Available online at http://scik.org
Adv. Fixed Point Theory, 2024, 14:16
https://doi.org/10.28919/afpt/8551
ISSN: 1927-6303

ENRICHED MULTIVALUED CONTRACTIONS ON DOUBLE CONTROLLED METRIC TYPE SPACES WITH AN APPLICATION

AMIT GANGWAR[1], SHIVAM RAWAT[2,*], HÜSEYIN IŞIK[3], R.C. DIMRI[1]

[1]H.N.B. Garhwal University, Srinagar (Garhwal), Uttarakhand 246174, India
[2]Department of Mathematics, Graphic Era (Deemed to be) University, Dehradun, Uttarakhand, 248002, India
[3]Department of Engineering Science, Bandırma Onyedi Eylül University, Bandırma 10200, Balıkesir, Turkey

Abstract. In this research, we define (λ,θ)-enriched multivalued contraction and λ-enriched multivalued nonexpansive mapping on a double controlled metric type space and propose some fixed point results for these introduced contractions. An example is given to support our proposed results. Also, we solve a problem of differential inclusion, which may be used to model controlled systems with discontinuities, as an application.

Keywords: double controlled metric type space; (λ,θ)-enriched multivalued contraction; differential inclusion problem; Pompeiu-Hausdorff metric.

2020 AMS Subject Classification: 47H09, 47H10, 54H25.

1. INTRODUCTION

In 1922, Banach [3] proved an important result for fixed point theory, which is known as the Banach contraction principle. The result states that a contraction self-mapping on a complete metric space (MS) has a unique fixed point. Because to its wide applicability, the result was extended and generalized in various ways (see, for instance [10, 14, 16, 17], and references

*Corresponding author
E-mail address: rawat.shivam09@gmail.com
Received March 19, 2024

therein). In 1969, Nadler [13] gave a popular and significant generalization by defining a multivalued mapping using Pompeiu-Hausdorff metric on a compact and bounded subset, which is useful in economics, differential equations, convex optimization, control theory, and so on. It is interesting to mention that translating a real-world problem into a fixed point problem is a suitable technique to find a subsequent solution.

In past few years, the study of enriched multivalued contractions has gained substantial attention due to its wide-ranging implications in various fields of mathematics and its diverse applications in real-world problems. The concept of enriched contractions, as introduced by Berinde ([4, 5]), has provided a significant framework for investigating the dynamics of nonlinear mappings and has been instrumental in the analysis of fixed point theory and related mathematical structures. This idea was subsequently exptended by Abbas et al. [1] to (λ,θ)-enriched and λ-enriched multivalued nonexpansive mappings. In the context of convex MSs, Rawat et al. [15] proposed interpolative enriched contractions of the Kannan type, Hardy-Rogers type, and Matkowski type using the technique of enrichment of mappings on existing interpolative contractions.

In 1993, the notion of a b-MS was introduced by Czerwik [9], as a generalization of MS. To enlarge the notion of b-MS, Kamran et al. [11] proposed the concept of an extended b-MS in 2017. Mlaiki et al. [12] proposed a novel type of controlled metric type space (CMS) as an extension of extended b-MS in 2018. Another generalisation of MS known as double controlled metric type space (DCMS) was initiated by Abdeljawad et al. [2] in the same year. This extension has been shown to be particularly useful in the investigation of various properties of MSs, offering a broader perspective for understanding the underlying geometric and topological characteristics.

Motivated by the significant developments in the respective fields of enriched multivalued contractions and DCMSs, this research article aims to bridge the gap between these two critical areas of study. By exploring the intricate relationship between enriched multivalued contractions and the DCMSs, we seek to provide a comprehensive understanding of the interplay between these two fundamental concepts. Furthermore, we intend to showcase the practical implications of this theoretical framework through an application to differential inclusion problem, which may be used to model controlled systems with discontinuities that highlights the applicability of our research in addressing real-world challenges.

Inspired by these results, we extend the work of Abbas et al. [1] and establish novel multivalued fixed point conclusions for (λ,θ)-enriched multivalued contraction and a λ-enriched multivalued nonexpansive mapping in the environment of DCMSs. Some nontrivial illustrations are provided to indicate the relevance of newly obtained conclusions. Also, an application to find the solutions to problem of differential inclusions is provided to vindicate our claims.

2. Preliminaries

Definition 1. [12] *Let* $\mathfrak{A} \neq \phi$ *and* $\mathfrak{d}^* : \mathfrak{A}\times\mathfrak{A} \to [0,\infty)$, $\sigma : \mathfrak{A}\times\mathfrak{A} \to [1,\infty)$ *be mappings. If for* $\mathfrak{a},\mathfrak{b},\mathfrak{c} \in \mathfrak{A}$,

(i) $0 \leq \mathfrak{d}^*(\mathfrak{a},\mathfrak{b})$ *and* $\mathfrak{d}^*(\mathfrak{a},\mathfrak{b}) = 0$ *if and only if* $\mathfrak{a} = \mathfrak{b}$;

(ii) $\mathfrak{d}^*(\mathfrak{a},\mathfrak{b}) = \mathfrak{d}^*(\mathfrak{b},\mathfrak{a})$;

(iii) $\mathfrak{d}^*(\mathfrak{a},\mathfrak{b}) \leq \sigma(\mathfrak{a},\mathfrak{c})\mathfrak{d}^*(\mathfrak{a},\mathfrak{c}) + \sigma(\mathfrak{c},\mathfrak{b})\mathfrak{d}^*(\mathfrak{c},\mathfrak{b})$.

Then, $(\mathfrak{A},\mathfrak{d}^*,\sigma)$ *is said to be a CMS.*

Definition 2. [2] *Let* $\mathfrak{A} \neq \phi$ *and* $\mathfrak{d}^* : \mathfrak{A}\times\mathfrak{A} \to [0,\infty)$ *and* $\alpha,\beta : \mathfrak{A}\times\mathfrak{A} \to [1,\infty)$ *be non-comparable functions. If for* $\mathfrak{a},\mathfrak{b},\mathfrak{c} \in \mathfrak{A}$,

(i) $0 \leq \mathfrak{d}^*(\mathfrak{a},\mathfrak{b})$ *and* $\mathfrak{d}^*(\mathfrak{a},\mathfrak{b}) = 0$ *if and only if* $\mathfrak{a} = \mathfrak{b}$;

(ii) $\mathfrak{d}^*(\mathfrak{a},\mathfrak{b}) = \mathfrak{d}^*(\mathfrak{b},\mathfrak{a})$;

(iii) $\mathfrak{d}^*(\mathfrak{a},\mathfrak{b}) \leq \alpha(\mathfrak{a},\mathfrak{c})\mathfrak{d}^*(\mathfrak{a},\mathfrak{c}) + \beta(\mathfrak{c},\mathfrak{b})\mathfrak{d}^*(\mathfrak{c},\mathfrak{b})$.

Then, $(\mathfrak{A},\mathfrak{d}^*,\alpha,\beta)$ *is said to be a DCMS.*

We see that if $\alpha(\mathfrak{a},\mathfrak{b}) = \beta(\mathfrak{a},\mathfrak{b}), \forall\, \mathfrak{a}, v \in \mathfrak{A}$, then a DCMS is simply a CMS. Also, if $\alpha(\mathfrak{a},\mathfrak{b}) = \beta(\mathfrak{a},\mathfrak{b}) = 1, \forall\, \mathfrak{a},\mathfrak{b} \in \mathfrak{A}$, then a DCMS reduces to a MS.

