0$ and

$$G(\tau,\tau_1,\tau_1)=G(f\tau,f\tau_1,f\tau_1)\leq KG(g\tau,g\tau_1,g\tau_1)=KG(\tau,\tau_1,\tau_1)<G(\tau,\tau_1,\tau_1),$$

a contradiction, therefore $\tau = \tau_1$. Hence uniqueness follows. □

Example 3.3. Let $X = [-1, 1]$ and let $G : X \times X \times X \to [0, \infty)$ be the generalized rectangular metric space defined as follows:
$G(\xi, \eta, \tau) = (|\xi - \eta| + |\eta - \tau| + |\tau - \xi|))$ for all $\xi, \eta, \tau \in X$. Then (X, G) is a generalized rectangular metric space. Define $f(\xi) = \frac{\xi}{6}$ and $g(\xi) = \frac{\xi}{2}$. Here it is observed that,

(1) $f(X) \subseteq g(X)$,
(2) g is continuous on X,
(3) $G(f\xi, f\eta, f\tau) \leq KG(g\xi, g\eta, g\tau)$ holds for all $\xi, \eta, \tau \in X, \frac{1}{3} \leq K < 1$.

However, the mapping f and g are commutative and $\xi = 0$ is unique fixed point of f and g. Hence all the condition of theorem 3.2 are satisfied.

4. Weakly Commuting Maps

Sessa [10] in 1982 introduced the concept of weakly commuting maps in metric spaces as follows:

Definition 4.1. Two self mappings f and g on a metric space (X, d) are said to be weakly commuting if $d(fg\xi, gf\xi) \leq d(f\xi, g\xi)$, for all $\xi \in X$.

We now introduce the notion of weakly commuting maps in generalized rectangular metric space.

Definition 4.2. Two self mappings f and g on a generalized rectangular metric space (X, G) are said to be weakly commuting if and only if $G(fg\xi, gf\xi, gf\xi) \leq G(f\xi, g\xi, g\xi)$ for all $\xi \in X$.

Theorem 4.3. *Let (X, G) be a complete generalized rectangular metric space and let f and g be weakly commuting mapping of X satisfying (3.2.1),(3.2.2) and (3.2.3). Then f and g have a unique common fixed point in X.*

Proof. From theorem 3.2 we conclude that $\{\eta_n\}$ is a cauchy sequence in X. Since (X, G) is a complete generalized rectangular metric space, there exist a point τ in X such that

$$\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = \tau.$$

Let us suppose that f is continuous. Therefore,

$$\lim_{n\to\infty} fg\xi_n = \lim_{n\to\infty} ff\xi_n = f\tau.$$

Since f and g are weakly commuting therefore,

$$G(fg\xi_n, gf\xi_n, gf\xi_n) \leq G(f\xi_n, g\xi_n, g\xi_n). \tag{8}$$

By letting $n \to \infty$, we have

$$\lim_{n\to\infty} fg\xi_n = \lim_{n\to\infty} gf\xi_n = f\tau$$

We now prove that $\tau = f\tau$. Suppose $\tau \neq f\tau$, then $G(\tau, f\tau, f\tau) > 0$. From (3.2.3), on letting $\xi = \xi_n, \eta = f\xi_n, \tau = f\xi_n$, we have

$$G(f\xi_n, ff\xi_n, ff\xi_n) \leq KG(g\xi_n, gf\xi_n, gf\xi_n).$$

Proceeding limit $n \to \infty$, we have

$$G(\tau, f\tau, f\tau) \leq KG(\tau, f\tau, f\tau) < G(\tau, f\tau, f\tau),$$

which is a contradiction. Therefore, $f\tau = \tau$. Since $f(X) \subseteq g(X)$, we can find τ_1 in X such that $\tau = f\tau = g\tau_1$. Now from (3.2.3), take $\xi = f\xi_n, \eta = \tau_1, \tau = \tau_1$, we have

$$G(ff\xi_n, f\tau_1, f\tau_1) \leq KG(gf\xi_n, g\tau_1, g\tau_1)$$

Taking limit $n \to \infty$, we get

$$G(f\tau, f\tau_1, f\tau_1) \leq KG(f\tau, g\tau_1, g\tau_1) = KG(f\tau, f\tau, f\tau) = 0,$$

which implies that $f\tau = f\tau_1$ i.e. $\tau = f\tau = f\tau_1 = g\tau_1$. Also by using definition of weakly commuting maps,

$$G(f\tau, g\tau, g\tau) = G(fg\tau_1, gf\tau_1, gf\tau_1) \leq G(f\tau_1, g\tau_1, g\tau_1) = 0, \tag{9}$$

which again implies that $f\tau = g\tau = \tau$. Thus τ is a common fixed point of f and g.

Uniqueness: We assume $\tau_1(\neq \tau)$ be another common fixed point of f and g. Then $G(\tau, \tau_1, \tau_1) > 0$ and

$$G(\tau, \tau_1, \tau_1) = G(f\tau, f\tau_1, f\tau_1) \leq KG(g\tau, g\tau_1, g\tau_1) = KG(\tau, \tau_1, \tau_1) < G(\tau, \tau_1, \tau_1),$$

a contradiction, therefore $\tau = \tau_1$. Hence uniqueness follows. □

Example 4.4. Let $X = [-1,1]$ and let $G : X \times X \times X \to [0,\infty)$ be the generalized rectangular metric space defined as follows:

$$G(\xi,\eta,\tau) = (|\xi-\eta| + |\eta-\tau| + |\tau-\xi|)$$

for all $\xi,\eta,\tau \in X$. Then (X,G) is a generalized rectangular metric space. Define $f(\xi) = \xi$ and $g(\xi) = 2\xi - 1$. Here we note that

(1) $f(X) \subseteq g(X)$,
(2) f is continuous on X,
(3) $G(f\xi, f\eta, f\tau) \leq KG(g\xi, g\eta, g\tau)$ holds for all $\xi,\eta,\tau \in X, \frac{1}{2} < K < 1$.

However, the mappings f and g are weakly commuting maps and $\xi = 1$ is unique common fixed point of f and g. Hence all the conditions of theorem 4.3 are satisfied.

Conflict of Interests

The authors declare that there is no conflict of interests.

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