Definition 3. [2] *Let* $(\mathfrak{A},\mathfrak{d}^*,\alpha,\beta)$ *be a DCMS and* $\{\mathfrak{a}_\mathfrak{p}\}$ *be a sequence in* $\mathfrak{A}$. *Then*

(1) $\{\mathfrak{a}_\mathfrak{p}\} \subset \mathfrak{A}$ *converges to a point* $\mathfrak{a}^*$ *if* $\lim_{\mathfrak{p}\to\infty} \mathfrak{d}^*(\mathfrak{a}_\mathfrak{p},\mathfrak{a}^*) = 0$.

(2) $\{\mathfrak{a}_\mathfrak{p}\} \subset \mathfrak{A}$ *is Cauchy if for each* $\varepsilon > 0$, *there exists a natural number* $N_\varepsilon \in \mathbb{N}$ *satisfying*

$$\mathfrak{d}^*(\mathfrak{a}_\mathfrak{p},\mathfrak{a}_\mathfrak{q}) < \varepsilon, \text{ where } \mathfrak{q},\mathfrak{p} \geq N_\varepsilon.$$

(3) $(\mathfrak{A}, \mathfrak{d}^*, \alpha, \beta)$ *is complete if every Cauchy sequence in* $\mathfrak{A}$ *is convergent.*

Definition 4. [13] *Let* $CB(\mathfrak{A})$ *be the collection of all nonempty, bounded and closed subsets of a MS* $(\mathfrak{A}, \mathfrak{d}^*)$*. Then* $H : CB(\mathfrak{A}) \times CB(\mathfrak{A}) \to [0, \infty)$*, which is a symmetric functional, such that*

$$H(\mathfrak{K}, \mathfrak{S}) = max\{D(\mathfrak{K}, \mathfrak{S}), D(\mathfrak{S}, \mathfrak{K})\},$$

where $D(\mathfrak{K}, \mathfrak{S}) = \sup_{\mathfrak{a} \in \mathfrak{K}} \inf_{\mathfrak{b} \in \mathfrak{S}} \mathfrak{d}^*(\mathfrak{a}, \mathfrak{b})$*, for all* $\mathfrak{K}, \mathfrak{S} \in CB(\mathfrak{A})$*, is known as the Pompeiu-Hausdorff metric.*

If we have Hausdorff metric $H(\mathfrak{K}, \mathfrak{S}) = 0$, then $\mathfrak{K}$ and $\mathfrak{S}$ have same closure.

Definition 5. [18] *Let* $\mathfrak{A} = (A_1, \mathfrak{d}_1^*)$ *and* $\mathfrak{B} = (A_2, \mathfrak{d}_2^*)$ *be two MSs. Let A be the set of bounded and continuous functions* $f : \mathfrak{A} \to \mathfrak{B}$*. Let* $\mathfrak{d}^* : A \times A \to [0, \infty)$ *be defined as*

$$\mathfrak{d}^*(f, g) = \sup_{x \in A_1} \mathfrak{d}_2^*(f(x), g(x)),\ f, g \in A.$$

Then, the metric $\mathfrak{d}^*$ *is known as the supremum metric on* $\mathfrak{A}$*.*

Definition 6. [13] *Let* $(\mathfrak{A}, \mathfrak{d}_1^*)$ *and* $(\mathfrak{B}, \mathfrak{d}_2^*)$ *be two MSs. A function* $f : \mathfrak{A} \to CB(\mathfrak{B})$ *is a multivalued Lipschitz mapping if*

$$H(f(\mathfrak{a}_1), f(\mathfrak{a}_2)) \leq k\mathfrak{d}_1^*(\mathfrak{a}_1, \mathfrak{a}_2), \text{ for all } \mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{A},$$

where $k \geq 0$ *is a fixed real number known as the Lipschitz constant. If* $k < 1$*, then* f *is a multivalued contraction.*

Definition 7. [13] *An element* $\mathfrak{a} \in \mathfrak{A}$ *is known as a fixed point for a given multivalued mapping* $f : \mathfrak{A} \to CB(\mathfrak{A})$*, if* $\mathfrak{a} \in f(\mathfrak{a})$*.*

Theorem 2.1. [13] *If* $f : \mathfrak{A} \to CB(\mathfrak{A})$ *is a multivalued contraction on a complete MS* $(\mathfrak{A}, \mathfrak{d}^*)$*, then* f *admits a fixed point.*

Lemma 2.2. [13] *Let* $A, B \in CB(\mathfrak{A})$*, where* $(\mathfrak{A}, \mathfrak{d}^*)$ *is a MS. Then, for every* $\varepsilon > 0$ *and* $\mathfrak{a} \in A$ *there exists an element* $\mathfrak{b} \in B$ *satisfying*

$$\mathfrak{d}^*(\mathfrak{a}, \mathfrak{b}) \leq H(A, B) + \varepsilon.$$

Lemma 2.3. [7] *Let $A, B \subset \mathfrak{A}$, where $(\mathfrak{A}, \mathfrak{d}^*)$ is a MS and $k \geq 1$. Then there is an element $\mathfrak{b}$ belonging to the set B for every $\mathfrak{a} \in A$ satisfying*

$$\mathfrak{d}^*(\mathfrak{a}, \mathfrak{b}) \leq kH(A, B).$$

Lemma (2.2) and (2.3) hold for the space defined in Definition 2 also. Abbas et al. [1] defined the following multivalued versions of enriched contraction in a standard MS.

Definition 8. [1] *Let $(\mathfrak{A}, \|.\|)$ be a linear normed space and there exist $\lambda \in [0, \infty)$ and $\theta \in [0, \lambda + 1)$. A multivalued mapping $f : \mathfrak{A} \to CB(\mathfrak{A})$ is known as an*
(a) enriched multivalued contraction if

$$H(\lambda\mathfrak{a} + f(\mathfrak{a}), \lambda\mathfrak{b} + f(\mathfrak{b})) \leq \theta\|\mathfrak{a} - \mathfrak{b}\| \textit{ for all } \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}.$$

(b) enriched multivalued nonexpansive mapping if

$$H(\lambda\mathfrak{a} + f(\mathfrak{a}), \lambda\mathfrak{b} + f(\mathfrak{b})) \leq (\lambda + 1)\|\mathfrak{a} - \mathfrak{b}\| \textit{ for all } \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}.$$

We also call f, as defined in (a) and (b), a (λ, θ)-enriched multivalued contraction and λ-enriched multivalued nonexpansive mapping, respectively.

Remark 2.4. [1] *Let $\mathfrak{A}$ be a linear space and A is a convex subset of $\mathfrak{A}$. Let $f : \mathfrak{A} \to CB(\mathfrak{A})$ be a mapping. For $\omega \in (0, 1)$, define $f_\omega : \mathfrak{A} \to CB(\mathfrak{A})$ as*

$$f_\omega(x) = (1 - \omega)x + \omega f x.$$

Then, $Fix(f_\omega) = Fix(f)$.

Definition 9. [8] *If $f : \mathfrak{A} \to \mathfrak{A}$ is a self-mapping on a MS $(\mathfrak{A}, \mathfrak{d}^*)$. The orbit of f at $\mathfrak{a}$ is defined as $O(\mathfrak{a}, f) = \{f^{\mathfrak{p}}(\mathfrak{a}) : \mathfrak{p} = 0, 1, 2, \ldots\}$.*

3. Main Results

We define a multivalued contraction condition for DCMSs as follows.

Definition 10. *A mapping $f : \mathfrak{A} \to CB(\mathfrak{A})$, defined on a DCMS $(\mathfrak{A}, \mathfrak{d}^*, \alpha, \beta)$, is a multivalued contraction if*

$$H(f(\mathfrak{a}), f(\mathfrak{b})) \leq \theta\mathfrak{d}^*(\mathfrak{a}, \mathfrak{b}), \textit{ for all } \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}, \tag{1}$$

where $\theta \in (0,1)$ is a fixed real number.

Theorem 3.1. *Let $f : \mathfrak{A} \to CB(\mathfrak{A})$ be a multivalued contraction on a complete DCMS $(\mathfrak{A}, \mathfrak{d}^*, \alpha, \beta)$. Suppose*

$$\sup_{\mathfrak{q} \geq i} \lim_{i \to \infty} \frac{\alpha(\mathfrak{a}_{i+1}, \mathfrak{a}_{i+2}).\beta(\mathfrak{a}_{i+1}, \mathfrak{a}_{\mathfrak{q}})}{\alpha(\mathfrak{a}_i, \mathfrak{a}_{i+1})} < \frac{1}{\theta},$$

where $\lim_{\mathfrak{p}\to\infty} \alpha(\mathfrak{a}, \mathfrak{a}_{\mathfrak{p}})$ and $\lim_{\mathfrak{p}\to\infty} \beta(\mathfrak{a}, \mathfrak{a}_{\mathfrak{p}})$ exists, $\forall\, \mathfrak{a} \in \mathfrak{A}$. Then, f admits a fixed point.

Proof. Let $\mathfrak{a}_0 \in \mathfrak{A}$ be arbitrary, then $f(\mathfrak{a}_0) \neq \phi$. Choose $\mathfrak{a}_1 \in f(\mathfrak{a}_0)$, then for $\theta \in (0,1)$, there exists an $\mathfrak{a}_2 \in f(\mathfrak{a}_1)$, satisfying

$$\begin{aligned} \mathfrak{d}^*(\mathfrak{a}_2, \mathfrak{a}_1) &\leq H(f(\mathfrak{a}_1), f(\mathfrak{a}_0)) + \theta \\ &\leq \theta \mathfrak{d}^*(\mathfrak{a}_1, \mathfrak{a}_0) + \theta. \end{aligned}$$

Similarly, there exists an $\mathfrak{a}_3 \in f(\mathfrak{a}_2)$ satisfying

$$\begin{aligned} \mathfrak{d}^*(\mathfrak{a}_3, \mathfrak{a}_2) &\leq H(f(\mathfrak{a}_2), f(\mathfrak{a}_1)) + \theta^2 \\ &\leq \theta^2 d(\mathfrak{a}_1, \mathfrak{a}_0) + 2\theta^2. \end{aligned}$$

Proceeding in this manner, we obtain $\mathfrak{a}_{\mathfrak{p}+1} \in f(\mathfrak{a}_{\mathfrak{p}})$ and $\theta^{\mathfrak{p}} \in (0,1)$, satisfying

$$\begin{aligned} \mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_{\mathfrak{p}}) &\leq H(f(\mathfrak{a}_{\mathfrak{p}}), f(\mathfrak{a}_{\mathfrak{p}-1})) + \theta^{\mathfrak{p}} \\ &\leq \theta^{\mathfrak{p}}\, \mathfrak{d}^*(\mathfrak{a}_1, \mathfrak{a}_0) + \mathfrak{p}\theta^{\mathfrak{p}}. \end{aligned}$$

Claim: $\{\mathfrak{a}_{\mathfrak{p}}\}$ is a Cauchy sequence.

For $\mathfrak{q} > \mathfrak{p}$, we have

$$\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{q}}) \leq \alpha(\mathfrak{a}_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}+1}) + \beta(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_{\mathfrak{q}})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_{\mathfrak{q}}).$$

Again, applying the triangle inequality on $\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_{\mathfrak{q}})$, we get

$$\begin{aligned} &\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{q}}) \\ &\leq \alpha(\mathfrak{a}_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}+1}) + \beta(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_{\mathfrak{q}})(\alpha(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_{\mathfrak{p}+2})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_{\mathfrak{p}+2}) + \beta(\mathfrak{a}_{\mathfrak{p}+2}, \mathfrak{a}_{\mathfrak{q}})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+2}, \mathfrak{a}_{\mathfrak{q}})) \\ &= \alpha(\mathfrak{a}_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}+1}) + \beta(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_{\mathfrak{q}}).\alpha(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_{\mathfrak{p}+2})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_{\mathfrak{p}+2}) \\ &+ \beta(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_{\mathfrak{q}}).\beta(\mathfrak{a}_{\mathfrak{p}+2}, \mathfrak{a}_{\mathfrak{q}})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+2}, \mathfrak{a}_{\mathfrak{q}})) \end{aligned}$$

Continuing in a similar manner, we obtain

$$\begin{aligned}\mathfrak{d}^*(\mathfrak{a}_\mathfrak{p},\mathfrak{a}_\mathfrak{q}) &\leq \alpha(\mathfrak{a}_\mathfrak{p},\mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}_\mathfrak{p},\mathfrak{a}_{\mathfrak{p}+1}) \\ &\quad + \sum_{i=\mathfrak{p}+1}^{\mathfrak{q}-2}\prod_{j=\mathfrak{p}+1}^{i}\beta(\mathfrak{a}_j,\mathfrak{a}_\mathfrak{q})\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})\mathfrak{d}^*(\mathfrak{a}_i,\mathfrak{a}_{i+1}) + \prod_{k=\mathfrak{p}+1}^{\mathfrak{q}-1}\beta(\mathfrak{a}_k,\mathfrak{a}_\mathfrak{q})d(\mathfrak{a}_\mathfrak{q},\mathfrak{a}_{\mathfrak{q}-1}) \\ &\leq \alpha(\mathfrak{a}_\mathfrak{p},\mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}_\mathfrak{p},\mathfrak{a}_{\mathfrak{p}+1}) + \sum_{i=\mathfrak{p}+1}^{\mathfrak{q}-1}\prod_{j=\mathfrak{p}+1}^{i}\beta(\mathfrak{a}_j,\mathfrak{a}_\mathfrak{q})\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})\mathfrak{d}^*(\mathfrak{a}_i,\mathfrak{a}_{i+1}) \\ &\leq \alpha(\mathfrak{a}_\mathfrak{p},\mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}_\mathfrak{p},\mathfrak{a}_{\mathfrak{p}+1}) + \sum_{i=\mathfrak{p}+1}^{\mathfrak{q}-1}\prod_{j=0}^{i}\beta(\mathfrak{a}_j,\mathfrak{a}_\mathfrak{q})\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})\mathfrak{d}^*(\mathfrak{a}_i,\mathfrak{a}_{i+1}) \\ &\leq \alpha(\mathfrak{a}_\mathfrak{p},\mathfrak{a}_{\mathfrak{p}+1})(\theta^\mathfrak{p}+\mathfrak{p}\theta^\mathfrak{p})\,\mathfrak{d}^*(\mathfrak{a}_0,\mathfrak{a}_1) + \sum_{i=\mathfrak{p}+1}^{\mathfrak{q}-1}\prod_{j=0}^{i}\beta(\mathfrak{a}_j,\mathfrak{a}_\mathfrak{q})\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})\,(\theta^i+i\theta^i)\,\mathfrak{d}^*(\mathfrak{a}_0,\mathfrak{a}_1)\end{aligned}$$

Now, by ratio test the series $S_\mathfrak{q} = \sum_{i=\mathfrak{p}}^{\mathfrak{q}-1}(\theta^i + i\theta^i)\,(\prod_{j=0}^{i}\beta(\mathfrak{a}_j,\mathfrak{a}_\mathfrak{q}))\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})$ converges if

$$\sup_{\mathfrak{q}\geq i}\lim_{i\to\infty}\frac{\alpha(\mathfrak{a}_{i+1},\mathfrak{a}_{i+2}).\beta(\mathfrak{a}_{i+1},\mathfrak{a}_\mathfrak{q})}{\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})} < \frac{1}{\theta}.$$

Letting $\mathfrak{q},\mathfrak{p}\to\infty$, we obtain

$$\lim_{\mathfrak{p}\to\infty}\mathfrak{d}^*(\mathfrak{a}_\mathfrak{p},\mathfrak{a}_\mathfrak{q}) = 0.$$

This gives that $\{\mathfrak{a}_\mathfrak{p}\}$ is a Cauchy sequence in $\mathfrak{A}$. The completeness of $\mathfrak{A}$ implies that there exists an element $\mathfrak{a}^*$ in $\mathfrak{A}$ such that $\lim_{\mathfrak{p}\to\infty}\mathfrak{a}_\mathfrak{p} = \mathfrak{a}^*$. f is continuous according to the contraction condition and also $\mathfrak{a}_n \in f(\mathfrak{a}_{\mathfrak{p}-1})$, therefore $\mathfrak{a}^* \in f(\mathfrak{a}^*)$, i.e., $\mathfrak{a}^*$ is a fixed point of f. □

Next, following Abbas et al. [1], we define (λ,θ)-enriched multivalued contraction as well as enriched multivalued nonexpansive mappings in DCMSs.

Definition 11. *For $\mathfrak{a},\mathfrak{b}\in\mathfrak{A}, \lambda\in[0,\infty)$, and $\theta\in[0,\lambda+1)$, a multivalued mapping $f:\mathfrak{A}\to CB(\mathfrak{A})$ on DCMS $(\mathfrak{A},\mathfrak{d}^*,\alpha,\beta)$ is an*

(a) enriched multivalued contraction if

$$H(\lambda\mathfrak{a}+f(\mathfrak{a}),\ \lambda\mathfrak{b}+f(\mathfrak{b})) \leq \ \theta\,\mathfrak{d}^*(\mathfrak{a},\mathfrak{b}), \tag{2}$$

(b) enriched multivalued nonexpansive mapping if

$$H(\lambda\mathfrak{a}+f(\mathfrak{a}),\ \lambda\mathfrak{b}+f(\mathfrak{b})) \leq (\lambda+1)\,\mathfrak{d}^*(\mathfrak{a},\mathfrak{b}). \tag{3}$$

We also call f, defined in (a) and (b), a (λ,θ)-enriched multivalued contraction and a λ-enriched multivalued nonexpansive mapping in a DCMS, respectively to emphasize the involved constants.

Noticeably, for $\theta \in [0,1), (2)$ reduces to Nadler's [13] contraction. In other words, we can say that any multivalued contraction [13] is $(0,\theta)$-enriched multi-valued contraction. Also, each multivalued nonexpansive mapping is a $(0,\theta)$-enriched multivalued nonexpansive mapping.

Theorem 3.2. *Let $f : \mathfrak{A} \to CB(\mathfrak{A})$ be a (λ,θ)-enriched multivalued contraction defined on a DCMS $(\mathfrak{A},\mathfrak{d}^*,\alpha,\beta)$. Then,*
(i) $Fix(f) \neq \phi$,
(ii) there exists an f_ω-orbital sequence $\{\mathfrak{a}_\mathfrak{p}\}$ around the initial point $\mathfrak{a}_0 \in \mathfrak{A}$ converging to the fixed point $\mathfrak{a}^*$ of f, if

$$\sup_{\mathfrak{q}\geq i} \lim_{i\to\infty} \frac{\alpha(\mathfrak{a}_{i+1},\mathfrak{a}_{i+2}).\beta(\mathfrak{a}_{i+1},\mathfrak{a}_\mathfrak{q})}{\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})} < \frac{1}{\mu}. \tag{4}$$

where $\mu = \dfrac{a\theta}{\lambda+1}$ for $a \geq 1$, $\omega = \dfrac{1}{\lambda+1}$, $\lim_{\mathfrak{p}\to\infty}\alpha(\mathfrak{a},\mathfrak{a}_\mathfrak{p})$ exists and $\lim_{\mathfrak{p}\to\infty}\beta(\mathfrak{a},\mathfrak{a}_\mathfrak{p})$ exists. Provided that for $\mathfrak{a}_0 \in \mathfrak{A}$, the f_ω-orbital subset $O(f_\omega,\mathfrak{a}_0)$ is a complete subset of $\mathfrak{A}$.

Proof. We have two subsequent cases:

Case I : Take $\lambda > 0$ and $\omega = \dfrac{1}{\lambda+1}$. So, $0 < \omega < 1$ and

$$H\left(\left(\frac{1}{\omega}-1\right)\mathfrak{a}+f(\mathfrak{a}), \left(\frac{1}{\omega}-1\right)\mathfrak{b}+f(\mathfrak{b})\right) \leq \theta\ \mathfrak{d}^*(\mathfrak{a},\mathfrak{b})$$

$$H((1-\omega)\mathfrak{a}+\omega f(\mathfrak{a}),(1-\omega)\mathfrak{b}+\omega f(\mathfrak{b})) \leq \theta\ \omega\ \mathfrak{d}^*(\mathfrak{a},\mathfrak{b}).$$

Equivalently, for $\mathfrak{a},\mathfrak{b} \in \mathfrak{A}$,

$$H(f_\omega(\mathfrak{a}),\ f_\omega(\mathfrak{b})) \leq c\ \mathfrak{d}^*(\mathfrak{a},\mathfrak{b}),$$

where $c = \theta\omega$. So, $c \in (0,1)$ and hence f_ω is a multivalued contraction in DCMS, as defined by equation (1).

Let $\mathfrak{a}_0 \in X$ and $\mathfrak{a}_1 \in f_\omega(\mathfrak{a}_0)$, if

$$H(f_\omega(\mathfrak{a}_0), f_\omega(\mathfrak{a}_1)) = 0$$
$$\implies f_\omega(\mathfrak{a}_0) = f_\omega(\mathfrak{a}_1)$$
$$\implies \mathfrak{a}_1 \in f_\omega(\mathfrak{a}_1),$$

and hence, $Fix(f_\omega) \neq \phi$.

Suppose that $H(f_\omega(\mathfrak{a}_0), f_\omega(\mathfrak{a}_1)) \neq 0$, then by Lemma (2.3), there exists an $\mathfrak{a}_2 \in f_\omega(\mathfrak{a}_1)$ and $a \geq 1$, such that

$$\mathfrak{d}^*(\mathfrak{a}_1, \mathfrak{a}_2) \leq aH(f_\omega(\mathfrak{a}_0), f_\omega(\mathfrak{a}_1)) \leq ac\, \mathfrak{d}^*(\mathfrak{a}_0, \mathfrak{a}_1).$$

If $H(f_\omega(\mathfrak{a}_1), f_\omega(\mathfrak{a}_2)) = 0$, then $f_\omega(\mathfrak{a}_1) = f_\omega(\mathfrak{a}_2)$, which further implies that $\mathfrak{a}_2 \in f_\omega(\mathfrak{a}_2)$.

Suppose that $H(f_\omega(\mathfrak{a}_1), f_\omega(\mathfrak{a}_2)) \neq 0$. Using a similar argument as earlier, there exists an $\mathfrak{a}_3 \in f_\omega(\mathfrak{a}_2)$, such that

$$\mathfrak{d}^*(\mathfrak{a}_2, \mathfrak{a}_3) \leq ac(\mathfrak{d}^*(\mathfrak{a}_1, \mathfrak{a}_2)).$$

Repeating the above process, we reach at a sequence $\{\mathfrak{a}_\mathfrak{p}\}_{\mathfrak{p}\in\mathbb{N}}$, in $\mathfrak{A}$ with $\mathfrak{a}_{\mathfrak{p}+1} \in f_\omega(\mathfrak{a}_\mathfrak{p})$ such that

$$\mathfrak{d}^*(\mathfrak{a}_\mathfrak{p}, \mathfrak{a}_{\mathfrak{p}+1}) \leq ac\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}-1}, \mathfrak{a}_\mathfrak{p}), \ . \tag{5}$$

By applying the principle of induction, we have

$$\mathfrak{d}^*(\mathfrak{a}_\mathfrak{p}, \mathfrak{a}_{\mathfrak{p}+1}) \leq (ac)^\mathfrak{p}\mathfrak{d}^*(\mathfrak{a}_0, \mathfrak{a}_1),$$

and

$$\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+\mathfrak{q}}, \mathfrak{a}_{\mathfrak{p}+\mathfrak{q}+1}) \leq (ac)^{\mathfrak{q}+1}\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}-1}, \mathfrak{a}_\mathfrak{p}), \text{ where } \mathfrak{q} \in \mathbb{N}, \mathfrak{p} \geq 1.$$

We can choose $a \geq 1$ such that $\mu = ac < 1$, and hence we get

$$\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+\mathfrak{q}}, \mathfrak{a}_{\mathfrak{p}+\mathfrak{q}+1}) \leq \mu^{\mathfrak{q}+1}\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}-1}, \mathfrak{a}_\mathfrak{p}). \tag{6}$$

If $\mathfrak{q} > \mathfrak{p}$, we have

$$\mathfrak{d}^*(\mathfrak{a}_\mathfrak{p}, \mathfrak{a}_\mathfrak{q}) \leq \alpha(\mathfrak{a}_\mathfrak{p}, \mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}_\mathfrak{p}, \mathfrak{a}_{\mathfrak{p}+1}) + \beta(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_\mathfrak{q})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+1}, \mathfrak{a}_\mathfrak{q})$$

Again, using the triangle inequality on $\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+1},\mathfrak{a}_{\mathfrak{q}})$, we have

$$\begin{aligned}\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{q}}) \leq\ & \alpha(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{p}+1})+\beta(\mathfrak{a}_{\mathfrak{p}+1},\mathfrak{a}_{\mathfrak{q}})(\alpha(\mathfrak{a}_{\mathfrak{p}+1},\mathfrak{a}_{\mathfrak{p}+2})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+1},\mathfrak{a}_{\mathfrak{p}+2})\\ &+\beta(\mathfrak{a}_{\mathfrak{p}+2},\mathfrak{a}_{\mathfrak{q}})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+2},\mathfrak{a}_{\mathfrak{q}}))\\ &=\alpha(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{p}+1})+\beta(\mathfrak{a}_{\mathfrak{p}+1},\mathfrak{a}_{\mathfrak{q}}).\alpha(\mathfrak{a}_{\mathfrak{p}+1},\mathfrak{a}_{\mathfrak{p}+2})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+1},\mathfrak{a}_{\mathfrak{p}+2})\\ &+\beta(\mathfrak{a}_{\mathfrak{p}+1},\mathfrak{a}_{\mathfrak{q}}).\beta(\mathfrak{a}_{\mathfrak{p}+2},\mathfrak{a}_{\mathfrak{q}})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}+2},\mathfrak{a}_{\mathfrak{q}}))\end{aligned}$$

On application of the triangle inequality in a similar fashion, we achieve

$$\begin{aligned}\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{q}}) \leq\ & \alpha(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{p}+1})\\ &+\sum_{i=\mathfrak{p}+1}^{\mathfrak{q}-2}\prod_{j=\mathfrak{p}+1}^{i}\beta(\mathfrak{a}_j,\mathfrak{a}_{\mathfrak{q}})\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})\mathfrak{d}^*(\mathfrak{a}_i,\mathfrak{a}_{i+1})+\prod_{k=\mathfrak{p}+1}^{\mathfrak{q}-1}\beta(\mathfrak{a}_k,\mathfrak{a}_{\mathfrak{q}})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{q}},\mathfrak{a}_{\mathfrak{q}-1})\\ &\leq \alpha(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}_n,\mathfrak{a}_{\mathfrak{p}+1})+\sum_{i=\mathfrak{p}+1}^{\mathfrak{q}-1}\prod_{j=\mathfrak{p}+1}^{i}\beta(\mathfrak{a}_j,\mathfrak{a}_{\mathfrak{q}})\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})\mathfrak{d}^*(\mathfrak{a}_i,\mathfrak{a}_{i+1})\\ &\leq \alpha(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{p}+1})+\sum_{i=\mathfrak{p}+1}^{\mathfrak{q}-1}\prod_{j=0}^{i}\beta(\mathfrak{a}_j,\mathfrak{a}_{\mathfrak{q}})\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})\mathfrak{d}^*(\mathfrak{a}_i,\mathfrak{a}_{i+1})\\ &\leq \alpha(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{p}+1})\mu^{\mathfrak{p}}\,\mathfrak{d}^*(\mathfrak{a}_0,\mathfrak{a}_1)+\sum_{i=\mathfrak{p}+1}^{\mathfrak{q}-1}\prod_{j=0}^{i}\beta(\mathfrak{a}_j,\mathfrak{a}_{\mathfrak{q}})\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})\ \mu^i\ \mathfrak{d}^*(\mathfrak{a}_0,\mathfrak{a}_1)\end{aligned}$$

Now, by ratio test the series $S_{\mathfrak{q}}=\sum_{i=\mathfrak{p}}^{\mathfrak{q}-1}\ \mu^i\ (\prod_{j=0}^{i}\beta(\mathfrak{a}_j,\mathfrak{a}_{\mathfrak{q}}))\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})$ converges if

$$\sup_{\mathfrak{q}\geq i}\ \lim_{i\to\infty}\ \frac{\alpha(\mathfrak{a}_{i+1},\mathfrak{a}_{i+2}).\beta(\mathfrak{a}_{i+1},\mathfrak{a}_{\mathfrak{q}})}{\alpha(\mathfrak{a}_i,\mathfrak{a}_{i+1})}<\frac{1}{\mu}.$$

Letting $\mathfrak{q},\mathfrak{p}\to\infty$, we get

$$\lim_{\mathfrak{p}\to\infty}\mathfrak{d}^*(\mathfrak{a}_{\mathfrak{p}},\mathfrak{a}_{\mathfrak{q}})=0.$$

Thus, $\{\mathfrak{a}_{\mathfrak{p}}\}$ is a Cauchy sequence in the subset $O(f_\omega,\mathfrak{a}_0)$ of $\mathfrak{A}$. Suppose an element $\mathfrak{a}^*$ exists in $O(f_\omega,\mathfrak{a}_0)$ such that $\lim_{\mathfrak{p}\to\infty}\mathfrak{a}_{\mathfrak{p}}=\mathfrak{a}^*$. So

$$\begin{aligned}D(\mathfrak{a}^*,f_\omega(\mathfrak{a}^*)) &\leq \alpha(\mathfrak{a}^*,\mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}^*,\mathfrak{a}_{\mathfrak{p}+1})+\beta(\mathfrak{a}_{\mathfrak{p}+1},f_\omega(\mathfrak{a}^*))D(\mathfrak{a}_{\mathfrak{p}+1},f_\omega(\mathfrak{a}^*))\\ &\leq \alpha(\mathfrak{a}^*,\mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}^*,\mathfrak{a}_{\mathfrak{p}+1})+\beta(\mathfrak{a}_{\mathfrak{p}+1},f_\omega(\mathfrak{a}^*))H(f_\omega(\mathfrak{a}_{\mathfrak{p}}),f_\omega(\mathfrak{a}^*))\\ &\leq \alpha(\mathfrak{a}^*,\mathfrak{a}_{\mathfrak{p}+1})\mathfrak{d}^*(\mathfrak{a}^*,\mathfrak{a}_{\mathfrak{p}+1})+c\beta(\mathfrak{a}_{\mathfrak{p}+1},f_\omega(\mathfrak{a}^*))\mathfrak{d}^*(\mathfrak{a}^*,\mathfrak{a}_{\mathfrak{p}}).\end{aligned}$$

As $\mathfrak{p} \to \infty$, we obtain $D(\mathfrak{a}^*, f_\omega(\mathfrak{a}^*)) = 0$. Since $f_\omega(\mathfrak{a}^*)$ is closed, $\mathfrak{a}^* \in f_\omega(\mathfrak{a}^*)$, i.e., $\mathfrak{a}^*$ is a fixed point of f_ω. Hence, $\mathfrak{a}^*$ is a fixed point of f also as $Fix(f) = Fix(f_\omega)$.

Case II: When $\lambda = 0$, the enriched multivalued contraction becomes

$$H(f(\mathfrak{a}), f(\mathfrak{b})) \leq \theta \, \mathfrak{d}^*(\mathfrak{a}, \mathfrak{b}), \, \forall \, \mathfrak{a}, \mathfrak{b} \in \mathfrak{A},$$

where $\theta \in (0,1)$, which is a multivalued contraction in a DCMS. Hence, by Theorem 3.1 $Fix(f) \neq \phi$. □

Example 3.3. *Let $\mathfrak{A} = [0,1]$ with usual metric $\mathfrak{d}^*$ and define two non-comparable functions $\alpha, \beta : \mathfrak{A} \times \mathfrak{A} \to [1, \infty)$ as*

$$\alpha(\mathfrak{a}, \mathfrak{b}) = 1 + |\mathfrak{b}|, \; \beta(\mathfrak{a}, b) = 1 + |\mathfrak{a}|.$$

Then, $(\mathfrak{A}, \mathfrak{d}^*, \alpha, \beta)$ is a complete DCMS.

Define $f : \mathfrak{A} \to CB(\mathfrak{A})$ as: $f(x) = \{-\mathfrak{a}, 1 - \mathfrak{a}\}$, for all $\mathfrak{a} \in \mathfrak{A}$.

In particular, taking $\lambda = 1$, consider

$$\begin{aligned} H\{\mathfrak{a} + f(\mathfrak{a}), \mathfrak{b} + f(\mathfrak{b})\} &= H\{\mathfrak{a} + \{-\mathfrak{a}, 1 - \mathfrak{a}\}, \mathfrak{b} + \{-\mathfrak{b}, 1 - \mathfrak{b}\}\} \\ &= H\{\{0,1\}, \{0,1\}\} \\ &= 0 \\ &\leq \frac{1}{k} \mathfrak{d}^*(\mathfrak{a}, \mathfrak{b}) \text{ for all } \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}. \end{aligned}$$

Hence, f is $(1,1)$-enriched multivalued contraction as $\omega = \frac{1}{\lambda + 1}$, i.e., $\omega = \frac{1}{2}$.

Let $\mathfrak{a}_0 \in \mathfrak{A}$ be fixed, then we have

$$\begin{aligned} f_{\frac{1}{2}}(\mathfrak{a}_0) &= \frac{1}{2}\mathfrak{a}_0 + \frac{1}{2} f(\mathfrak{a}_0), \\ &= \frac{1}{2}(0) + \frac{1}{2}\{0,1\}, \\ &= \{0, \frac{1}{2}\} \end{aligned}$$

Let $\mathfrak{a}_1 = 0 \in f_{\frac{1}{2}}(\mathfrak{a}_0) \implies f_{\frac{1}{2}}(\mathfrak{a}_1) = \{0, \frac{1}{2}\}$.

Choose $\mathfrak{a}_2 = 0$ *where* $\mathfrak{a}_2 \in f_{\frac{1}{2}}(\mathfrak{a}_1)$. *Repeating this, we get* $\mathfrak{a}_{\mathfrak{p}+1} \in f_{\frac{1}{2}}(\mathfrak{a}_{\mathfrak{p}})$ *where* $\mathfrak{a}_{\mathfrak{p}+1} = 0$. *Thus, we reach at a sequence* $\{\mathfrak{a}_{\mathfrak{p}}\}$ *in* $O(f_\omega, \mathfrak{a}_0) = \{0, \frac{1}{2}, \frac{1}{4}, \ldots\}$, *converging to* 0, *and* $O(f_\omega, \mathfrak{a}_0)$ *is complete subset of* $(\mathfrak{A}, \mathfrak{d}^*, \alpha, \beta)$. *Hence,* 0 *is the fixed point of* f.

Corollary 3.4. *Let* $f : \mathfrak{A} \to CB(\mathfrak{A})$ *be a multivalued contraction on a complete DCMS* $(\mathfrak{A}, \mathfrak{d}^*, \alpha, \beta)$ *which satisfies (4), then* $Fix(f) \neq \phi$.

Proof. The proof follows by considering $\lambda = 0$ in Theorem 3.2. □

Corollary 3.5. *Let* $(\mathfrak{A}, \mathfrak{d}^*, \alpha, \beta)$ *be a complete DCMS. Define* $f : \mathfrak{A} \to \mathfrak{A}$ *as*

$$\mathfrak{d}^*(\lambda \mathfrak{a} + f(\mathfrak{a}),\ \lambda \mathfrak{b} + f(\mathfrak{b})) \leq \theta\ \mathfrak{d}^*(\mathfrak{a}, \mathfrak{b}) \text{ for all } \mathfrak{a}, \mathfrak{b} \in \mathfrak{A},$$

where $\lambda \in [0,1)$ *and* $\theta \in [0, \lambda + 1)$. *Then,* $Fix(f) \neq \phi$, *and the fixed point of* f *is unique.*

Noticeably, Corollary 3.5 is an extension of Berinde and Pacurar [4] to DCMS.

4. Application

Let $\sigma > 0$, $\mathfrak{R} = [0, \sigma]$ and $\mathfrak{A} = C(\mathfrak{R})$ denotes the collection of functions which are continuous and equipped with a supremum metric $\mathfrak{d}^*$. Taking $\alpha(\mathfrak{a}, \mathfrak{b}) = \beta(\mathfrak{a}, \mathfrak{b}) = 2$, for all $\mathfrak{a}, \mathfrak{b} \in C(\mathfrak{R})$, $(\mathfrak{A}, \mathfrak{d}^*, \alpha, \beta)$ is a complete DCMS. We find a solution for the differential inclusion problem:

$$\text{Solve for } \mathfrak{a} \in C(\mathfrak{R}), \text{ which satisfies } \mathfrak{a}'(t) \in \mathscr{F}(t, \mathfrak{a}(t)),\ t \in \mathfrak{R}, \tag{7}$$

here the mapping $\mathscr{F} : \mathfrak{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is multivalued and the collection of Lebesgue integrable functions is defined as $S_{\mathscr{F}}(\mathfrak{a}) = \{g \in L^1(\mathfrak{R}, \mathbb{R}) : g(t) \in \mathscr{F}(t, \mathfrak{a}(t))\}$, for $t \in \mathfrak{R}$.

Definition 12. [1]*The problem* (7) *is said to have a solution* $\mathfrak{a} \in C(\mathfrak{R})$, *if* $h \in S_{\mathscr{F}}(x)$ *and moreover* $\mathfrak{a}'(t) = h(t)$, *where* $t \in \mathfrak{R}$.

Theorem 4.1. *Suppose,*

(1) $S_{\mathscr{F}}(\mathfrak{a}) \neq \phi, \forall \mathfrak{a} \in C(\mathfrak{R})$;

(2) For every $\mathfrak{a} \in C(\mathfrak{R})$ *and for sequence* $\{h_{\mathfrak{p}}\} \in S_{\mathscr{F}}(\mathfrak{a})$, *we get a subsequence* $\{h_{\mathfrak{p}_j}\}$ *of* $\{h_{\mathfrak{p}}\}$ *which converges to a function* $h \in L^1(\mathfrak{R}, \mathbb{R})$ *as* $j \to \infty$, *for* $t \in \mathfrak{R}$ *and*

$$\int_0^t h_{\mathfrak{p}_j}(s)ds \to \int_0^t h(s)ds,\ as\ j \to \infty;$$

(3) $\mathscr{F}(t,\mathfrak{a})$ *is closed,* $(t,\mathfrak{a}) \in \mathfrak{R} \times C(\mathfrak{R})$*;*

(4) For all $\mathfrak{a} \in C(\mathfrak{R}), \mathscr{F}(.,\mathfrak{a}(.))$ *is bounded on* $\mathfrak{R}$*;*

(5) $\phi \in L^1(\mathfrak{R},\mathbb{R})$ *with* $\sup_{t\in\mathfrak{R}} \int_0^t |\phi(s)|ds < 1$*, and*

$$0 \leq \mathfrak{d}^*(h_\mathfrak{a}, h_\mathfrak{b}) \leq |\phi(t)|\, \mathfrak{d}^*(\mathfrak{a}, v),$$

where $t \in \mathfrak{R}$*,* $\mathfrak{a},\mathfrak{b} \in C(\mathfrak{R})$*,* $h_\mathfrak{a} \in S_F(\mathfrak{b}), h_v \in S_F(v)$*.*

Then, a solution exists for the differential inclusion problem (7).

Proof. The multivalued mapping f on $\mathfrak{A}$ is defined as:

$$f(\mathfrak{a}) = \{g \in \mathfrak{A} : g(t) = \int_0^t h(s)ds,\ t \in \mathfrak{R},\ h \in S_{\mathscr{F}}(\mathfrak{a})\}. \tag{8}$$

As $S_{\mathscr{F}}(\mathfrak{a}) \neq \phi$, for $\mathfrak{a} \in \mathfrak{A}$, therefore f is a well-defined mapping.

If $\mathfrak{a} \in f(\mathfrak{a})$, then $\mathfrak{a}(t) = \int_0^t h(s)ds$, i.e.

$$\mathfrak{a}'(t) = h(t), t \in \mathfrak{R}.$$

As a result problem (7) is comparable to the subsequent inclusion,

$$\mathfrak{a}(t) \in f(\mathfrak{a}(t)),\ t \in \mathfrak{R}.$$

Now, we demonstrate that the postulates of Theorem 3.2 are satisfied by the multivalued mapping f. Also, $f(\mathfrak{a})$ is non-empty.

Claim: $f(\mathfrak{a}) \in CB(\mathfrak{A})$.

We pick a fixed $\mathfrak{a} \in \mathfrak{A}$ and $\{h_\mathfrak{p}\}$ be a sequence in $f(\mathfrak{a})$, so that $h_\mathfrak{p}$ converges to $h \in X$ as $\mathfrak{p} \to \infty$. Consequently, we obtain a subsequence $\{h_\mathfrak{p}\}$ in $S_{\mathscr{F}}(\mathfrak{a})$ such that

$$g_\mathfrak{p}(t) = \int_0^t h_\mathfrak{p}(s)ds,\ t \in \mathfrak{R}.$$

Based on the given assumption, we obtain a subsequence $\{h_{\mathfrak{p}_k}\}$ of $\{h_\mathfrak{p}\}$, so that $\{h_{\mathfrak{p}_k}\} \to h \in L^1(\mathfrak{R},\mathbb{R})$ as $i \to \infty$, for $t \in \mathfrak{R}$ and

$$\int_0^t h_{\mathfrak{p}_k}(s)ds \to \int_0^t h_\mathfrak{p}(s)ds,\ \text{as } k \to \infty.$$

As $\mathscr{F}(t,\mathfrak{a}(t))$ is closed, $t \in \mathfrak{R}$, *and* $h(t) \in \mathscr{F}(t,x(t))$. So $h \in S_{\mathscr{F}}(\mathfrak{a})$. Noticeably

$$f(t) = \lim_{\mathfrak{p}\to\infty} g_\mathfrak{p}(t) = \lim_{\mathfrak{p}\to\infty} \int_0^t h_\mathfrak{p} ds = \int_0^t h_{\mathfrak{p}_k}(s)ds = \int_0^t h(s)ds,\ k \to \infty.$$

Hence, $h \in f(\mathfrak{a})$, i.e., $f(\mathfrak{a})$ is closed. Also, for $\mathfrak{a} \in \mathfrak{A}$, $\mathscr{F}(.,\mathfrak{a}(.))$ is bounded on $\mathfrak{R}$. There exists $\mathfrak{q} > 0$ satisfying $|h(t)| \leq \mathfrak{q}$, for $t \in \mathfrak{R}$, and $h \in S_{\mathscr{F}}(\mathfrak{a})$. Furthermore, $h(t) \in \mathscr{F}(t,\mathfrak{a}(t))$. Consequently, $h \in f(\mathfrak{a})$, and

$$\sup_{t\in\mathfrak{R}}|g(t)| \leq \sup_{t\in\mathfrak{R}} \int_0^t |h(s)ds| \leq \mathfrak{q}a.$$

Therefore, f is bounded as well, from which we conclude that $f : \mathfrak{A} \to CB(\mathfrak{A})$.

Now, we will show that f is an enriched multivalued contraction. Suppose, $\lambda > 0$ is fixed and $A = \lambda\mathfrak{a} + f(\mathfrak{a}), B = \lambda\mathfrak{b} + f(\mathfrak{b})$ for fixed $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{A}$. So

$$H(\lambda\mathfrak{a}+f(\mathfrak{a}),\ \lambda\mathfrak{b}+f(\mathfrak{b})) = max\{D(\lambda\mathfrak{a}+f(\mathfrak{a}),\ \lambda\mathfrak{b}+f(\mathfrak{b})), D(\lambda\mathfrak{b}+f(b),\ \lambda\mathfrak{a}+f(\mathfrak{a}))\}, \tag{9}$$

where,

$$\begin{aligned} D(\lambda\mathfrak{a}+f(\mathfrak{a}),\lambda\mathfrak{b}+f(\mathfrak{b})) &= \sup_{g_{\mathfrak{a}}\in A}\ \inf_{g_{\mathfrak{b}}\in B} \mathfrak{d}^*(g_{\mathfrak{a}}, g_{\mathfrak{b}}) \\ &= \sup_{g_{\mathfrak{a}}\in A}\ \inf_{g_{\mathfrak{b}}\in B} \sup_{t\in\mathfrak{R}} \mathfrak{d}_2^*(g_{\mathfrak{a}(t)}, g_{\mathfrak{b}(t)}),\ \text{(Using Definition 5)} \\ &= \sup_{g_{\mathfrak{a}}\in A}\ \inf_{g_{\mathfrak{b}}\in B}\ \sup_{t\in\mathfrak{R}} \mathfrak{d}_2^*(\lambda\mathfrak{a}(t) + \int_0^t h_{\mathfrak{a}}(s)ds, \lambda\mathfrak{b}(t) + \int_0^t h_{\mathfrak{b}}(s)ds). \end{aligned}$$

Now, as

$$\begin{aligned} \mathfrak{d}_2^*(\lambda\mathfrak{a}(t) + \int_0^t h_{\mathfrak{a}}(s)ds,\ \lambda\mathfrak{b}(t) + \int_0^t h_{\mathfrak{b}}(s)ds) &\leq \mathfrak{d}_2^*(\lambda\mathfrak{a}(t), \lambda\mathfrak{b}(t)) + \mathfrak{d}_2^*(\int_0^t h_{\mathfrak{a}}(s)ds, \int_0^t h_{\mathfrak{b}}(s)ds) \\ &= |\lambda|\mathfrak{d}_2^*(\mathfrak{a},\mathfrak{b}) + \int_0^t \mathfrak{d}_2^*(h_{\mathfrak{a}}, h_{\mathfrak{b}})ds \\ &\leq \lambda\mathfrak{d}_2^*(\mathfrak{a},\mathfrak{b}) + \int_0^t |\phi(s)| d_2(\mathfrak{a}(s), \mathfrak{b}(s))ds \\ &\leq \mathfrak{d}_2^*(\mathfrak{a},\mathfrak{b})(\lambda + \int_0^t \phi(s)ds). \end{aligned}$$

Hence,

$$\mathfrak{d}_2^*(\lambda\mathfrak{a}(t) + \int_0^t h_{\mathfrak{a}}(s)ds,\ \lambda\mathfrak{b}(t) + \int_0^t h_{\mathfrak{b}}(s)ds) \leq (\lambda + \int_0^t |\phi(s)|ds) d_2(\mathfrak{a},\mathfrak{b}).$$

Utilizing the above inequalities, we obtain

$$\begin{aligned} D(\lambda\mathfrak{a}+f(\mathfrak{a}),\ \lambda\mathfrak{b}+f(\mathfrak{b})) &\leq \sup_{g_{\mathfrak{a}}\in A}\ \inf_{g_{\mathfrak{b}}\in B}\ \sup_{t\in\mathfrak{R}} (\lambda + \int_0^t |\phi(s)|ds) d_2(\mathfrak{a},\mathfrak{b}) \\ &< (\lambda+1)\mathfrak{d}^*(\mathfrak{a},\mathfrak{b}). \end{aligned}$$

That is,

$$D(\lambda\mathfrak{a}+f(\mathfrak{a}),\ \lambda\mathfrak{b}+f(\mathfrak{b}))\leq\ \theta\ \mathfrak{d}^*(\mathfrak{a},\mathfrak{b}).$$

Similarly,

$$D(\lambda\mathfrak{b}+f(\mathfrak{b}),\ \lambda\mathfrak{a}+f(\mathfrak{a}))\leq\theta\mathfrak{d}^*(\mathfrak{a},\mathfrak{b}).$$

So, (9) becomes

$$H((\lambda\mathfrak{a}+f(\mathfrak{a}),\ \lambda\mathfrak{b}+f(\mathfrak{b}))\leq\theta\mathfrak{d}^*(\mathfrak{a},\mathfrak{b}).$$

Hence, all the postulates of Theorem 3.2, if $\theta<\frac{1}{4}$, are validated. Therefore, the given problem has a solution. □

Acknowledgement

The first author would like to thank UGC, New Delhi, India, for financial support.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] M. Abbas, R. Anjum, V. Berinde, Enriched multivalued contractions with applications to differential inclusions and dynamic programming, Symmetry. 13 (2021), 1350. https://doi.org/10.3390/sym13081350.

[2] T. Abdeljawad, N. Mlaiki, H. Aydi, N. Souayah, Double controlled metric type spaces and some fixed point results, Mathematics. 6 (2018), 320. https://doi.org/10.3390/math6120320.

[3] S. Banach, Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales, Fund. Math. 3 (1922), 133–181.

[4] V. Berinde, M. Pacurar, Approximating fixed points of enriched contractions in Banach spaces, J. Fixed Point Theory Appl. 22 (2020), 38. https://doi.org/10.1007/s11784-020-0769-9.

[5] V. Berinde, Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces, Carpathian J. Math. 35 (2019), 293–304.

[6] N. Bourbaki, Topologie generale, Herman, Paris, 1974.

[7] L.B. Ćirić, Fixed point theory: contraction mapping principle, FME Press, Beograd, Serbia, 2003.

[8] L.B. Ćirić, Generalized contractions and fixed-point theorems, Publ. Inst. Math. 12 (1971), 19–26.

[9] S. Czerwik, Contraction mappings in *b*-metric spaces, Acta Math. Inform. Univ. Ostr. 1 (1993), 5–11. https://eudml.org/doc/23748.

[10] A. Gangwar, S. Rawat, R.C. Dimri, Solution of differential inclusion problem in controlled S-metric spaces via new multivalued fixed point theorem, J. Anal. 31 (2023), 2459–2472. https://doi.org/10.1007/s41478-023-00574-7.

[11] T. Kamran, M. Samreen, Q. UL Ain, A generalization of *b*-metric space and some fixed point theorems, Mathematics. 5 (2017), 19. https://doi.org/10.3390/math5020019.

[12] N. Mlaiki, H. Aydi, N. Souayah, et al. Controlled metric type spaces and the related contraction principle, Mathematics. 6 (2018), 194. https://doi.org/10.3390/math6100194.

[13] S.B. Nadler, Jr., Multi-valued contraction mappings, Pac. J. Math. 30 (1969), 415–487.

[14] S. Rawat, R.C. Dimri, A. Bartwal, F-bipolar metric spaces and fixed point theorems with applications, J. Math. Computer Sci. 26 (2021), 184–195. https://doi.org/10.22436/jmcs.026.02.08.

[15] S. Rawat, A. Bartwal, R.C. Dimri, Approximation and existence of fixed points via interpolative enriched contractions, Filomat. 37 (2023), 5455–5467. https://doi.org/10.2298/fil2316455r.

[16] I.A. Rus, Principles and applications of the fixed point theory, Editura Dacia, Cluj-Napoca, Romania, 1979.

[17] I.A. Rus, Basic problems of the metric fixed point theory revisited (II), Stud. Univ. Babes-Bolyai Math. 36 (1991), 81–99.

[18] W.A. Sutherland, Introduction to metric and topological spaces, Clarendon Press, Oxford, 1975.